Symmetry Properties of Generalized Gas Dynamics Equations

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Abstract
We describe a class of generalized gas dynamics equations invariant under the extended Galilei algebra $A\tilde{G}(1, n)$.

Symmetry properties of the gas dynamics equations

\[
\begin{align*}
\ddot{u}_0 + (\ddot{u} \cdot \vec{\nabla}) \ddot{u} + \frac{1}{\rho} \vec{\nabla} p &= 0, \\
\rho_0 + \text{div} (\rho \ddot{u}) &= 0, \\
p &= f(\rho),
\end{align*}
\]

where $\ddot{u}$ is the velocity, $\rho$ is the density, $p$ is the pressure of gas, were investigated in [1].

As has been shown in [1], the system (1) has the widest symmetry when $f(\rho) = \lambda \rho^{\frac{n+2}{n}}$ ($\lambda = \text{const}, n$ is the quantity of space variables $\vec{x} \in R_n$). In this case a basis of the maximum invariant algebra of the equation (1) is represented by the operators

\[
\begin{align*}
\partial_0, \partial_a, J_{ab} &= x_a \partial_b - x_b \partial_a + u^a \partial_{u^b} - u^b \partial_{u^a}, \\
D_1 &= x_0 \partial_0 + x_a \partial_a, \quad D_2 = x_0 \partial_0 - n \rho \partial_\rho - u^a \partial_{u^a}, \\
G_a &= x_0 \partial_a + \partial_{u^a}, \quad \Pi = x_0 (x_0 \partial_0 + x_a \partial_a - n \rho \partial_\rho - u^a \partial_{u^a}),
\end{align*}
\]

where $a, b = 1, n$.

We shall call the algebra (2) the extended Galilean algebra and designate it by $A\tilde{G}(1, n)$.

Other models of gas conduct are well-known except the system (1) (see, for example, [2]). Usually the first and second equations of the system (1) are invariables and the third equation has any form. For this reason we have the problem of finding the function

\[
S = S (x_0, \vec{x}, \ddot{u}, \rho, p, \ddot{u}_0, \ddot{u}_a, \rho_0, \rho_a, p_0, p_a),
\]

where $a = 1, n, \ddot{u}_a = \{ \ddot{u}_1, \ddot{u}_2, \ldots, \ddot{u}_n \}$, $\rho_a = \vec{\nabla} \rho, p_a = \vec{\nabla} p, \ddot{u}_a = \frac{\partial \ddot{u}}{\partial x_a}$, with which the system

\[
\begin{align*}
\ddot{u}_0 + (\ddot{u} \cdot \vec{\nabla}) \ddot{u} + \frac{1}{\rho} \vec{\nabla} p &= 0, \\
\rho_0 + \text{div} (\rho \ddot{u}) &= 0, \\
S &= 0
\end{align*}
\]
is invariant with respect to the algebra $A\tilde{G}(1, n)$. It follows from the invariance with respect to the operators $\partial_0, \partial_a$ that this system has the form

$$
\begin{align*}
\vec{u}_0 + (\vec{u} \cdot \vec{\nabla}) \vec{u} + \frac{1}{\rho} \vec{\nabla} p &= 0, \\
\rho_0 + \text{div} (\rho \vec{u}) &= 0, \\
\rho_0 + F (\vec{u}, \rho, p, \vec{u}_a, \rho_a, p_a) &= 0.
\end{align*}
$$

The infinitesimal operator of the algebra $A\tilde{G}(1, n)$ has the following form

$$
X = d\mu \partial_\mu + c_{ab} J_{ab} + g_a G_a + \kappa_1 D_1 + \kappa_2 D_2 + a\Pi + \eta (x_0, \vec{x}, \vec{u}, \rho, p) \partial_\rho
$$

Using invariance of the first equation of the system (5) with respect to the operator (6), we obtain

$$
\eta = - (n + 2) (ax_0 + \kappa_2) p.
$$

This means that all operators of the algebra (2) must have such a form except

$$
D_2' = D_2 - (n + 2)p\partial_p, \quad \Pi' = \Pi - (n + 2)x_0 p\partial_p.
$$

Demanding that the third equation of the system (5) be invariant with respect to the Galilean operators, we obtain

$$
\frac{\partial F}{\partial \vec{u}^a} = p_a.
$$

Hence

$$
F = (\vec{u} \vec{\nabla}) p + \Phi (\rho, p, \vec{u}_a, \rho_a, p_a).
$$

We assume that

$$
\Phi = \Phi (\rho, p, \vec{\nabla} \vec{u}, \vec{\nabla} \rho, \vec{\nabla} p).
$$

It follows from invariance with respect to the rotations $J_{ab}$ that

$$
\Phi = \Phi (\rho, p, w_1, w_2, w_3, w_4),
$$

where

$$
w_1 = \vec{\nabla} \vec{u}, w_2 = (\vec{\nabla} \rho)^2, w_3 = \left(\vec{\nabla} \rho\right) \left(\vec{\nabla} p\right), w_4 = \left(\vec{\nabla} p\right)^2.
$$

Substituting (10)–(13) in the third equation of the system (5), we have

$$
p_0 + (\vec{u} \cdot \vec{\nabla}) p + \Phi (\rho, p, w_1, w_2, w_3, w_4) = 0.
$$

Now let us consider the invariance of the equation (14) with respect to the operators $D_1, D_2', \Pi'$. For each of them we obtain the equation for the function $\Phi$

$$
\begin{align*}
D_1 : & \quad w_1 \Phi_1 + 2w_2 \Phi_2 + 2w_3 \Phi_3 + 2w_4 \Phi_4 - \Phi = 0, \\
D_2' : & \quad n \rho \Phi_p + (n + 2)p \Phi_p + w_1 \Phi_1 + 2nw_2 \Phi_2 + 2(n + 1)w_3 \Phi_3 + 2(n + 2)w_4 \Phi_4 - (n + 3)\Phi = 0; \\
\Pi' : & \quad n \rho \Phi_p + (n + 2)p \Phi_p + 2w_1 \Phi_1 + 2(n + 1)w_2 \Phi_2 + 2(n + 2)w_3 \Phi_3 + 2(n + 3)w_4 \Phi_4 - (n + 4)\Phi + n \Phi_1 - (n + 2)p = 0.
\end{align*}
$$
The function
\[ \Phi = \frac{n+2}{n} p \text{div} \vec{u} - |\vec{\nabla} p| p^{\frac{1}{n+2}} H \left( \rho p^{\frac{2}{n+2}}, \frac{\vec{\nabla}\rho\vec{\nabla}p}{\vec{\nabla}\rho}, \frac{|\vec{\nabla} p|}{\vec{\nabla}\rho}, p^{\frac{2}{n+2}} \right) \] (16)

is a general solution of the system (15).

Thus we have proved the following

**Theorem.** The system (5) is invariant with respect to the extended Galilean algebra \( A\tilde{G}(1,n) \) (2), (8) when it has the form

\[
\begin{align*}
\tilde{u}_0 + (\tilde{u} \cdot \vec{\nabla})\tilde{u} + \frac{1}{H} \vec{\nabla} p &= 0, \\
\rho_0 + \text{div} (\rho \tilde{u}) &= 0, \\
p_0 + (\tilde{u} \cdot \vec{\nabla}) p + \frac{n+2}{n} p \text{div} \tilde{u} &= |\vec{\nabla} p| p^{\frac{1}{n+2}} H \left( \rho p^{\frac{2}{n+2}}, \frac{\vec{\nabla}\rho\vec{\nabla}p}{\vec{\nabla}\rho}, \frac{|\vec{\nabla} p|}{\vec{\nabla}\rho}, p^{\frac{2}{n+2}} \right)
\end{align*}
\] (17)

where \( H \) is an arbitrary smooth function.

**Notation 1.** At \( H = 0 \) the result of the theorem is the same as one obtained by Ovsyannikov in [2].

**Notation 2.** By substitution

\[ p = \lambda \frac{n}{n+2} P^{\frac{n+2}{n}}, \quad \lambda = \text{const} \] (18)

the system (17) reduces to

\[
\begin{align*}
\tilde{u}_0 + (\tilde{u} \cdot \vec{\nabla})\tilde{u} + \frac{1}{H} P^{\frac{n}{n+2}} \vec{\nabla} P &= 0, \\
\rho_0 + \text{div} (\rho \tilde{u}) &= 0, \\
P_0 + \text{div} (P \tilde{u}) &= P^{\frac{n}{n+2}} |\vec{\nabla} P| f \left( P^{\frac{n}{n+2}}, \frac{\vec{\nabla}\rho\vec{\nabla}P}{\vec{\nabla}\rho}, \frac{|\vec{\nabla} P|}{\vec{\nabla}\rho} \right).
\end{align*}
\] (19)

**References**
