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New Spherically Symmetric Solutions of Nonlinear Schrödinger Equations

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Abstract

New soliton-like spherically symmetric solutions for nonlinear generalizations of the Schrödiner equation are constructed. A new nonlinear projective invariant Schrödiner equation is suggested and formulae of multiplication of its solutions are found.

1 Introduction

At the present time there are no efficient analytic methods for solving nonlinear partial differential equations (PDEs). Construction of particular exact solutions to these equations remains an important problem. Finding exact solutions that have physical interpretation is of fundamental importance. Soliton solutions seem to be just as needed, since they describe the movement of a traveling solitary wave.

In many papers (see, for example, [1–4]) there are constructed and investigated soliton–like ones of the nonlinear Schrödiner equation (NSchE)

$$i\Psi_t + \Delta\Psi + \lambda|\Psi|^{2d}\Psi = 0, \tag{1}$$

where $\Psi_t = \partial\Psi/\partial t, \Delta = \partial^2/\partial(x_1)^2 + ... + \partial^2/\partial(x_n)^2, \Psi = \Psi(t,\vec{x}), \vec{x} = (x_1, ..., x_n),\ \lambda =$ const, $d \in \Re$. But almost all of these solutions are solutions to the one-dimensional NSchEs ($n = 1$) or their simple generalizations to the n-dimensional case. For example, the well-known soliton solution [1]

$$\Psi = (\lambda/2)^{1/2}\alpha_1 \frac{\exp[(-iv/2)(x_1 + (v/2 - 2(\alpha_1)^2/v)t)]}{\cos h(\alpha_1 x_1 + \alpha_1 vt)} \tag{2}$$

to the NSchE

$$i\Psi_t + \Psi_{x_1x_1} + \lambda|\Psi|^2\Psi = 0 \tag{3}$$

can be generalized to the n-dimensional case by substituting $\alpha_1 x_1 \to \alpha_1 x_1 + \cdots + \alpha_n x_n$, $\alpha_a, v \in \Re, a = 1, ..., n$. In the case of many-dimensional NSchEs, it is of great importance to find spherically symmetric solutions (SSSs).

In the present paper (Section 2) we construct soliton-like SSSs to the NSchEs of the form

$$\Delta\Psi + \lambda h_1(|\vec{x}|^2)|\Psi|^{2d}\Psi + \alpha h_2(|\vec{x}|^2)\Psi = 0, \qquad (4)$$

where h_1 and h_2 are arbitrary differentiable functions, λ and α are constants, $d \in \Re, \Psi = \Psi(\vec{x})$ and $|\vec{x}|^2 = x_1^2 + ... + x_n^2$.

Note that nonstationary NSchEs of the form

$$i\Psi_{1,t} + \Delta\Psi_1 + \lambda h_1(|\vec{x}|^2)|\Psi_1|^{2d}\Psi_1 = 0, \Psi_{1,t} = \partial\Psi_1/\partial t \qquad (5)$$

are reduced, by the substitution

$$\Psi_1(t, \vec{x}) = \exp(-i\alpha t)\Psi(\vec{x}), \qquad (6)$$

to Eq. (4) for $h_2 = 1$. Consequently, solutions of the NSchEs (5) can be obtained from those of the stationary NSchE (4) by the substitution (6).

In Section 3 new exact solutions of a NSchE of the form (5) are constructed.

In Section 4 the ansätze to Eq.(4) for obtaining periodic solutions and solutions with a singularity are presented.

2 Soliton-like solutions of the nonlinear Schrödiner equations (4)

Since we are constructing spherically symmetric solutions (SSSs), we will always have

$$\Psi(\vec{x}) = \phi(|\vec{x}|^2). \qquad (7)$$

Substituting the ansatz (7) in Eq. (4), we obtain the following nonlinear ordinary differential equation (ODE):

$$4\omega\phi_{\omega\omega} + 2n\phi_\omega + \lambda h_1(\omega)|\phi|^{2d}\phi + \alpha h_2(\omega)\phi = 0, \omega = |\vec{x}|^2 \qquad (8)$$

(indices denote derivatives with respect to ω). It is easily seen that the substitution

$$\phi = \rho(\omega)\exp(iW(\omega)), \qquad (9)$$

where ρ and W are real functions, reduces the ODE (8) to the following nonlinear system: (10)

$$W_\omega = c\omega^{-n/2}\rho^{-2}, \qquad c \in \Re, \qquad (10a)$$

$$4\omega\rho_{\omega\omega} + 2n\rho_\omega + \lambda h_1(\omega)\rho^{2d+1} - 4c^2\omega^{1-n}\rho^{-3} + \alpha h_2(\omega)\rho = 0. \qquad (10b)$$

In the special case, where

$$d = -2, \quad h_1(\omega) = \omega^{1-n}, \quad \lambda = 4c^2, \quad h^2(\omega) = 1,$$

Eq. (10b) reduces to the Bessel equation, and we obtain its general solution in the form [6]

$$\begin{aligned} \rho(\omega) &= \omega^{1/2-n/4}Z_\nu((\alpha\omega)^{1/2}), \quad \nu = n/2 - 1, \\ W(\omega) &= c\textstyle\int \omega^{-1}[Z_\nu((\alpha\omega)^{1/2})]^{-2}d\omega + c_0, \end{aligned} \qquad (11)$$

where Z_ν are cylindrical functions.

Thus, the formulas (7), (9), (11) give the family of the SSSs to the NSchE

$$\Delta\Psi + \lambda|\vec{x}|^{2-2n}|\Psi|^{-4}\Psi + \alpha\Psi = 0. \tag{12}$$

In the general case the system (10) cannot be integrated. For this reason we will construct particular exact solutions to this system. Consider the following generalization of soliton-like solutions:

$$\rho(\omega) = af(\omega)[\cos h(g(\omega))]^{-1/d}, \quad \omega = |\vec{x}|^2, \tag{13}$$

where f and g are arbitrary twice differentiable real functions except for a finite number of points, and a is an arbitrary constant.

Note that it is easily seen that the soliton-like solution (2) can be obtained from the formula (13) for $n = 1$ and $d = 1$ if we apply the Galilei transformations (for details see [3]).

Theorem 1 *Let a NSchE be of the form (4). Then*

$$\Psi(|\vec{x}|^2) = af(|\vec{x}|^2)\left[\cos h\left(g(|\vec{x}|^2)\right)\right]^{-1/d} \tag{14}$$

is a soliton-like SSS of this equation if

$$\begin{aligned} &f = c_1|\vec{x}|^{2-n} + c_0, \quad n \neq 2 \quad or \quad f = c_1 \ln|\vec{x}|^2 + c_0, \quad n = 2, \\ &g = c_2 \textstyle\int \omega^{-n/2} f^{-2} d\omega, \quad \omega = |\vec{x}|^2, \\ &h_1 = (1+d)b^2\lambda^{-1}a^{-2d}|\vec{x}|^{2-2n}f^{-4-2d}, \quad h_2 = -b^2\alpha^{-1}|\vec{x}|^{2-2n}f^{-4}, \end{aligned} \tag{15}$$

where $b^2 = 4(c_2)^2/d^2, \quad \alpha\lambda \neq 0, \quad d > 0, \quad c_0, c_1, c_2 \in \Re.$

The proof of Theorem 1 is realized in [5].

Remark 1. In case $d < 0$ the formula (14) gives the solution of the NSchE (4) but it is not a soliton-like solution.

Consider some examples, which are corollaries of Theorem 1.

Example 1. The case $n = 1 \quad (\vec{x} = x_1)$. For $c_1 = 0$ (see (15)) we obtain the well–known soliton–like solution

$$\Psi = a[\cos h(d\alpha^{1/2}x_1 + c)]^{-1/d} \tag{16}$$

of the stationary NSchE

$$\Psi_{x_1x_1} + \lambda|\Psi|^{2d}\Psi - \alpha\Psi = 0$$

(indices denote derivatives with respect to x_1). Here and subsequently, $a^{2d} = (1 + d)\alpha/\lambda, \ \ \alpha > 0, \ \ \lambda > 0, \ \ c \in \Re$.

From the solution (16), using the formula (6) for $\alpha \to -\alpha$ and the Galilei transformations [3], it is easily to obtain the following soliton solution:

$$\Psi = a\frac{\exp[(-iv/2)(x_1 + (v/2 - 2\alpha/v)t)]}{[\cos h(d\alpha^{1/2}(x_1 + vt) + c)]^{1/d}}$$

to the nonstationary NSchE (5) for $h_1 = 1$.

If $c_0 = 0$ (see (15)), then we obtain the solution

$$\Psi = ax_1[\cos h(d\alpha^{1/2}/x_1 + c)]^{-1/d} \quad (17)$$

to the NSchE

$$\Psi_{x_1x_1} + \lambda|x_1|^{-4-2d}|\Psi|^{2d}\Psi - \alpha|x_1|^{-4}\Psi = 0.$$

Example 2. The case $n = 2$ $(\vec{x} = (x_1, x_2))$. For $c_1 = 0$ (see (15)) we obtain the soliton-like solution

$$\Psi = a[\cos h(d\alpha^{1/2}\ln|\vec{x}| + c)]^{-1/d} \quad (18)$$

of the stationary NSchE

$$\Delta_2\Psi + (\lambda/|\vec{x}|^2)|\Psi|^{2d}\Psi - (\alpha/|\vec{x}|^2)\Psi = 0,$$

d being a positive constant.

Example 3. The case $n = 3$ $(\vec{x} = (x_1, x_2, x_3))$. For $c_1 = 0$ (see (15)) we find the soliton-like solution

$$\Psi = a[\cos h(d(\alpha/|\vec{x}|^2)^{1/2} + c)]^{-1/d} \quad (19)$$

of the stationary NSchE

$$\Delta_3\Psi + (\lambda/|\vec{x}|^4)|\Psi|^{2d}\Psi - (\alpha/|\vec{x}|^4)\Psi = 0,$$

d being a positive constant.

In case $c_0 = 0$ we obtain the soliton-like SSS

$$\Psi = a|\vec{x}|^{-1}[\cos h\,(d(\alpha|\vec{x}|^2)^{1/2} + c)]^{-1/d} \quad (20)$$

of the stationary NSchE

$$\Delta_3\Psi + \lambda|\vec{x}|^{2d}|\Psi|^{2d}\Psi - \alpha\Psi = 0,$$

d being a positive constant.

From the solution (20) we can obtain (see (6)) the solution

$$\Psi_1 = a|\vec{x}|^{-1}\exp(i\alpha t)[\cos h(d(\alpha|\vec{x}|^2)^{1/2} + c)]^{-1/d}. \quad (21)$$

to the nonstationary NSchE

$$i\Psi_{1,t} + \Delta_3\Psi_1 + \lambda(|\vec{x}||\Psi_1|)^{2d}\Psi_1 = 0. \quad (22)$$

Note that for the solutions (20)–(21) possessing discontinuity at the point $|\vec{x}|^2 = 0$ the integrals are well defined in the vicinity of $\vec{x} = 0$, since we work in the space $\Re^3$.

Example 4. The case $n = 4$ $(\vec{x} = (x_1, x_2, x_3, x_4))$. For $c_1 = 0$ (see (15)) we find the soliton-like solution

$$\Psi = a[\cos h\,(d\alpha^{1/2}/(2|\vec{x}|^2) + c)]^{-1/d} \quad (23)$$

of the NSchE

$$\Delta_4\Psi + (\lambda/|\vec{x}|^6)|\Psi|^{2d}\Psi - (\alpha/|\vec{x}|^6)\Psi = 0,$$

d being a positive constant.

In case $c_0 = 0$ we obtain the soliton-like SSS

$$\Psi = a|\vec{x}|^{-2}[\cos h(d\alpha^{1/2}|\vec{x}|^2/2 + c)]^{-1/d}$$

of the stationary NSchE

$$\Delta_4\Psi + \lambda|\vec{x}|^{2+4d}|\Psi|^{2d}\Psi - \alpha|\vec{x}|^2\Psi = 0 \tag{24}$$

d being a positive constant.

Observe that Eq. (24) is a nonlinear generalization of the harmonic oscillator equation

$$\Delta\Psi - \alpha|\vec{x}|^2\Psi = 0.$$

Remark 2. The $\vec{x} = 0$ is not a point of discontinuity to the solutions (17)–(19), (23) for $d > 0$. So, the integrals are well defined in the vicinity of $\vec{x} = 0$.

Theorem 1 admits the following generalization [5].

Theorem 2 *The soliton-like solution (14) is a solution of the NSchE of the form (4) if*

$$g = c_2 \int \omega^{-n/2} f^{-2} d\omega, \quad \omega = |\vec{x}|^2, \tag{25a}$$

$$h_1 = (1 + d)b^2\lambda^{-1}a^{-2d}|\vec{x}|^{2-2n}f^{-4-2d}, \tag{25b}$$

$$h_2 = -(\alpha f)^{-1}\left[b^2|\vec{x}|^{2-2n}f^{-3} + 4\omega f_{\omega\omega} + 2nf_\omega\right], \tag{25c}$$

where $f(|\vec{x}|^2)$ is an arbitrary twice differentiable real functions except for a finite number of points, $b^2 = 4(c_2)^2/d^2$, $\alpha\lambda \neq 0$, $d > 0$, $c_2 \in \Re$.

From Theorem 2 we can obtain the following corollary .

Corollary. *The soliton-like solution (14) is a solution of the NSchE of the form*

$$\Delta\Psi + \lambda h_1(|\vec{x}|^2)|\Psi|^{2d}\Psi + \alpha\Psi = 0,$$

if $f = Re(-\omega)^{1/2-n/4}Z_\nu((\alpha\omega)^{1/2})$, $Im(-\omega)^{1/2-n/4}Z_\nu((\alpha\omega)^{1/2}) \neq 0$, and the functions g and h are determined with the help of the formulas (25a), (25b). Here Z_ν are cylindrical functions and $\nu = |n/2 - 1|$, $n = 1, 3, 5...$

The proof of this corollary is realized in [5].

3 Soliton-like solutions of nonlinear Schrödiner equations of the form (5)

The NSchE (22) for $d = 2$ has a wide Lie symmetry. In fact, using the Lie method [7–9] we can prove that in this case Eq. (22) is invariant under the projective transformations

$$t' = t/(1 - pt), \quad x' = x/(1 - pt), \quad p \in \Re, \tag{26a}$$

$$\Psi_1' = \Psi_1(1 - pt)^{3/2} \exp\frac{ip|\vec{x}|^2}{4(1 - pt)}, \tag{26b}$$

the scale transformations

$$t' = m^2 t, \quad x' = mx, \quad \Psi_1' = m^{-3/2}\exp(iq)\Psi_1, \quad m, q \in \Re, \tag{27}$$

the translations in variable t, and rotations in the space $\Re^3$.

Using the transformations (26) to the solutions (21) for $d = 2$, we obtain the following soliton-like SSS

$$\Psi_1 = a[|\vec{x}|^2(1-pt)]^{-1/2} \exp \frac{i(4\alpha t - p|\vec{x}|^2)}{4(1-pt)} \left[\cos h \left(\frac{2(\alpha|\vec{x}|^2)^{1/2}}{1-pt} + c \right) \right]^{-1/2} \tag{28}$$

of the NSchE

$$i\Psi_{1,t} + \Delta_3 \Psi_1 + \lambda(|\vec{x}||\Psi_1|)^4 \Psi_1 = 0. \tag{29}$$

Note that the solution (28) is not a solution of the form (6). Moreover, using the transformations (26), (27) and the translations in variable t (for details see [3, 4]), we can construct the following formula for obtaining new solutions of the NSchE (29)

$$\Psi_1(t, \vec{x}) = \frac{1}{t^{3/2}} \exp\left(\frac{i|\vec{x}|^2}{4t} \right) V\left(-\frac{1}{t}, \frac{\vec{x}}{t} \right) \tag{30}$$

where $V(t, \vec{x})$ is an arbitrary fixed solution of this equation.

Note that the formula (30) is valid as well for any other (1+3)-dimensional nonlinear equation, which is invariant with respect to the transformations (26), (27) and the translations in variable t.

4 Discussion

We have thus proved Theorems 1 and 2 that allow us to construct soliton-like SSSs of the form (14) to the NSchE (4). This is well illustrated by the examples 1–4. In particular cases, where we have solutions to the NSchE (4) for $h_2 = \text{const}$, exact solutions can be found to the nonstationary NSchE (5). The well-known soliton-like solutions of the (1+1)-dimensional Schrödinger equation with a power nonlinearity, for example, Zakharov–Shabat‘s solution, can be easily obtained from the formula (14) by the Galilean transformations.

It should be observed that the ansatz (13) is a particular case of one for Lie‘s solutions (see, for example, [7–9]) and of the ansatz for non-Lie‘s solutions [9, 10]. But here we determine first the ‘outer‘ function $F = (\cos h)^{-1/d}$ and then obtain a system of ODEs for the unknown functions f and g. According to the Lie method [7–9] we find first the functions f and g, whereupon we find the unknown function F.

As concerns the direct method [10], it does not provide a clear algorithm for obtaining the ‘outer‘ function F and it does not connect this function with the Lie solutions.

Here, for obtaining the function F, we have used the form of the Lie solutions constructed in [3]. Moreover, using the other Lie solutions from the papers [3], we can find new ones to the NSchE (4). For example we can obtain periodic SSSs of the form

$$\Psi(|\vec{x}|^2) = af(|\vec{x}|^2) \left[\cos(g(|\vec{x}|^2)) \right]^{-1/d}, \tag{31}$$

and solutions with a singularity for $d > 0$ of the form

$$\Psi(|\vec{x}|^2) = af(|\vec{x}|^2) \left[\sin h\, (g(|\vec{x}|^2)) \right]^{-1/d}. \tag{32}$$

For ansätze (31)–(32) we can obtain theorems that are analogous to Theorem 1, 2.

Thus, a wide class of the Lie solutions to the nonlinear Schrödiner equation enables us to determine a structure of exact solutions to a generalization of this equation.

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