

Conditional Symmetry of Equations of Nonstationary Filtration and of the Nonlinear Heat Equation

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Abstract

Conditional symmetry of the nonlinear gas filtration equation is studied. The operators obtained enabled to construct ansatzes reducing this equation to ordinary differential equations and to obtain its exact solutions.

In describing filtration processes of gas, the following nonlinear equation is widely used [1]

$$\frac{\partial v}{\partial x_0} + \frac{\partial^2 \varphi(v)}{\partial x_1^2} + \frac{N}{x_1} \frac{\partial \varphi(v)}{\partial x_1} = \Phi(v), \quad (1)$$

where $v = v(x)$, $x = (x_0, x_1) \in R_2$, $N = \text{const}$, $\varphi(v), \Phi(v)$ are given smooth functions. Substitution $u = \varphi(v)$ reduces equation (1) to the equivalent equation

$$H(u)u_0 + u_{11} + \frac{N}{x_1}u_1 = F(u), \quad (2)$$

where $u_0 = \frac{\partial u}{\partial x_0}$, $u_1 = \frac{\partial u}{\partial x_1}$, $u_{11} = \frac{\partial^2 u}{\partial x_1^2}$.

Lie symmetry of equation (2) under $N = 0$ was studied in [2,3] and its conditional symmetry was studied in [4, 5].

In present paper we study conditional symmetry of equation (2) with $N \neq 0$. Operators of conditional symmetry are used to construct ansätze which reduce (2) to ordinary differential equations (ODE). By means of this method we obtain exact solutions of equations (2) and then exact solutions of a multidimensional nonlinear heat equation. Below we will use terms and definitions given in [4, 5].

Theorem 1 *Equation (2) is Q-conditionally invariant under the operator*

$$Q = A(x, u)\partial_0 + B(x, u)\partial_1 + C(x, u)\partial_u, \quad (3)$$

iff functions A, B, C satisfy the following system of equations:

Case I. $A \neq 0$ (without loss of generality one can put $A = 1$),

$$\begin{aligned} B_{uu} &= 0, \quad C_{uu} = 2 \left(B_{1u} + HBB_u - \frac{N}{x_1}B_u \right), \\ 3B_u F &= 2(c_{1u} + HB_u C) - \left(HB_0 + B_{11} - \frac{N}{x_1}B_1 + \frac{N}{x_1^2}B + 2HBB_1 + \dot{H}BC \right), \\ C\dot{F} - (C_u - 2B_1)F &= HC_0 + C_{11} + \frac{N}{x_1}C_1 + 2HCB_1 + \dot{H}C^2; \end{aligned} \quad (4)$$

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Case II. $A = 0, B = 1,$

$$C\dot{F} - \left(C_u + \frac{\dot{H}}{H}C \right) F = HC_0 + C_{11} + \frac{N}{x_1}C_1 - \frac{N}{x_1^2}C + 2CC_{1u} + C^2C_{uu} -$$

$$C\frac{\dot{H}}{H} \left(CC_u + C_1 + \frac{N}{x_1}C \right). \quad (5)$$

In formulas (4), (5) and everywhere below, subscripts mean differentiation with respect to corresponding arguments.

To prove the theorem one should use the method described in [4, 5].

To find the general solution of equations (4), (5) is impossible, but we succeeded in obtaining several partial solutions. The list of reduction is present in the Table.

$H(u)$	$F(u)$	Operator Q	Ansatz	Reduced eq.
$\lambda_1 u^{\frac{2}{1-N}} + \lambda_2$ $\lambda_2 \neq 0$	$\lambda_3 u^{\frac{3-N}{1-N}}$ $N \neq 1; 3$	$\lambda_2 x_1^2 \partial_0 + (3-N) \times$ $x_1 \partial_1 + (3-N) \times$ $\times (1-N)u \partial_u$	$u = x_1^{1-N} \varphi(\omega),$ $\omega = \frac{x_0}{\lambda_2} + \frac{x_1^2}{2(N-3)}$	$\frac{\lambda_1^{1-N}}{\lambda_2} \dot{\varphi} + \frac{\ddot{\varphi}}{(N-3)^2}$ $= \lambda_3 \varphi^{\frac{3-N}{1-N}}$
$\frac{\lambda_1}{u}, N = 3$	λ_3	$x_1^2 \partial_0 + x_1 \partial_1 -$ $2u \partial_u$	$u = x_1^{-2} \varphi \left(x_0 - \frac{x_1^2}{2} \right)$	$\lambda_1 \frac{\dot{\varphi}}{\varphi} + \ddot{\varphi} = 3$
$\lambda_1 e^u + \lambda_2$ $\lambda_2 \neq 0$	$\lambda_3 e^u$ $N = 1$	$\lambda_2 x_1^2 \partial_0 + 2x_1 \partial_1 +$ $+4 \partial_u$	$u = \varphi(\omega) + 2 \ln x_1,$ $\omega = \frac{x_0}{\lambda_2} - \frac{x_1^2}{4}$	$\frac{\lambda_1}{\lambda_2} e^\varphi \dot{\varphi} + \frac{1}{4} \ddot{\varphi} =$ $= \lambda_3 e^\varphi$
$\frac{\lambda_1}{W^2} (N -$ $-\frac{W\ddot{W}}{W^2})$	$\frac{\lambda_2}{W\dot{W}} (N -$ $-\frac{W\ddot{W}}{W^2})$	$x_1 \partial_1 + \frac{W}{\dot{W}} \partial_u$	$W(u) = x_1 \varphi(x_0)$	$\lambda_1 \dot{\varphi} = \lambda_3 \varphi - \varphi^3$
$W(u),$ $N \neq -1$	$\lambda_1 \omega + \lambda_2,$ $\lambda_2 \neq 0$	$\partial_1 + \frac{\lambda_2}{N+1} x_1 \partial_u$	$u = \varphi(x_0) - \frac{\lambda_2 x_1^2}{2(N+1)}$	$\dot{\varphi} = \lambda_1$
$W(u),$ $N = -1$	$\lambda_1 W(u)$	$\partial_1 + \lambda_3 x_1 \partial_u$	$u = \varphi(x_0) + \lambda_3 \frac{x_1^2}{2}$	$\dot{\varphi} = \lambda_1$
1	$\lambda_3 u \ln u$	$\partial_1 + \frac{\lambda_3}{2} x_1 u \partial_u$	$u = \varphi(x_0) e^{\frac{\lambda_3 x_1^2}{4}}$	$\dot{\varphi} + \lambda_3 \frac{N+1}{2} \varphi =$ $= \lambda_3 \varphi \ln \varphi$

Theorem 2 Equation (2) is Q-conditionally invariant under operator (3) with $H(u) = 1, A = 1, B_0 \neq 0$ iff it is locally equivalent to the equation

$$u_0 + u_{11} + \frac{3}{2x_1} u_1 = \lambda u^3 \quad (\lambda = \text{const}), \quad (6)$$

and in this case operator (3) takes the form

$$Q = \partial_0 + \frac{3}{2} \left(\sqrt{2\lambda} u + \frac{1}{x_1} \right) \partial_1 + \frac{3}{4} u \left(2\lambda u^2 - \frac{1}{x_1^2} \right) \partial_u. \quad (7)$$

To prove the theorem, one has to solve equations (4) under $H(u) = 1, B(u) \neq 0$. By means of operator (7) we construct an implicit ansatz

$$15 \left(x_0 - \frac{x_1^2}{3} \right) \omega + 4\sqrt{2\lambda} x_1^{\frac{5}{2}} = \varphi(\omega), \quad \omega = \frac{1 + \sqrt{2\lambda} u x_1}{u \sqrt{x_1}} \quad (8)$$

which reduces equation (6) to the ODE. Having solved the latter and taking into account (8), we obtain the following solution of equation (6), $u(x_0, x_1)$ being a new solution:

$$15 \left(x_0 - \frac{x_1^2}{3} \right) \omega + 4\sqrt{2\lambda}x_1^{\frac{5}{2}} = \varphi(\omega), \quad \omega = \frac{1 + \sqrt{2\lambda}ux_1}{u\sqrt{x_1}} \quad (c_1 = \text{const}). \tag{9}$$

All inequivalent ansätze of the Lie type are given by one of the formulae

$$u = \varphi(x_1), \quad u = x_0^{-\frac{1}{2}}\varphi(x_0^{-\frac{1}{2}}x_1). \tag{10}$$

It is obvious that (9) does not belong to (10).

The above solutions of equation (6) can be multiplied by means of formulae generating solutions by using the Lie symmetry:

$$u(x_0, x_1) = \theta_1 f(\theta_1^2 x_0 + \theta_0, \theta_1, x_1), \tag{11}$$

where θ_0, θ_1 are group parameters, $F(x_0, x_1)$ is a known solution of equation (6), $u(x_0, x_1)$ is a new solution.

Theorem 3 *Equation*

$$\frac{1}{u}u_0 + u_{11} + \frac{N}{x_1}u_1 = \frac{1}{u}(\lambda_1 u + \lambda_2)(\lambda_1, \lambda_2 - \text{const}) \tag{12}$$

is Q -conditionally invariant under the operator

$$Q = \partial_0 + (N + 1)\frac{u}{x_1}\partial_1 + (\lambda_1 u + \lambda_2)\partial_u. \tag{13}$$

Proof. To prove Theorem, it is sufficient to show that the following relation holds true

$$\tilde{Q}S = \tilde{\lambda}_1 S + \tilde{\lambda}_2 Qu, \tag{14}$$

where

$$S = \frac{1}{u}u_0 + u_{11} + \frac{N}{x_1}u_1 - \frac{1}{u}(\lambda_1 u + \lambda_2), \quad Qu = u_0 + (N + 1)\frac{u}{x_1}u_1 - (\lambda_1 u + \lambda_2),$$

\tilde{Q} is a corresponding prolongation of the operator Q , $\tilde{\lambda}_1, \tilde{\lambda}_2$ are some functions.

On acting the operator \tilde{Q} on S we get after rather tedious calculations

$$\tilde{Q}S = \left[\lambda_1 + \frac{N + 1}{x_1^2}(2u + 3x_1u_1) \right] S - \left[\frac{N + 1}{x_1u}u_1 - \frac{N + 1}{x_1^2u}(2u + 3x_1u_1) - \frac{\lambda_1 u + \lambda_2}{u^2} \right] Qu.$$

So, Theorem is proved.

Operator (13) results in the ansatz

$$\frac{x_1^2}{2(N + 1)} - \int \frac{udu}{\lambda_1 u + \lambda_2} = \varphi(\omega), \quad \omega = x_0 - \int \frac{du}{\lambda_1 u + \lambda_2}, \tag{15}$$

which reduces equation (12) to the ODE

$$-\ddot{\varphi} = \lambda_1 \dot{\varphi} + \lambda_2. \tag{16}$$

Note that substituting $x_1 \rightarrow r = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$ and putting $N = n - 1$, we find that equation (2) coincides with the reduced nonlinear heat equation

$$H(u)u_0 + \Delta U = F(u), \quad (17)$$

where $u = u(x_0, \vec{x})$, $\Delta = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2}$. Equation (17) is reduced to (2) by means of the $O(n)$ -invariant ansatz $u = u(x_0, r)$. Therefore, many results obtained above for equation (2) can be used straightforwardly for finding operators of conditional symmetry and corresponding solutions of multidimensional equation (19). We summarize them in the following statement.

Theorem 4 *Nonlinear heat equation (17) is Q -conditionally invariant under the set of operators $AO(n), Q$ if:*

- 1) $H(u) = \lambda_1 u^{\frac{2}{2-n}} + \lambda_2$, $F(u) = \lambda_3 u^{\frac{4-n}{2-n}}$,
 $Q = \lambda_2 \vec{x}^2 \partial_0 + (4-n)x_a \partial_a + (4-n)(2-n)u \partial_u$, $\lambda_2 \neq 0, n \neq 2; 4$;
- 2) $H(u) = \frac{\lambda_1}{u}$, $F(u) = \lambda_3$, $Q = \vec{x}^2 \partial_0 + x_a \partial_a - 2u \partial_u$, $n = 4$;
- 3) $H(u) = \lambda_1 \exp u + \lambda_2$; $F(u) = \lambda_3 \exp u$,
 $Q = \lambda_2 \vec{x}^2 \partial_0 + 2x_a \partial_a + 4 \partial_u$, $n = 2$, $\lambda_2 \neq 0$;
- 4) $H(u) = 1$, $F(u) = \lambda_3 u \ln u$, $Q = x_a \partial_a + \frac{\lambda_3}{2} \vec{x}^2 u \partial_u$;
- 5) $H(u) = \frac{1}{u}$, $F(u) = \frac{1}{u}(\lambda_1 u + \lambda_2)$, $Q = \partial_0 + \frac{n}{\vec{x}^2} u x_a \partial_a + (\lambda_1 u + \lambda_2) \partial_u$.

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