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Reduction and Exact Solutions of the Monge-Ampere Equation

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Abstract

On the basis of a subgroup structure of the Poincaré group $P(1,3)$ the ansatzes which reduce the Monge–Ampere equation to differential equations with fewer independent variables have been constructed. The corresponding symmetry reduction has been done. By means of the solutions of the reduced equations some classes of exact solutions of the investigated equation have been found.

The Monge-Ampere equation in different-dimensional spaces is widely applicable [1–5]. In [6, 7] the symmetry properties of the multidimensional Monge-Ampere equation have been studied. In the same papers, multiparameter families of exact solutions of the equation have been constructed, by means of special ansatzes.

Let us consider the equation:

$$\det(u_{\mu\nu}) = 0, \tag{1}$$

where $u = u(x)$, $x = (x_0, x_1, x_2) \in \mathbf{R}_3$, $u_{\mu\nu} \equiv \dfrac{\partial^2 u}{\partial x_\mu \partial x_\nu}$, $\mu, \nu = 0, 1, 2$.

The invariance group [6, 7] of the equation (1) includes the Poincaré group $P(1,3)$ as a subgroup. Using the invariants [8, 9] of the subgroups of the group $P(1,3)$, we construct ansatzes which reduce the equation (1) to differential equations with fewer independent variables. The corresponding symmetry reduction has been done. Using the solutions of the reduced equations, we have found some classes of exact solutions of the Monge-Ampere equation.

1. Below we present ansatzes which reduce the equation (1) to ordinary differential equation (ODE), and we list the ODEs obtained as well as some exact solutions of the Monge-Ampere equation.

1. $u^2 = \varphi^2(\omega) - x_2^2$, $\omega = \left(x_0^2 - x_1^2\right)^{1/2}$, $\varphi'' = 0$,
   $u = \left(\left(C_1 \left(x_0^2 - x_1^2\right)^{1/2} + C_2\right)^2 - x_2^2\right)^{1/2}$.

2. $u = \varphi(\omega)$, $\omega = \left(x_0^2 - x_1^2 - x_2^2\right)^{1/2}$, $\varphi'' = 0$, $u = C_1 \left(x_0^2 - x_1^2 - x_2^2\right)^{1/2} + C_2$.

3. $u^2 = \varphi^2(\omega) - x_1^2 - x_2^2$, $\omega = x_0$, $\varphi'' = 0$, $u = \left((C_1 x_0 + C_2)^2 - x_1^2 - x_2^2\right)^{1/2}$.

4. $u^2 = \varphi(\omega) + \dfrac{x_0 + x_1 - 1}{x_0 + x_1}\left(x_0^2 - x_1^2 - x_2^2\right)$, $\omega = x_0 + x_1$,

$$\left(\omega^4-2\omega^3+\omega^2\right)\varphi''-2\left(2\omega^3-3\omega^2+\omega\right)\varphi'+2\left(3\omega^2-3\omega+1\right)\varphi=0,$$

$$u=\left(\frac{x_0+x_1-1}{x_0+x_1}\left(x_0^2-x_1^2-x_2^2\right)+(x_0+x_1)\left(C_1(x_0+x_1)^2+C_2(x_0+x_1)-C_1-C_2\right)\right)^{1/2}.$$

The ansatzes (1)–(4) reduce the equation (1) to the linear ODEs.

5. $u^2=-\varphi^2(\omega)+x_0^2-x_1^2-x_2^2,\ \ \omega=x_0+x_1,\ \ \omega^2\left(\varphi''\varphi+\varphi'^2\right)-2\omega\varphi'\varphi+\varphi^2=0,$

$$u=\left(x_0^2-x_1^2-x_2^2-\left(C_1\left(x_0+x_1\right)^{1/2}+C_2\left(x_0+x_1\right)\right)^2\right)^{1/2}.$$

6. $u^2=-\varphi^2(\omega)+x_0^2-x_1^2,\ \ \omega=x_2+a\ln(x_0+x_1),\ \ \varphi''\varphi^2-a\varphi''\varphi'\varphi-a\varphi'^3=0,$

$$u=\left(x_0^2-x_1^2-C(x_0+x_1)\exp(x_2/a)\right)^{1/2}.$$

2. Next we present ansatzes which reduce the equation (1) to two-dimensional partial differential equations (PDE) and we list the PDEs obtained.

1. $u=\varphi(\omega_1,\omega_2),\ \ \omega_1=x_0,\ \ \omega_2=\left(x_1^2+x_2^2\right)^{1/2};$

$$\det\varphi=0,\quad \det\varphi=\begin{vmatrix}\varphi_{11} & \varphi_{12}\\ \varphi_{21} & \varphi_{22}\end{vmatrix},\quad \varphi_{ij}\equiv\frac{\partial^2\varphi}{\partial\omega_i\partial\omega_j},\quad i,j=1,2.$$

2. $u^2=-\varphi(\omega_1,\omega_2)+x_0^2,\ \ \omega_1=x_1,\ \ \omega_2=x_2;\ \ \det\varphi=0.$

3. $u=\varphi(\omega_1,\omega_2)+x_0^2,\ \ \omega_1=x_0+x_1,\ \ \omega_2=\left(x_0^2-x_1^2-x_2^2\right)^{1/2};$

$$\omega_1^2\omega_2\det\varphi-\omega_1^2\varphi_2\varphi_{11}-\omega_2^2\varphi_2\varphi_{22}-2\omega_1\omega_2\varphi_2\varphi_{12}=0,\ \ \varphi_i\equiv\frac{\partial\varphi}{\partial\omega_i},\ \ i=1,2.$$

4. $u=\varphi(\omega_1,\omega_2),\ \ \omega_1=x_0+\arcsin\dfrac{x_1}{\sqrt{x_1^2+x_2^2}},\ \ \omega_2=\left(x_1^2+x_2^2\right)^{1/2};$

$$\omega_2^3\varphi_2\det\varphi-\varphi_1^2\varphi_{11}=0.$$

5. $u=\varphi(\omega_1,\omega_2),\ \ \omega_1=x_2+a\ln(x_0+x_1),\ \ \omega_2=\left(x_0^2-x_1^2\right)^{1/2};$

$$\omega_2(a\varphi_1+\omega_2\varphi_2)\det\varphi-a\varphi_1\varphi_2\varphi_{11}=0.$$

6. $u=\varphi(\omega_1,\omega_2),\ \ \omega_1=x_2+\frac{1}{4}\left(x_0+x_1\right)^2,\ \ \omega_2=x_0-x_1+(x_0+x_1)x_2+\frac{1}{6}\left(x_0+x_1\right)^3;$

$$\varphi_1\det\varphi-2\varphi_2^2\varphi_{22}=0.$$

7. $u=\varphi(\omega_1,\omega_2)-a\cdot\arcsin\dfrac{x_1}{\sqrt{x_1^2+x_2^2}},\ \ \omega_1=x_0,\ \ \omega_2=\left(x_1^2+x_2^2\right)^{1/2};$

$$\omega_2^3\varphi_2\det\varphi-a^2\varphi_{11}=0.$$

8. $u=\dfrac{1}{\varepsilon(x_0+x_1)}\varphi(\omega_1,\omega_2)-\dfrac{x_2}{\varepsilon(x_0+x_1)},\ \ \omega_1=x_0+x_1,\ \ \omega_2=\left(x_0^2-x_1^2-x_2^2\right)^{1/2};$

$$2\omega_1^2\varphi_2\det\varphi-\omega_1^2\varphi_2^2\varphi_{11}-\left(\omega_2^2\varphi_2^2+2\omega_1\omega_2\varphi_1\varphi_2-2\omega_2\varphi\varphi_2-\omega_2^2\right)\varphi_{22}+$$
$$\varphi_2\left(\omega_1\omega_2\varphi_2^2-2\varphi\varphi_2+2\omega_1\varphi_1\varphi_2-\omega_2\right)=0,\ \ \varepsilon=\pm1.$$

The ansatzes (1)–(8) can be written in the form:

$$h(u)=f(x)\cdot\varphi(\omega_1,\,\omega_2)+g(x),$$

where $h(u)$, $f(x)$, $g(x)$ are given functions, $\varphi(\omega_1, \omega_2)$ is an unknown function. The new variables $\omega_1 = \omega_1(x)$, $\omega_2 = \omega_2(x)$ are invariants of subgroups of the group $\mathrm{P}(1,3)$.

9. $\arcsin \dfrac{x_2}{\omega_1} = \varphi(\omega_1, \omega_2) - a\ln(x_0 + x_1)$, $\omega_1 = \left(x_2^2 + u^2\right)^{1/2}$, $\omega_2 = \left(x_0^2 - x_1^2\right)^{1/2}$;

$$\left(\frac{1}{\omega_1}\varphi_2 - \frac{a^2}{\omega_2}\varphi_1 - \frac{a}{\omega_1\omega_2}\right)\det\varphi + \left(\varphi_1\varphi_2^2 + \frac{a^2}{\omega_2^2}\varphi_1 - \frac{3a}{\omega_2}\varphi_1\varphi_2 + \frac{a}{\omega_1\omega_2^2}\right)\varphi_2\varphi_{11} +$$

$$\left(\varphi_1^3\varphi_2 - \frac{2a}{\omega_1^2\omega_2}\varphi_1 + \frac{2}{\omega_1^2}\varphi_1\varphi_2 - \frac{a}{\omega_2}\varphi_1^3 + \frac{a}{\omega_1^3\omega_2}\right)\varphi_{22} +$$

$$2\varphi_2\left(\frac{2a}{\omega_2}\varphi_1^2 - \varphi_1^2\varphi_2 - \frac{1}{\omega_1^2}\varphi_2 + \frac{2a}{\omega_1^2\omega_2}\right)\varphi_{12} +$$

$$\varphi_2\left(\frac{a}{\omega_2^2}\varphi_1^3 + \frac{3a}{\omega_1^3\omega_2}\varphi_2 - \frac{1}{\omega_1^3}\varphi_2^2 + \frac{2a}{\omega_1^2\omega_2^2}\varphi_1 - \frac{a^2}{\omega_1^3\omega_2^2}\right) = 0.$$

10. $\arcsin \dfrac{x_2}{\omega_2} = \varphi(\omega_1, \omega_2) - \varepsilon(x_0 - x_1)$, $\omega_1 = x_0 + x_1$, $\omega_2 = \left(x_2 + u^2\right)^{1/2}$;
$\varphi_2 \det\varphi - \omega_2\varphi_{11} = 0.$

The ansatzes (9) and (10) can be written in the form:

$h(\omega, x) = f(x)\cdot\varphi(\omega) + g(x)$,

where $h(\omega, x)$, $f(x)$, $g(x)$ are given functions, $\varphi(\omega)$ is an unknown function. $\omega = (\omega_1(x,u), \omega_2(x,u))$ are invariants of subgroups of the group $\mathrm{P}(1,3)$.

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