Nonlinear Mathematical Physics 1997, V.4, N 1–2, 209–213.

Two-Point Boundary Optimization Problem for Bilinear Control Systems

Alla V. VINOGRADSKAYA

Kyïv University, Department of Cybernetics

Abstract

This paper presents a new approach to the optimization problem for the bilinear system

$$\dot{x} = \{x, \omega\} \tag{1}$$

based on the well-known method of continuous parametric group reconstruction using of its structure constants defined by the Brockett equation

$$\dot{z} = \{z, \omega\}. \tag{2}$$

Here x is the system state vector, $\{\cdot, \cdot\}$ are the Lie brackets, $z = \{x, y\}$, y is the vector of cojoint variables, $\omega = A^{-1}z$ is the control vector, A is the inertion matrix.

The quadratic control functional has to reach an extremum at the optimal solution of the equation (2) and the boundary optimization problem is to find such z_0 that solution (2) makes evolution from the state $x(t_0) = x_0$ up to the final state $x(t_1) = x_1$ during the time delay $T = t_1 - t_0$. Therefore it is necessary to define a transformation group of the state space which is parametrized by components of the vector and then to solve the Cauchy problem for an arbitrary smooth curve joining $x(t_0)$ with $x(t_0)$.

Key words. Bilinear system, Lie group, optimization, boundary problem, structure constants.

1 Introduction

Optimization problem with a quadratic quality criterion for smooth a dynamic system

$$\dot{x} = f(x, u), \quad x(t_0) = x_0, \quad x(t_1) = x_1, \tag{3}$$

in many important cases [1, 2] can be reduced to the bilinear form as follows: to find such a control $u : R \to R^m$, $u = u(t)$ for the system

$$\dot{x} = \left(\sum_{\mu=1}^{m} H_\mu u_\mu \right) x, \tag{4}$$

where H_μ are matrices generating the Lie group G defined by $f(x, u)$, that the state vector x varies from $x_0 = x(t_0)$ to $x_1 = x(t_1)$ and a loss functional riches a minimum. Brockett (1973) in [2] proposed instead of the equation for adjoint variable y another one for the commutator $z = \{x, y\}$ as follows

$$\dot{z} = \{z, A^{-1}z\}, \tag{5}$$

where a matrix A can be expressed in terms of H_μ. Eliminating u_μ on the basis on the Pontryagin maximum principle and expressing it via z yield the next two-point boundary problem. To find such $z = z(t)$ that the system

$$\dot{x} = \{x, A^{-1}z\} \tag{6}$$

in the force of (4) brings the state vector x from $x(t_0) = x_0$ to $x(t_1) = x_1$ during the time delay $T = t_1 - t_0$ which depends on coefficients of the quadratic loss functional.

An approach explained below gives global optimum in the case of a compact G, otherwise a final compact approximation is necessary. Note that an usual linearization procedure applied to (3) gives only local optimum in all cases.

2 Main results

The optimization of bilinear system (6) is based on the well-known restoration method of continuous parametric group involving its structure constants defined from the Brockett equation (5).

Accordingly, we are to define a transformation group of the state space which is parametrized by the components of vector z_0. In the basis matched with the structure of the Lie algebra, we obtain that the equation (5) has the following form

$$\dot{z_\alpha} = \sum_{\beta,\gamma=1}^{n} \frac{C_\alpha^{\beta\gamma}}{I_\gamma} z_\beta z_\gamma, \tag{7}$$

where $C_\alpha^{\beta\gamma}$ are structural constants, I_γ are eigenvalues of the matrix A.

The linear system, together with (7)

$$\dot{x_\alpha} = \sum_{\beta=1}^{n} \sum_{\gamma=1}^{n} \frac{C_\alpha^{\beta\gamma}}{I_\gamma} x_\beta z_\gamma, \tag{8}$$

is considered.

Under given $z_j^0 = z_j(t_0)$, $\tilde{x}_j^0 = x_j(t_0)$ one can represent a partial solution of a system (8) in the form

$$x_\alpha(t) = \sum_{\beta=1}^{n} S_{\alpha\beta}(t, t_0; z_\gamma^0) x_\beta^0; \tag{9}$$

where $S_{\alpha\beta}(t, t_0; z_\gamma^0)$ are elements of a fundamental matrix. The transformation (9) preserves a scalar product being a space rotation. If $\tilde{x}_\alpha^0 = z_\alpha^0$, the solution of a system (7) has the similar form

$$z_\alpha(t) = \sum_{\beta=1}^{n} S_{\alpha\beta}(t, t_0; z_\gamma^0) z_\beta^0. \tag{10}$$

As fixed z^0_γ ($\gamma = \overline{1,n}$), equation (7) defines variable coefficients of equation (8) and the fundamental matrix. Changing t, we obtain a one-parameter set of rotation of a space over a fixed point, the origin of coordinates. Fundamental matrices satisfy the group relations

$$\sum_{\beta_1=1}^{n} S_{\alpha\beta_1}(t_2, t_1; z^0_\gamma) S_{\beta_1\beta}(t_1, t_0; z^0_\gamma) = S_{\alpha\beta}(t_2, t_0; z^0_\gamma), \quad S_{\alpha\beta}(t, t_0; z^0_\gamma) = \delta_{\alpha\beta} \tag{11}$$

and create a one-parameter Lie group according to time t. We note that system (7), (8) is invariant under the change of variables

$$t = \tau T, \qquad z_\alpha = \frac{\zeta_\alpha}{T} \tag{12}$$

and, consequently, its fundamental matrix

$$S_{\alpha\beta}\left(\tau T, \tau_0 T; \frac{\zeta^0_\gamma}{T}\right) = S_{\alpha\beta}(t, t_0; z^0_\gamma)$$

does not change.

If we take $\delta_{\beta\beta_1}$ instead of x^0_β, then after substitution (9) and (10) for (8) we obtain

$$\frac{\partial}{\partial t} S_{\alpha\beta_1}(t, t_0; z^0_{\gamma_2}) = \sum_{\beta=1}^{n}\sum_{\gamma=1}^{n} \frac{C^{\beta\gamma}_\alpha}{I_\gamma} S_{\beta\beta_1}(t, t_0; z^0_{\gamma_2}) \sum_{\gamma_1=1}^{n} S_{\gamma\gamma_1}(t, t_0; z_{\gamma^0_2}) z^0_{\gamma_1}. \tag{13}$$

The variety of fundamental matrices $\parallel S_{\alpha\beta}(t, t_0; z^0_\gamma) \parallel$ under all possible $z^0_\gamma \in R^n$ and fixed $t = t_1 = t_0 + T$ forms a subgroup of the group $SO(n)$ i.e., the group of rotation of n-dimensional space.

By virtue of the change (13), it is sufficient to prove that the subgroup of $SO(n)$ is formed by matrices $S_{\alpha\beta}(t, t_0; z^0_\gamma)$ under every $t \in R$, $\quad z^0_\gamma \in S^n$, where S^n is a unit sphere in R^n:

$$\sum_{\gamma=1}^{n} (z^0_\gamma)^2 = 1.$$

Let z^0_γ be directive cosines of a unit vector in R^n. Fixing $\vec{\zeta}$ and changing t, we get the one-parameter set of matrices

$$\{\parallel S_{\alpha\beta}(t, t_0; z^0_\gamma) \parallel\}. \tag{14}$$

Since $\sum\limits_{\beta_1=1}^{n} S_{\alpha\beta_1}(t_0, t_1; z^0_\gamma) S_{\beta_1\beta}(t_1, t_0; z^0_\gamma) = \delta_{\alpha\beta}$, then the variety of matrices (14) forms a group G isomorphic to the group $SO(n)$. Choose $\vec{\zeta_\mu}$ as a unit vector with components $\zeta_{\mu\gamma_1} = \delta_{\mu\gamma_1}$ ($\mu, \gamma_1 = \overline{1,n}$).

Then by (13) the infinitesimal matrices of corresponding one-parameter groups will have the following elements

$$I^\mu_{\alpha\beta_1} = \lim_{t\to t_0} \frac{\partial}{\partial t} S_{\alpha\beta_1}(t, t_0; \vec{\zeta_\mu}) = \sum_{\beta=1}^{n}\sum_{\gamma=1}^{n} \frac{C^{\beta\gamma}_\alpha}{I_\gamma} \delta_{\beta\beta_1} \sum_{\gamma_1=1}^{n} \delta_{\gamma\gamma_1}\delta_{\mu\gamma_1} = \frac{C^{\beta_1\mu}_\alpha}{I_\mu}. \tag{15}$$

Compose a commutator and determine the structural constants of the group G

$$\sum_{\delta=1}^{n}(I_{\alpha\beta}^{\gamma_1}I_{\delta\beta}^{\gamma_2}-I_{\alpha\delta}^{\gamma_2}I_{\delta\beta}^{\gamma_1})=\frac{1}{I_{\gamma_1}I_{\gamma_2}}\sum_{\delta=1}^{n}(C_{\gamma_1}^{\alpha\delta}C_{\gamma_2}^{\delta\beta}-C_{\gamma_2}^{\alpha\delta}C_{\gamma_1}^{\delta\beta})=\frac{1}{I_{\gamma_1}I_{\gamma_2}}\sum_{\delta=1}^{n}(C_{\gamma_1}^{\alpha\delta}C_{\delta}^{\beta\gamma_2}-$$

$$C_{\gamma_1}^{\beta\alpha}C_{\delta}^{\gamma_2\alpha})=-\frac{1}{I_{\gamma_1}I_{\gamma_2}}\sum_{\delta=1}^{n}C_{\gamma_1}^{\gamma_2\delta}C_{\delta}^{\alpha\beta}=\sum_{\gamma_3=1}^{n}\frac{I_{\gamma_3}}{I_{\gamma_1}I_{\gamma_2}}C_{\gamma_3}^{\alpha\beta}\times\frac{C_{\gamma_3}^{\gamma_1\gamma_2}}{I_3}=\sum_{\gamma_3=1}^{n}A_{\gamma_3}^{\gamma_1\gamma_2}I_{\alpha\beta}^{\gamma_3},$$

$$A_{\gamma_3}^{\gamma_1\gamma_2}=\frac{I_{\gamma_3}}{I_{\gamma_1}I_{\gamma_2}}C_{\gamma_3}^{\gamma_1\gamma_2}.$$

For them Jacobi's identity is fulfilled

$$\sum_{s=1}^{n}(A_p^{is}A_s^{jk}+A_p^{js}A_s^{ki}+A_p^{ks}A_s^{ij})=\sum_{s=1}^{n}\frac{I_p}{I_sI_i}C_p^{is}C_s^{jk}\frac{I_s}{I_jI_k}+\frac{I_p}{I_jI_s}C_p^{js}C_s^{ki}\frac{I_s}{I_kI_i}+$$

$$\frac{I_p}{I_kI_s}C_p^{ks}C_s^{ij}\frac{I_s}{I_jI_i}=\frac{I_p}{I_iI_jI_k}\sum_{s=1}^{n}(C_p^{is}C_s^{jk}+C_p^{js}C_s^{ki}+C_p^{ks}C_s^{ij})=0. \quad (16)$$

These infinitesimal operators form a dimensional Lie algebra. Its corresponding group is the n-parametrized Lie group G with $z_{\gamma_0}^0$ $(\gamma=\overline{1,n})$ as parameters.

A matrix $V_{\alpha\beta}(z_\gamma^0)$ of the adjoint representation of a group formed by fundamental matrices $\|\ S_{\alpha\beta}(t_0+T,t_0;z_\gamma^0)\ \|$ is determined according to [3] by the solution $W_{\alpha\beta}$ of the following linear system of differential equations with constant coefficients

$$dW_{\alpha\beta}/dt=\delta_{\alpha\beta}+\sum_{i=1}^{n}\sum_{j=1}^{n}z_i^0A_\alpha^{ij}W_{j\beta} \quad (17)$$

under the initial condition $W_{\alpha\beta}(t,z_\gamma^0)=0$, $(t=0)$ $(\alpha,\beta,\gamma=\overline{1,n})$, where $V_\beta^\alpha(z_j^0)=W_{\alpha\beta}(T,z_\gamma^0)$. For restoration of a n-parameter group by means of structural constants, it is necessary to solve the Cauchy problem for a system (17).

According to [4] for the solution of an initial boundary-value problem, we need to solve also the second Cauchy problem for a system of linear equations in partial derivatives

$$\partial r^\alpha(\vec{\zeta})/\partial\zeta^\beta=\sum_{\beta=1}^{n}\sum_{\mu=1}^{n}V_\gamma^\mu(\vec{\zeta})I_\beta^\alpha(\mu)r^\beta(\vec{\zeta}),\qquad \vec{r}(\vec{\zeta})|_{\zeta=0}=x_0,\qquad \vec{\zeta}=\vec{\zeta}(S). \quad (18)$$

For this, the trajectory connecting $\vec{x_0}$ and $\vec{x_1}$ in R^n is given and a Riemann connexity is introduced

$$\Gamma_{\gamma\beta}^\alpha(\vec{\zeta})=-\sum_{\mu=1}^{n}I_\beta^\alpha(\mu)V_\gamma^\mu(\vec{\zeta}).$$

Then the Cauchy problem for equation (18) can be reduced to the definition $\vec{\zeta}(s)$, $s\in[0,1]$, from the equation

$$\frac{dr^\alpha(S)}{dS}-\sum_{\beta=1}^{n}\sum_{\gamma=1}^{n}\Gamma_{\beta\gamma}^\alpha(\vec{\zeta})r^\gamma(S)\frac{d\zeta^\beta}{dS}=0;\qquad \vec{\zeta}(0)=0. \quad (19)$$

The solution of a boundary-value optimization problem is obtained by integrating a system (6) with the initial condition $z(\vec{0})=\zeta(\vec{1})$. The approach proposed uses no iterative procedures and is applicable for solving the optimal control problems in a real time scale.

3 Conclusion

The analysis fulfilled above of the system with a multiplicative control demonstrated the following possibilities.

1. Construction of the Lie group representation basis with a minimum dimension.
2. Reduction of the two-point boundary optimization problem to Cauchy one for an auxiliary system which has to be integrated along a smooth fixed trajectory joining given points in the state space of the system.
3. Practically such a method is applicable for a real-time on-board control.

References

[1] Wong Wei, Brockhaus R., Xiao Shunda, Nonlinear multi-point modelling and parameter estimation of Do 28 research aircraft, 11-th IFAC World congress, Tallin, Estonia, USSR, August 13–17, V.5, 1990.

[2] Brockett Roger W., Lie algebras and Lie groups in control theory, in: Mathematical methods in system theory, Moscow, Mir, 1979.

[3] Pontryagin L.S., Continuous Groups, Moscow, Nauka, 1984 (in Russian).

[4] Smirnov V.I., Course of High Mathematics, Moscow, ITTL, 1951, V.3, part 1, 340 (in Russian).