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Reduction and Some Exact Solutions of the Multidimensional Liouville Equation

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Abstract

Exact solutions of the multidimensional Liouville equation are constructed.

Let us consider the multidimensional Liouville equation

$$\Box u + \lambda \exp u = 0, \tag{1}$$

where $u = u(x)$ is a scalar function of the variable $x = (x_1, x_2, \ldots, x_{n+2})$, $\Box u = u_{11} + u_{22} - u_{33} - \cdots - u_{n+2,n+2}$, $u_{ab} = \dfrac{\partial^2 u}{\partial x_a \partial x_b}$; $a, b = 1, 2, \ldots, n+2$. The equation(1) is invariant under the extended Poincaré algebra $A\tilde{P}(2,n)$, which is generated by the following vector fields

$$P_a = \partial_a, \quad J_{ab} = g^{ac} x_c \partial_b - g^{bc} x_c \partial_a, \quad D = -x^a \partial_a + 2\partial_u,$$

where $\partial_a \equiv \dfrac{\partial}{\partial x_a}$, $\partial_u \equiv \dfrac{\partial}{\partial u}$, $g_{11} = g_{22} = -g_{33} = \cdots = -g_{n+2,n+2} = 1$, $g_{ab} = 0$ when $a \neq b$; $a, b = 1, 2, \ldots, n+2$. Using subalgebras of the rank 3 of the algebra $A\tilde{P}(2,2)$ in the paper [1] the symmetry ansatzes reducing the equation (1) to ordinary differential equations are built. With the help of the reducing equations some classes of exact solutions of equation (1) are constructed.

In the present paper classes of exact solutions of equation (1) are constructed for arbitrary n.

Let us consider the symmetry ansatz $u = u(\omega_1, \omega_2, \omega_3)$, where $\omega_1 = x_1 - x_4$, $\omega_2 = x_1^2 + x_2^2 - x_3^2 - \cdots - x_{n+2}^2$, $\omega_3 = (x_1 - x_4)(x_2 - x_3)^{-1}$. The ansatz reduces the equation (1) to the equation

$$4\omega_1 u_{12} + 4\omega_2 u_{22} + 2(n+2)u_2 + \lambda \exp u = 0. \tag{2}$$

Let us investigate symmetry of the equation (2).

Theorem 1 *The maximal invariance algebra of the equation (2) in Lie sense is infinitely-dimensional Lie algebra $A_\infty(3)$, which is generated by such operators $\alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3$, where*

$$X_1 = \omega_1\frac{\partial}{\partial\omega_1} + \omega_2\frac{\partial}{\partial\omega_2} - \frac{\partial}{\partial u}, \quad X_2 = \omega_2\frac{\partial}{\partial\omega_2} - \frac{\partial}{\partial u}, \quad X_3 = \omega_1\frac{\partial}{\partial\omega_2},$$

$\alpha_1, \alpha_2, \alpha_3$ *are arbitrary smooth functions of the variable* ω_3.

Let us consider the 3-dimensional subalgebra $A(3) =< X_1, X_2, X_3 >$ of the algebra $A_\infty(3)$ and carry out reduction of the equation (1) by 1-dimensional subalgebras of the algebra $A(3)$. Let L be an one-dimensional subalgebra of the algebra $A(3)$. Then L is conjugated to one of the following algebras: $L_1 =< X_1 + \alpha X_2 >$ $(\alpha \in R)$, $L_2 =< X_2 >$, $L_3 =< X_1 + \varepsilon X_3 >$ $(\varepsilon = \pm 1)$, $L_4 =< X_3 >$.

1) Subalgebra $L_1 =< X_1 + \alpha X_2 >$ $(\alpha \in R)$. The ansatz $u = \varphi(\omega) - \ln \omega_1^{\alpha+1}$, $\omega = \omega_2\omega_1^{-\alpha-1}$ reduces equation (2) to

$$4\alpha\omega\ddot{\varphi} + 2(n+2)\dot{\varphi} + \lambda \exp\varphi = 0.$$

2) Subalgebra $L_3 =< X_1 + \varepsilon X_3 >$ $(\varepsilon = \pm 1)$. The ansatz $u = \varphi(\omega) - \ln\omega_1$, $\omega = \frac{\omega_2}{\omega_1} - \varepsilon\ln\omega_1$ reduces equation (2) to

$$-4\varepsilon\ddot{\varphi} + 2n\dot{\varphi} + \lambda\exp\varphi = 0.$$

Now let us consider a symmetry ansatz $u = u(x_1 - x_{n+2}, x_2, \dots, x_{n+1})$. The ansatz reduces the equation (1) to the equation

$$\frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_3^2} - \dots - \frac{\partial^2 u}{\partial x_{n+1}^2} + \lambda\exp u = 0. \tag{3}$$

Theorem 2 *Algebra of invariance of the equation (3) in Lie sense is infinitely-dimensional Lie algebra* $AP_\infty(1, n-1)$ *which is generated by such operators*

$$\alpha_2 P_2 + \dots + \alpha_{n+1}P_{n+1} + \sum_{a,b}\beta_{a,b}J_{a,b} + \gamma D,$$

where

$$P_a \equiv \frac{\partial}{\partial x_a}, \qquad J_{2a} = x_2\frac{\partial}{\partial x_a} + x_a\frac{\partial}{\partial x_2},$$
$$J_{ab} = x_b\frac{\partial}{\partial x_a} - x_a\frac{\partial}{\partial x_b}, \qquad D = -x_2\frac{\partial}{\partial x_2} - \dots - x_{n+1}\frac{\partial}{\partial x_{n+1}} + 2\frac{\partial}{\partial u},$$

$a, b = 3, \dots, n+1$; $\alpha_2, \dots, \alpha_{n+1}, \beta_{a,b}, \gamma$ *are arbitrary smooth functions of the variable* $x_1 - x_{n+2}$.

Let us carry out reduction of the equation (3) on subalgebras of the rank $n-1$ of the algebra $AP(1, n-1) =< P_2, \dots, P_{n+1}, J_{23}, \dots, J_{n,n+1}, D >$. We write out the ansatzes corresponding to such subalgebras.

1) $u = \varphi(\omega) - 2\ln(x_1 + x_{n+1}),\ \omega = \dfrac{x_1}{x_{n+1}};$

2) $u = \varphi(\omega) + \dfrac{2\alpha}{1-\alpha}\ln(x_1 + x_{n+1}),$

$\omega = (1+\alpha)\ln(x_1 + x_{n+1}) + (1-\alpha)\ln(x_1 - x_{n+1});$

3) $u = \varphi(\omega) - \ln(x_1 - x_{n+1}), \quad \omega = x_1 + x_{n+1} + \ln(x_1 - x_{n+1});$

4) $u = \varphi(\omega) - 2\ln x_1, \quad \omega = \dfrac{x_3^2 + \cdots + x_m^2}{x_2^2}, \quad m = 3, \ldots, n+1;$

5) $u = \varphi(\omega) - 2\ln(x_1 - x_{n+1}),\ \omega = \dfrac{(x_3^2 - \cdots - x_m^2 - x_{n+1}^2)^{1/2}}{x_2 - x_{n+1}},\ m = 3, \ldots, n;$

6) $u = -\ln\left[-x_1^2 + \dfrac{x_1 - x_{n+1}}{x_1 - x_{n+1} + \lambda_3}x_3^2 + \dfrac{x_1 - x_{n+1}}{x_1 - x_{n+1} + \lambda_t}x_t^2 + x_{n+1}^2\right] + \varphi(\omega),$

where $t \le n, \lambda_3, \ldots, \lambda_t \in R$.

Using solutions of reduced equations we find exact solutions of the Liouville equation. Let us adduce some of them

$$u = -\ln\left\{-\frac{1}{2m\theta}\left[x_2 - x_{n+1} - \theta(x_3^2 - \cdots - x_m^2 - x_{n+1}^2)\right]\right\},$$

where θ is an arbitrary twice differentiable function of the variable $x_1 - x_{n+2}$; $m = 3, \ldots, n$;

$$u = \ln\frac{1 + \dfrac{\omega}{\omega + \lambda_3} + \cdots + \dfrac{\omega}{\omega + \lambda_t}}{\lambda\left[-x_2^2 + \dfrac{\omega}{\omega + \lambda_3}\,x_3^2 + \cdots + \dfrac{\omega}{\omega + \lambda_t}\,x_t^2 + x_{n+1}^2\right]},$$

where $\omega = x_2 - x_{n+1}$, $\lambda_3, \ldots, \lambda_t$ are arbitrary functions of the variable $x_1 - x_{n+2}$; $t \le n$.

References

[1] Fushchych W.I., Barannyk L.F. and Barannyk A.F., Subgroup Analysis of the Galilei and Poincaré Groups and Reduction of Nonlinear Equations, Kiev, Naukova Dumka, 1991, 304p.