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Variational Symmetry in Non-integrable Hamiltonian Systems

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Abstract

We consider the variational symmetry from the viewpoint of the non-integrability criterion towards dynamical systems. That variational symmetry can reduce complexity in determining non-integrability of general dynamical systems is illustrated here by a new non-integrability result about Hamiltonian systems with many degrees of freedom.

1 Variational Symmetry in the Non-integrability Criterion

Monodromy matrices with the eigenvalues $\left(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1}, \cdots, \sigma_{n-1}, \sigma_{n-1}^{-1}\right)$ for Hamiltonian systems with n degrees of freedom are called *non-resonant* when they cannot satisfy the following relation for all integers m_i (except for the trivial case $m_1 = m_2 = \cdots = m_{n-1} = 0$):

$$\sigma_1^{m_1} \sigma_2^{m_2} \cdots \sigma_{n-1}^{m_{n-1}} = 1.$$

One of the crucial conditions to apply Ziglin's test of non-integrability is that the monodromy matrix of a normal variational equation for a particular solution must be non-resonant [6, 1]. It is clear that this non-resonance condition does not include the degenerate case and also does not include the case of a power root of unity for eigenvalues. However, there is a case that a particular solution

$$\Gamma : (\boldsymbol{q}, \boldsymbol{p}) = \left(C_1\phi(t), \cdots, C_n\phi(t), C_1\dot{\phi}(t), \cdots, C_n\dot{\phi}(t)\right)$$

is invariant under an exchange operator (i, j) for canonical variables $\{q_i, p_i\}$, where

$$(i, j)Q(\cdots, q_i, \cdots, q_j, \cdots, p_i, \cdots, p_j, \cdots) = Q(\cdots, q_j, \cdots, q_i, \cdots, p_j, \cdots, p_i, \cdots).$$

In this case, the system naturally admits the variational symmetry and consequently the eigenvalues of monodromy matrices of the variational equations along this symmetric solutions must be degenerate. Recently, the author gave many examples of Hamiltonian dynamical systems which have no particular solution whose monodromy matrices are not degenerate in Ref. [3]. There, the author gave a new criterion using a weaker sufficient condition than the non-resonance condition of Ziglin and Yoshida to prove the non-integrability. In the above criterion, we use the following condition for symplectic monodromy matrices. We call a symplectic monodromy matrix *non-resonantly-degenerate* [3] if eigenvalues may be degenerate but all the different representations of eigenvalues satisfy the non-resonance condition. In other words,

Definition 1.1 *We call a monodromy matrix* **non-resonantly-degenerate** *[3] when the eigenvalues* $(\sigma_1, \cdots, \sigma_{n-1})$ *degenerate into* $d (\leq n-1)$ *groups as follows:*

$$
\begin{aligned}
&\sigma'_1 = \sigma_1 = \cdots = \sigma_{i_1} \\
&\sigma'_2 = \sigma_{i_1+1} = \cdots = \sigma_{i_2} \\
&\cdots\cdots\cdots \\
&\sigma'_d = \sigma_{i_{d-1}+1} = \cdots = \sigma_{n-1} \\
&1 \leq i_1 < i_2 < i_{d-1} < n-1 \qquad d \leq n-1
\end{aligned}
$$

but the representatives $(\sigma'_1, \cdots, \sigma'_d)$ *satisfy the non-resonance condition:*

$$
\begin{aligned}
&\sigma_1'^{m_1} \sigma_2'^{m_2} \cdots \sigma_d'^{m_d} = 1 \\
&\quad \Rightarrow (m_1, m_2, \cdots, m_d) = (0, 0, \cdots, 0)
\end{aligned}
$$

for all integers m_i*.*

When $d = n-1$, the non-resonantly-degenerate condition is nothing but the non-resonance condition. The main purpose of the present paper is to clarify the relationship between variational symmetry and non-resonantly-degenerate monodromy in the non-integrability criterion, because in the series of recent papers about the non-integrability criterion including the author's paper [3], this relation has been obscure.

1.1 Main Theorem

Here, we have the following theorem which connects the variational symmetry and non-resonantly-degenerate monodromy in the non-integrability criterion:

Theorem 1 *Consider the following symmetric Hamiltonian systems of n degrees of freedom with a homogeneous polynomial potential function whose degree L is not* $0, 2$*, or* -2*:*

$$H_n = \frac{1}{2} \sum_{i=1,n} p_i^2 + V(q_1, q_2, \cdots, q_n),$$

where

$$(i,j) H_n = H_n \quad \textit{for} \quad i \neq j.$$

Assume that the system has an additional global analytic integral $\Phi(\boldsymbol{q}, \boldsymbol{p})$ *besides the Hamiltonian itself*

$$(i,j)\Gamma = \Gamma.$$

If there exist two different non-resonantly-degenerate matrices

$$m_1 = \operatorname{diag}[m_1(\lambda_1), \cdots, m_1(\lambda_{n-1})]$$

and

$$m_2 = \mathrm{diag}\,[m_2(\lambda_1), \cdots, m_2(\lambda_{n-1})]$$

in the monodromy group for a particular solution

$$\Gamma : q_i = C_i\phi(t), p_i = C_i\dot{\phi(t)} \quad \Big(\prod_{i=1,n} |C_i| \neq 0\Big),$$

which admits a symmetry

$$(i,j)\Gamma = \Gamma,$$

then it is necessary that $m_1(\lambda_i)$ *commute with* $m_2(\lambda_i)$ *for some* i*.*

1.2 Proof of Theorem

For Hamiltonian systems with n degrees of freedom, with a homogeneous potential function $V(\boldsymbol{q})$ of degree L,

$$H_n = \frac{1}{2}\boldsymbol{p}^{\,2} + V(\boldsymbol{q}\,)\,,$$

we can always get a straight-line solution as follows:

$$z(t) = (\boldsymbol{q} = \boldsymbol{C}\phi\,(t)\,, \boldsymbol{p} = \boldsymbol{C}\dot{\phi}\,(t)), \tag{1.1}$$

where

$$\frac{d^2}{dt^2}\phi(t) + \phi(t)^{L-1} = 0$$

and

$$\boldsymbol{C} = \frac{\partial V(\boldsymbol{C})}{\partial \boldsymbol{C}}.$$

If we consider a slight perturbation around the particular solution (1.1) such as

$$\boldsymbol{q'} = \boldsymbol{q} + \boldsymbol{\xi}, \quad \boldsymbol{p'} = \boldsymbol{p} + \boldsymbol{\eta},$$

we obtain the following linear variational equation:

$$\frac{d\boldsymbol{\xi}}{dt} = \boldsymbol{\eta}, \qquad \frac{d\boldsymbol{\eta}}{dt} = -\phi(t)^{L-2}\boldsymbol{V_{CC}\xi} \tag{1.2}$$

where $\boldsymbol{\xi} = \delta\boldsymbol{q}, \boldsymbol{\eta} = \delta\boldsymbol{p}$ and $\boldsymbol{V_{CC}}$ is the symmetric Hessian matrix of $V(\boldsymbol{C})$. This system is again a Hamiltonian system with a time-dependent Hamiltonian

$$H(t) = \frac{1}{2} < \boldsymbol{\eta}, \boldsymbol{\eta} > +\phi(t)^{L-2} < \boldsymbol{\xi}, \boldsymbol{V_{CC}\xi} > .$$

Using an orthogonal transformation

$$\boldsymbol{\xi} = \boldsymbol{O\xi'}, \qquad \boldsymbol{\eta} = \boldsymbol{O\eta'}$$

with $\boldsymbol{O}$ being an orthogonal matrix, the linear variational equation (1.2) can be rewritten in the diagonalized form:

$$\frac{d\boldsymbol{\xi}'}{dt} = \boldsymbol{\eta}', \qquad \frac{d\boldsymbol{\eta}'}{dt} = -\phi(t)^{L-2}\mathrm{diag}\,(\lambda_1, \cdots, \lambda_n)\,\boldsymbol{\xi}',$$

where $\lambda_1, \cdots \lambda_n$ are the eigenvalues of $\boldsymbol{V_{CC}}$ and $\lambda_n = L-1$. Since the variational equation corresponded to the last eigenvalue λ_n has an integral

$$\boldsymbol{\eta} \cdot \boldsymbol{p} + \boldsymbol{\xi} \cdot \boldsymbol{V_q},$$

we can reduce the $(2n-1)$-dimensional variational equations into the $(2n-2)$-dimensional vector form of variational equations:

$$\frac{d\boldsymbol{\xi}'}{dt} = \boldsymbol{\eta}', \qquad \frac{d\boldsymbol{\eta}'}{dt} = -\phi(t)^{L-2}\mathrm{diag}\,(\lambda_1, \cdots, \lambda_{n-1})\,\boldsymbol{\xi}' \tag{1.3}$$

where $\boldsymbol{\xi}' = (\xi'_1, \cdots, \xi'_{n-1})$ and $\boldsymbol{\eta}' = (\eta'_1, \cdots, \eta'_{n-1})$. These decoupled vector equations (1.3) are called NVE (normal variational equations), and it is known by Whittaker's theorem that these normal variational equations have also the symplectic property. Assume that there exists an additional holomorphic integral $\Phi(\boldsymbol{q}, \boldsymbol{p})$ in a connected neighborhood of the phase curve $\Gamma = \{z(t)\}$ where t runs on a connected Riemann surface $\boldsymbol{X}$ making $z(t)$ single-valued. We expand Φ as

$$\Phi(z + \zeta) = \Phi_0(z) + \sum_{m=1}^{\infty} \Phi_m(\zeta, t), \tag{1.4}$$

where Φ_m is a homogeneous polynomial in $\zeta = (\xi_1, \cdots, \xi_{n-1}, \eta_1, \cdots, \eta_{n-1})$. Each homogeneous term of the series in Eq. (1.4) is easily checked to be an integral invariant of the normal variational equation (1.3). There is a possibility that $\Phi_1 \cdots \Phi_{m-1}$ vanish along the curve Γ. However, it is known that there is a non-zero homogeneous form of the series and we define the first non-zero homogeneous term Φ_m as $I^*(\boldsymbol{\xi'}, \boldsymbol{\eta'}, t)$ which is clearly an integral of the normal variational equation.

Along the closed loop $\gamma \in \Gamma$ on the Riemann surface $w = \phi(t)$ with period T, a symplectic monodromy mapping (an element of the monodromy group) can be naturally defined. Then an integral $I^*(\boldsymbol{\xi'}, \boldsymbol{\eta'}, t)$ of NVE must be invariant under the action of the monodromy matrix $m(\gamma(T))$, because $I^*(\boldsymbol{\xi'}, \boldsymbol{\eta'}, t)$ itself depends on t only through $\phi(t)$ and $\dot\phi(t)$ and $\phi(0) = \phi(T)$ and $\dot\phi(0) = \dot\phi(T)$. This means that this polynomial function I^* with $\xi'_1, \cdots, \xi'_{n-1}, \eta'_1, \cdots, \eta'_{n-1}$ being standing for the arguments of $I*$ is invariant under the symplectic monodromy matrix m. We now assume that there exists a *non-resonantly-degenerate* monodromy matrix m_1. The monodromy matrix m_1 can be expressed as a pure diagonal matrix in a proper basis as

$$m_1 = \mathrm{diag}(\sigma_1, \sigma_1^{-1}, \cdots, \sigma_{n-1}, \sigma_{n-1}^{-1}),$$

while the invariant polynomial $I^*(\boldsymbol{\xi}, \boldsymbol{\eta}, t)$ is written in the form

$$I_{n-1}(u_1, \cdots, u_{n-1}, v_1, \cdots, v_{n-1}) = \sum C^{k_1,\cdots,k_{n-1}}_{j_1,\cdots,j_{n-1}} u_1^{j_1} v_1^{k_1} \cdots u_{n-1}^{j_{n-1}} v_{n-1}^{k_{n-1}}$$

where $\sum_i j_i = \sum_i k_i = l$ and the coefficients $C^{k_1,\cdots,k_{n-1}}_{j_1,\cdots,j_{n-1}}$ are holomorphic functions of $t \in \boldsymbol{X}$.

From the invariance property of I_{n-1} under the monodromy action m_1,

$$I'_{n-1} = \sum C^{k_1,\cdots,k_{n-1}}_{j_1,\cdots,j_{n-1}} \sigma_1^{j_1-k_1} \cdots \sigma_{n-1}^{j_{n-1}-k_{n-1}} u_1^{j_1} v_1^{k_1} \cdots u_{n-1}^{j_{n-1}} v_{n-1}^{k_{n-1}} = I_{n-1},$$

a further restriction must be obeyed:

$$C^{k_1,\cdots,k_{n-1}}_{j_1,\cdots,j_{n-1}} = 0 \quad \text{unless} \quad \sum_{i=1}^{i_1} j_i = \sum_{i=1}^{i_1} k_i, \cdots, \sum_{i=i_{d-1}+1}^{n-1} j_i = \sum_{i=i_{d-1}+1}^{n-1} k_i, \tag{1.5}$$

where d is the number of representative eigenvalues. Moreover, since this degeneracy ($\sigma_i = \sigma_j$) of the eigenvalues of the monodromy matrix comes from the degeneracy of the eigenvalues ($\lambda_i = \lambda_j$) of the Hessian matrix $\boldsymbol{V_{CC}}$ along the phase curve Γ which has a variational symmetry

$$(i,j)\Gamma = \Gamma,$$

NVE (1.3) is invariant under the following exchange of variables:

$$(i,j)\text{NVE} = \text{NVE}.$$

Since the Hamiltonian is assumed to be invariant under the change of variables (i,j), we can always have a symmetric integral

$$(i,j)\tilde{\Phi} = \tilde{\Phi}$$

from the existence of an integral Φ. We have the following relations in the series

$$(i,j)\tilde{\Phi}(z+\zeta) = (i,j)\tilde{\Phi}(z) + \sum_{m=1}^{\infty} (i,j)\tilde{\Phi}_m(\zeta,t) = \tilde{\Phi}(z) + \sum_{m=1}^{\infty} \tilde{\Phi}_m(\zeta,t).$$

Thus, it is concluded that each homogeneous term $\tilde{\Phi}_m$ must be symmetric:

$$(i,j)\tilde{\Phi}_m = \tilde{\Phi}_m.$$

Thus, an integral $\tilde{I}$ of NVE is also symmetric under the (i,j):

$$(i,j)\tilde{I} = \tilde{I}.$$

Because the normal variational equations in Eq. (1.3) happens to be separated from the homogeneous form of the Hamiltonian, any monodromy matrix $m_2 \equiv m$ other than m_1 can be expressed in the block diagonal form as

$$m_2 = m = \text{diag}[m(\lambda'_1), m(\lambda'_1), \cdots, m(\lambda'_2), \cdots, m(\lambda'_d)],$$

where the eigenvalues of the Hessian $\boldsymbol{V_{CC}}$ are regrouped according to their degeneracy:

$$\begin{aligned} &\lambda'_1 = \lambda_1 = \cdots = \lambda_{i_1} \\ &\lambda'_2 = \lambda_{i_1+1} = \cdots = \lambda_{i_2} \\ &\cdots \\ &\lambda'_d = \lambda_{i_{d-1}+1} = \cdots = \lambda_{n-1}. \end{aligned} \tag{1.6}$$

With the use of condition (1.5), $\tilde{I}$ can be interpreted as:

$$\tilde{I} = \sum_{Cr} \tilde{I}_1(Cr)\tilde{I}_2(Cr)\cdots\tilde{I}_d(Cr)$$

where Cr runs over the allowable configurations of $j_1,\cdots,j_{n-1},k_1,\cdots,k_{n-1}$ satisfying condition (1.5) and each $\tilde{I}_l(Cr)(1 \le l \le d)$ is a homogeneous polynomial in $u_{i_{l-1}+1},\cdots,$ u_{i_l}, $v_{i_{l-1}+1},\cdots,v_{i_l}$.

Clearly, each $m(\tilde{I}_j(C))$ must also be homogeneous; otherwise $m(\tilde{I})$ does not have a homogeneous form. Using the regrouping property of eigenvalues (1.6), we can consider the following further-reduced variational equations:

$$\frac{d\boldsymbol{\xi}''}{dt} = \boldsymbol{\eta}'', \qquad \frac{d\boldsymbol{\eta}''}{dt} = -\phi(t)^{L-2}\mathrm{diag}\,(\lambda'_1,\cdots,\lambda'_d)\,\boldsymbol{\xi}''$$

where $\boldsymbol{\xi}'' = (\xi''_1,\cdots,\xi''_d)$, $\boldsymbol{\eta}'' = (\eta''_1,\cdots,\eta''_d)$, and

$$\begin{aligned}
&\xi''_1 = \xi'_1 = \cdots = \xi'_{i_1}, \eta''_1 = \eta'_1 = \cdots = \eta'_{i_1}\\
&\xi''_2 = \xi'_{i_1+1} = \cdots = \xi'_{i_2}\eta''_2 = \eta'_{i_1+1} = \cdots = \eta'_{i_2}\\
&\cdots\\
&\xi''_d = \xi'_{i_{d-1}+1} = \cdots = \xi'_{n-1}, \eta''_d = \eta'_{i_{d-1}+1} = \cdots = \eta'_{n-1}.
\end{aligned}$$

For this d dimensional normal variational equations reduced from the original $n-1$ dimensional normal variational equations, the original non-resonantly-degenerate monodromy matrix of the $n-1$ dimensional variational equations corresponds to the non-resonant monodromy matrix of the d dimensional normal variational equations. Let us consider the phase manifold $M' \in M$ as

$$\begin{aligned}
M' = \Big\{ & z \in M | (l_1, m_1)z \ne z, 1 \le l_1 < m_1 \le i_1, \cdots,\\
& (l_d, m_d)z \ne z, i_{d-1}+1 \le l_d < m_d \le n-1 \Big\}.
\end{aligned}$$

According to the variational symmetry, we can consider the symmetry group G. Then the projection $M \to M'/G \equiv \hat{M}$ corresponds to the above reduction about the monodromy, and, furthermore, because it is known [6] that the existence of an additional analytic and functionally independent integral on M induces the existence of an additional analytic and functionally independent integral on $\hat{M}$, we can use the Ziglin-Yoshida analysis using a non-resonant monodromy matrix concerning the normal variational equations reduced by the variational symmetry for a particular solution on $\hat{M}$ [6, 5]. Thus, the conclusion of the present theorem about the commuting property of block monodromy matrices as a necessary condition of the existence of an additional integral follows by the standard argument [6, 5, 4].

2 Non-integrability Results

2.1 Non-resonantly Degenerate and Non-commuting Monodromy

The main theorem shows that the non-existence of additional analytic integrals besides the Hamiltonian itself can be proved if the monodromy matrices associated with a particular solution are checked to be *non-resonantly degenerate* and non-commuting. By changing the variable as

$$z = (\phi(t))^L,$$

the normal variational equation NVE (1.3)

$$\frac{d^2\xi_j}{dt^2} + \lambda_j \phi(t)^{L-2}\xi_j = 0 \tag{2.1}$$

becomes the **Gauss hypergeometric equation** [5]:

$$z(1-z)\frac{d^2\xi_j}{dz^2} + \left[\left(1-\frac{1}{L}\right) - \left(\frac{3}{2}-\frac{1}{L}\right)z\right]\frac{d\xi_j}{dz} + \frac{\lambda_j}{2L}\xi_j = 0.$$

The two fundamental monodromy matrices $\boldsymbol{M}(\gamma_0), \boldsymbol{M}(\gamma_1)$ for the Gauss hypergeometric equations can be expressed as follows:

$$\begin{aligned} \boldsymbol{M}(\gamma_0) &= \begin{pmatrix} 1 & e^{-2\pi i b} - e^{-2\pi i c} \\ 0 & e^{-2\pi i c} \end{pmatrix}, \\ \boldsymbol{M}(\gamma_1) &= \begin{pmatrix} e^{2\pi i(c-a-b)} & 0 \\ 1 - e^{2\pi i(c-a)} & 1 \end{pmatrix}, \end{aligned} \tag{2.2}$$

where

$$a + b = \frac{1}{2} - \frac{1}{L}, \quad ab = -\frac{\lambda_j}{2L}, \quad c = 1 - \frac{1}{L}.$$

Monodromy matrices $\boldsymbol{m}$ of NVE (2.1) can be generated by the above matrices $\boldsymbol{M}(\gamma_0)$, $\boldsymbol{M}(\gamma_1)$ as follows:

$$\boldsymbol{m} = \boldsymbol{M}(\gamma_0)^{i_1}\boldsymbol{M}(\gamma_1)^{i_2}\boldsymbol{M}(\gamma_0)^{i_3}\cdots$$

where i_k are integers. We can obtain the monodromy matrices

$$\boldsymbol{m}_1 = \boldsymbol{M}(\gamma_1)\boldsymbol{M}(\gamma_0)\boldsymbol{M}(\gamma_1)\boldsymbol{M}(\gamma_0)^{-1}$$

$$\boldsymbol{m}_2 = \boldsymbol{M}(\gamma_0)^{-1}\boldsymbol{M}(\gamma_1)\boldsymbol{M}(\gamma_0)\boldsymbol{M}(\gamma_1).$$

Their traces are given by

$$\mathrm{Tr}(\boldsymbol{m}_{1,2}(\lambda_i)) = 2\cos\left(\frac{2\pi}{L}\right) + 4\cos^2\left[(\frac{\pi}{2L})\sqrt{[(L-2)^2 + 8L\lambda_i]}\right].$$

It is known from the explicit expression of the traces that if $\lambda \equiv \lambda_i (1 \le i \le n-1)$ is in the following region [5]:

$$\begin{aligned} S_L = \Big\{&\lambda < 0, 1 < \lambda < L-1, L+2 < \lambda < 3L-2, \\ &\cdots j(j-1)L/2 + j < \lambda < j(j+1)L/2 - j, \cdots\Big\}, \end{aligned} \tag{2.3}$$

where L is the degree of a potential function and $L \ge 3$, then $\boldsymbol{m}_1(\lambda_i)$ and $\boldsymbol{m}_2(\lambda_i)$ do not commute each other and both of $\boldsymbol{m}_1(\lambda_i), \boldsymbol{m}_2(\lambda_i)$ are non-resonant. This means that the eigenvalues of

$$\begin{aligned} \boldsymbol{m}_1 &= \mathrm{diag}[\boldsymbol{m}_1(\lambda_1), \cdots, \boldsymbol{m}_1(\lambda_{n-1})] \quad \text{and} \\ \boldsymbol{m}_2 &= \mathrm{diag}[\boldsymbol{m}_2(\lambda_1), \cdots, \boldsymbol{m}_2(\lambda_{n-1})] \end{aligned}$$

satisfy the *non-resonantly degenerate condition* proposed here to prove non-integrability because of their degeneracy

$$\lambda = \lambda_1 = \lambda_2 = \cdots = \lambda_{n-1}.$$

2.2 Examples

We consider the following Hamiltonian systems

$$H_\alpha = \sum_{i=1,n} (\frac{1}{2}p_i^2 + q_i^4) + \alpha \sum_{<i,j>}^{<n>} q_i^2 q_j^2 \tag{2.4}$$

with a real parameter α. If $\alpha = 0$, this system corresponds to *separable systems* which is trivially complete integrable. If $\alpha = 2$, then we can rewrite the above Hamiltonian as

$$H_{\alpha=2} = \sum_{i=1,n} \frac{1}{2}p_i^2 + (q_1^2 + \cdots + q_n^2)^2.$$

Thus, the system happens to have a centrifugal potential function and the Hamiltonian $H_{\alpha=2}$ commutes with angular momenta:

$$\{H_{\alpha=2}, p_i q_j - p_j q_i\} = 0$$

because of the rotational symmetry. How about the integrability of the case with $0 < \alpha < 2$? In the case of two degrees of freedom, Yoshida obtained that the relevant systems are non-integrable except for $\alpha = 0, \alpha = 2$, and $\alpha = 6$ [5]. In the case of the systems (2.4) with three or more degrees of freedom, we obtain here the following theorem.

Theorem 2.1 *For the case of* $0 < \alpha < 2$*, Hamiltonian systems* H_α *with three or more degrees of freedom cannot have additional analytic integrals other than the Hamiltonian itself.*

We consider the particular solutions of

$$\Gamma : q_i = C\phi(t) \quad \text{for} \quad 1 \le i \le n,$$

where

$$\frac{d^2}{dt^2}\phi(t) + \phi(t)^3 = 0.$$

Thus, the coefficient C satisfies the following relation

$$C = 2(2 + \alpha(n-1))C^3.$$

This phase curve Γ clearly admits the variational symmetry. Therefore, the eigenvalues λ_i of the $n \times n$ symmetric matrix $\boldsymbol{V}_{CC}$ are given by

$$\lambda(\alpha) \equiv \lambda_i = \frac{6 + \alpha(n-3)}{2 + \alpha(n-1)} \quad (1 \le i \le n-1),$$

$$\lambda_i = 3 \quad (i = n).$$

If $\lambda(\alpha) \in S_{L=4}$ (2.3), the two monodromy matrices $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$ in (2.2) are *non-resonantly degenerate.*

From the relation

$$1 < \lambda(\alpha) = \frac{6 + \alpha(n-3)}{2 + \alpha(n-1)} < 3,$$

we can easily obtain $0 < \alpha < 2$, whose monodromy matrices m_1, m_2 are non-resonantly degenerate. We remark here that

$$\lambda(\alpha = 0) = 3, \quad \lambda(\alpha = 2) = 1,$$

which are the end points of the region of $S_{L=4}$ of (2.3). This completes the proof on the non-existence of additional analytic conserved quantities in Hamiltonian systems $H_{0<\alpha<2}$ with three or more degrees of freedom. To summarize, we can say that variational symmetry is an effective tool in proving the non-integrability of Hamiltonian dynamical systems with many degrees of freedom [3, 4].

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