Nonlinear Mathematical Physics 1997, V.4, N 1–2, 78–88.

The Symmetry Reduction of Nonlinear Equations of the Type $\Box u + F(u, \underset{1}{u})u_0 = 0$ to Ordinary Differential Equations

Leonid BARANNYK and Halyna LAHNO

Department of Mathematics, Pedagogical University,
2 Ostrogradsky Street, 314003, Poltava, Ukraïna

Abstract

The reduction of two nonlinear equations of the type $\Box u + F(u, \underset{1}{u})u_0 = 0$ with respect to all rank three subalgebras of a subdirect sum of the extended Euclidean algebras $A\tilde{E}(1)$ and $A\tilde{E}(3)$ is carried out. Some new invariant exact solutions of these equations are obtained.

1 Introduction

Fushchych and Serova [1] have described equations of the type

$$\Box u + F(u, \underset{1}{u})u_0 = 0$$

which are invariant under subdirect sums of the extended Euclidean algebras $A\tilde{E}(1) = < P_0, D_1 >$ and $A\tilde{E}(3) =< P_1, P_2, P_3 > \niplus (AO(3) \bigoplus < D_2 >)$. Such, in particular, are the equations

$$\frac{\partial^2 u}{\partial x_0^2} - \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_3^2} + \lambda u \frac{\partial u}{\partial x_0} = 0, \tag{1.1}$$

$$\frac{\partial^2 u}{\partial x_0^2} - \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_3^2} + \lambda \frac{\partial u}{\partial x_0} \exp(u) = 0, \tag{1.2}$$

where λ is an arbitrary nonzero real constant. It is known [1] that the maximal invariance algebra of equation (1.1) in Lie sense is the algebra $F_{(1)}$ generated the by vector fields

$$P_0 = \frac{\partial}{\partial x_0}, \quad P_a = \frac{\partial}{\partial x_a}, \quad J_{ab} = x_a \frac{\partial}{\partial x_b} - x_b \frac{\partial}{\partial x_a},$$
$$D = x_0 \frac{\partial}{\partial x_0} + x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} - u \frac{\partial}{\partial u},$$

where $a, b = 1, 2, 3$. The maximal invariance algebra of equation (1.2) is the algebra $F_{(2)}$ generated by the vector fields P_0, P_a, J_{ab} $(a, b = 1, 2, 3)$ and field

$$D = x_0 \frac{\partial}{\partial x_0} + x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} - \frac{\partial}{\partial u}.$$

Some exact solutions of equation (1.1) were found in [2] (for a two-dimensional case where $\lambda = 2$) and in [1, 3] (for three- and four-dimensional cases). Some particular solutions of equation (1.2) are obtained in [1, 3].

In present paper, complete lists of subalgebras of the algebras $F_{(1)}$ and $F_{(2)}$ with respect to conjugation have been found and new exact solutions of the investigated equations are constructed by solutions of ordinary differential equations obtained as a result of reduction on rank three subalgebras (as regards the concepts and results used here, see [4, 5] as well).

Equations (1.1) and (1.2) being invariant, the transformation $(x_0, x_1, x_2, x_3, u) \to (x_0, -x_1, x_2, x_3, u)$ under their reducing subalgebras of algebras $F_{(i)}$ can be regarded with respect to conjugation determining by group $G_{(i)}$ generated by inner automorphisms of the algebra $F_{(i)}$ $(i = 1, 2)$ and the discrete automorphism $P_0 \to P_0, P_1 \to -P_1, \quad P_2 \to P_2, \quad P_3 \to P_3, J_{12} \to -J_{12}, \quad J_{13} \to -J_{13}, \quad J_{23} \to J_{23}, \quad D \to D$.

Applying a general method suggested in [6] and complemented by a number of propositions in [7], we carry out the required classification of all subalgebras of the algebra $F_{(i)}$. The complete list of required subalgebras is given in Sec.2.

Let ω, ω' be a system of functionally independent invariants of rank three subalgebra L of the algebra $F_{(i)}$. Then the ansatz

$$\omega' = \varphi(\omega) \tag{1.3}$$

reduces equation (1.1) or (1.2) to a differential equation involving only $\omega, \varphi, \dot{\varphi}, \ddot{\varphi}$. Such reduction is called symmetry reduction. It is presented in Sec.3 and 4. For each of rank three subalgebras, we point out the corresponding ansatz (1.3) solved for u, the invariant ω as well as the reduced equation which is obtained by means of this ansatz. In the cases where a reduced equation can be solved we point out the corresponding invariant solutions of equations (1.1) and (1.2).

We denote the real Lie algebra with generators $X_1, \ldots, X_s$ by $< X_1, \ldots, X_s >$, sequence of algebras $U_1 \dotplus K, \ldots, U_m \dotplus K$ by $K : U_1, \ldots, U_m$.

2 Classification of subalgebras of the invariance algebra

We restrict ourselves to consideration of the case of subalgebras of the algebra $F = F_{(1)}$ as for as classification of subalgebras of the algebra $F_{(2)}$ does not differ from classification of subalgebras of the algebra $F_{(1)}$. Obtained subalgebras can be interpreted just as subalgebras of the algebra $F_{(2)}$ if we use properly the representation of their generators.

Among subalgebras of the algebra F possessing the same invariants, there exists the subalgebra contaning other subalgebras. We call it I-maximal. To carry out the symmetry reduction of equations (1.1), it is sufficient to classify I-maximal subalgebras of the algebra F up to conjugacy under G, where $G = G_{(1)}$.

It is known that the algebra $AO(3) =< J_{12}, J_{13}, J_{23} >$ has with respect to inner automorphisms only three subalgebras: $0, < J_{12} >, AO(3)$. The algebra $AO(3)$ is a simple algebra. Applying the Lie-Goursat classification method for subalgebras of algebraic sums of Lie algebras [6, 7], we come to the conclusion that up to $O(3)$-conjugacy subalgebras of the algebra $AO(3) \bigoplus < D >$ are exhausted by the following subalgebras:

$$\begin{aligned} &0, < D >, < J_{12} >, < J_{12} + \alpha D > \ (\alpha \in R, \alpha > 0), \\ &< J_{12}, D >, AO(3), AO(3) \bigoplus < D > . \end{aligned} \tag{2.1}$$

Let K be one of the subalgebras (2.1) and $\hat{K}$ be such a subalgebra of the algebra F that its projection onto $AO(3) \bigoplus < D >$ coincides with K. If the projection of K onto $< D >$ is nonzero, then K annuls only a zero subspace in the space $U =< P_0, P_1, P_2, P_3 >$. Since K is a completely reducible algebra of linear transformations of this space, then in view of Theorem I.5.3 [7], the algebra $\hat{K}$ is a splitting one, i.e., it is conjugated with an algebra of the form $V \subset\!\!\!\!+ K$, where $V \subset U$. Let $\pi(K)$ be the projection of K onto $AO(3)$. If $\pi(K) =< J_{12} >$, then in view of Theorem III.4.1 [7] the algebra $\hat{K}$ contains its projection onto $< P_1, P_2 >$, and if $\pi(K) = AO(3)$, then $AO(3) \subset \hat{K}$ and $\hat{K}$ contains its projection onto $< P_1, P_2, P_3 >$.

In view of Witt's mapping theorem [8] and the Lie-Goursat classification method, nonzero subspaces of the space U are exhausted with respect to $O(3)$-conjugation by the subspaces:

$$\begin{aligned} &< P_0 >, < \alpha P_0 + P_1 >, < P_0, P_1 >, < \alpha P_0 + P_1, P_2 > \\ &< P_0, P_1, P_2 >, < \alpha P_0 + P_1, P_2, P_3 >, < P_0, P_1, P_2, P_3 >, \end{aligned} \tag{2.2}$$

where $\alpha \in R$ and $\alpha \geq 0$. Using the automorphism corresponding to the element $\exp(\theta D)$, we can assume that $\alpha \in \{0, 1\}$.

It is proved in the work [9] that the algebra $AE(3) =< P_1, P_2, P_3 > \subset\!\!\!\!+ AO(3)$ has with respect to inner automorphisms such and only such nonzero subalgebras:

$$\begin{aligned} &< P_1 >, < P_1, P_2 >, < P_1, P_2, P_3 >; \\ &< J_{12} >: 0, < P_3 >, < P_1, P_2 >, < P_1, P_2, P_3 >; \\ &< J_{12} + \alpha P_3 >: 0, < P_1, P_2 > (\alpha \neq 0); \\ &< J_{12}, J_{13}, J_{23} >: 0, < P_1, P_2, P_3 > . \end{aligned}$$

For description of subalgebras of the algebra $< P_0 > \bigoplus AE(3)$, it should be used the Lie-Goursat classification method and remarks made previosly.

According to what has been said, it is not difficult to receive that the algebra F has with respect to G-conjugation only such I-maximal subalgebras:

A. Subalgebras having zero projections onto $AO(3)$:

$$< P_0 >, < \alpha P_0 + P_1 >, < P_0, P_1 >, < \beta P_0 + P_1, P_2 >,$$

where $\alpha \geq 0, \beta > 0$;

$$< D >: 0, < P_0 >, < \alpha P_0 + P_1 >, < P_0, P_1 >, < \beta P_0 + P_1, P_2 >,$$

where $\alpha \geq 0, \beta > 0$.

B. Subalgebras having zero projections onto $< D >$ and nonzero projections onto $AO(3)$:

$< J_{12} >: 0, < P_0 >, < \alpha P_0 + P_3 >, < P_0, P_3 >, < P_1, P_2 >,$

$< P_0, P_1, P_2 >, < \beta P_0 + P_3, P_1, P_2 >,$ where $\alpha \geq 0, \beta > 0;$

$< J_{12} + P_0 >: 0, < \alpha P_0 + P_3 > (\alpha \geq 0);$

$< J_{12} + \alpha P_0 + P_3 > (\alpha \geq 0);$

$< J_{12}, J_{13}, J_{23} >: 0, < P_0 >, < P_1, P_2, P_3 >, < P_0, P_1, P_2, P_3 > .$

C. Subalgebras having nonzero projections onto $AO(3)$ and $< D >$:

$< J_{12} + \beta D >: 0, < P_0 >, < \alpha P_0 + P_3 >, < P_0, P_3 >,$ where $\alpha \geq 0, \beta > 0;$

$< J_{12}, D >: 0, < P_0 >, < \alpha P_0 + P_3 >, < P_0, P_3 >, < P_1, P_2 >,$

$< P_0, P_1, P_2 >, < \alpha P_0 + P_3, P_1, P_2 >, (\alpha \geq 0);$

$AO(3) \bigoplus < D >: 0, < P_0 >, < P_1, P_2, P_3 >, < P_0, P_1, P_2, P_3 > .$

3 Reduction of equation (1.1) to ordinary differential equations

If $u = u(x_1, x_2, x_3)$ is the solution of equation (1) or (2), then u is a solution of the Laplace equation $\Delta u = 0$. In this connection, let us restrict ourselves to subalgebras of the algebra F that don't contain P_0.

3.1. $< \alpha P_0 + P_1, P_2, P_3, J_{23} > (\alpha \geq 0) : u = \varphi(\omega),\ \omega = x_0 - \alpha x_1,$

$$(1 - \alpha^2)\ddot{\varphi} + \lambda \varphi \dot{\varphi} = 0. \tag{3.1}$$

If $\alpha = 1$, then $\varphi = C$. Let $\alpha \neq 1$. Equation (3.1) is equivalent to

$$\int \frac{d\varphi}{\varphi^2 + C_1} = \frac{\lambda}{2(\alpha^2 - 1)}\omega + C_2.$$

For $C_1 = a^2 > 0$ we have $\varphi = a \tan\left\{\frac{\lambda a \omega}{2(\alpha^2 - 1)} + C_2\right\}$. The corresponding solution of equation (1.1) has the form

$$u = a \tan\left\{\frac{\lambda a(x_0 - \alpha x_1)}{2(\alpha^2 - 1)} + C_2\right\}.$$

For $C_1 = 0$ we have $\varphi = \dfrac{2(1 - \alpha^2)}{\lambda\omega + C}$, and therefore $u = \dfrac{2(1 - \alpha^2)}{\lambda(x_0 - \alpha x_1) + C}$. For $C_1 = -a^2 < 0$ we find that $\varphi = \dfrac{a\left(1 + C\exp\left\{\frac{\lambda a}{\alpha^2 - 1}\omega\right\}\right)}{1 - C\exp\left\{\frac{\lambda a}{\alpha^2 - 1}\omega\right\}}$. The corresponding solution of equation (1.1) will be written in the form

$$u = \frac{a\left(1 + C\exp\left\{\frac{\lambda a}{\alpha^2 - 1}(x_0 - \alpha x_1)\right\}\right)}{1 - C\exp\left\{\frac{\lambda a}{\alpha^2 - 1}(x_0 - \alpha x_1)\right\}}.$$

3.2. $<\alpha P_0+P_1,P_3,D>$ $(\alpha\geq 0)$: $u=\dfrac{1}{x_0-\alpha x_1}\varphi(\omega),\omega=\dfrac{x_2}{x_0-\alpha x_1}$,

$$((1-\alpha^2)\omega^2-1)\ddot{\varphi}+4(1-\alpha^2)\omega\dot{\varphi}+2(1-\alpha^2)\varphi-\lambda\varphi^2-\lambda\omega\varphi\dot{\varphi}=0. \quad (3.2)$$

We integrate equation (3.2) and obtain

$$\omega[(1-\alpha^2)\omega^2-1]\dot{\varphi}+[(1-\alpha^2)\omega^2+1]\varphi-\frac{\lambda}{2}\omega^2\varphi^2=C_1. \quad (3.3)$$

For $C_1=0$ equation (3.3) is the Bernoulli equation. Depending on values of α, we receive such its solutions:

$$\varphi=\frac{6\omega}{\lambda\omega^3+C},\ \text{for } \alpha=1;$$

$$\varphi=\frac{8\omega}{\lambda\beta\left\{2\beta\omega+(\omega^2-\beta^2)\left[\ln\left|\frac{\omega+\beta}{\omega-\beta}\right|+C\right]\right\}},\ \text{for } \frac{1}{1-\alpha^2}=\beta^2>0;$$

$$\varphi=\frac{4\omega}{\lambda\beta\left\{-\beta\omega+(\omega^2+\beta^2)\left[\arctan\frac{\omega}{\beta}+C\right]\right\}},\ \text{for } \frac{1}{1-\alpha^2}=-\beta^2<0.$$

Corresponding solutions of equation (1.1) are:

$$u=\frac{6x_2(x_0-x_1)^2}{\lambda x_2^3+C(x_0-x_1)^3},\ \text{for } \alpha=1;$$

$$u=\frac{8x_2(x_0-\alpha x_1)}{\lambda\beta\left\{2\beta x_2(x_0-\alpha x_1)+(x_2^2-\beta^2(x_0-\alpha x_1)^2)\left(\ln\left|\frac{x_2+\beta x_0-\alpha\beta x_1}{x_2-\beta x_0+\alpha\beta x_1}\right|+C\right)\right\}},$$

for $\dfrac{1}{1-\alpha^2}=\beta^2>0$,

$$u=\frac{4x_2(x_0-\alpha x_1)}{\lambda\beta\left\{-\beta x_2(x_0-\alpha x_1)+(x_2^2+\beta^2(x_0-\alpha x_1)^2)\left(\arctan\frac{x_2}{\beta(x_0-\alpha x_1)}+C\right)\right\}},$$

for $\dfrac{1}{1-\alpha^2}=-\beta^2<0$.

Let $C_1\neq 0$ in equation (3.3). If $\alpha\neq 1$ and $\dfrac{1}{1-\alpha^2}=\beta^2$, then equation (3.3) can be written in the form

$$\omega(\omega^2-\beta^2)\dot{\varphi}+(\omega^2+\beta^2)\varphi-\mu\omega^2\varphi^2=C_1,\ \mu=\frac{\lambda}{2(1-\alpha^2)}. \quad (3.4)$$

A solution of equation (3.4) is looked for in the form $\varphi=\dfrac{C_1}{\omega\psi(\omega)+\beta^2}$. The function ψ is defined by the equation

$$\frac{d\psi}{\psi^2-\beta^2+\mu C_1}=\frac{d\omega}{-(\omega^2-\beta^2)}. \quad (3.5)$$

Depending on values of $\beta^2 - \mu C_1$, we receive the following solutions of equation (3.5):

$$\psi = \gamma \frac{|\omega-\beta|^{\frac{\gamma}{\beta}} + C_2|\omega+\beta|^{\frac{\gamma}{\beta}}}{|\omega-\beta|^{\frac{\gamma}{\beta}} - C_2|\omega+\beta|^{\frac{\gamma}{\beta}}}, \quad \text{for} \quad \beta^2 - \mu C_1 = \gamma^2 > 0;$$

$$\psi = 2\beta \left[\ln|\frac{\omega+\beta}{\omega-\beta}| + C_2\right]^{-1}, \quad \text{for} \quad \beta^2 - \mu C_1 = 0;$$

$$\psi = \gamma \tan\left\{\frac{\gamma}{2\beta}\ln\left|\frac{\omega+\beta}{\omega-\beta}\right| + C_2\right\}, \ \text{for } \beta^2 - \mu C_1 = -\gamma^2 < 0.$$

Corresponding solutions of equation (3.4) have the form:

$$\varphi = C_1\left\{\gamma\omega \frac{|\omega-\beta|^{\frac{\gamma}{\beta}} + C_2|\omega+\beta|^{\frac{\gamma}{\beta}}}{|\omega-\beta|^{\frac{\gamma}{\beta}} - C_2|\omega+\beta|^{\frac{\gamma}{\beta}}} + \beta^2\right\}^{-1}, \quad \text{for} \quad \beta^2 - \mu C_1 = \gamma^2 > 0;$$

$$\varphi = C_1\left\{2\beta\omega\left(\ln\left|\frac{\omega+\beta}{\omega-\beta}\right| + C_2\right)^{-1} + \beta^2\right\}^{-1}, \ \text{for } \beta^2 - \mu C_1 = 0;$$

$$\varphi = C_1\left\{\gamma\omega\tan\left\{\frac{\gamma}{2\beta}\ln\left|\frac{\omega+\beta}{\omega-\beta}\right| + C_2\right\} + \beta^2\right\}^{-1}, \ \text{for } \beta^2 - \mu C_1 = -\gamma^2 < 0.$$

Corresponding solutions of equation (1.1) are:

$$u = C_1\left\{\gamma x_2\left(\frac{|x_2-\beta x_0+\alpha\beta x_1|^{\frac{\gamma}{\beta}} + C_2|x_2+\beta x_0-\alpha\beta x_1|^{\frac{\gamma}{\beta}}}{|x_2-\beta x_0+\alpha\beta x_1|^{\frac{\gamma}{\beta}} - C_2|x_2+\beta x_0-\alpha\beta x_1|^{\frac{\gamma}{\beta}}} + \beta^2(x_0-\alpha x_1)\right)\right\}^{-1},$$

for $\beta^2 - \mu C_1 = \gamma^2 > 0$;

$$u = C_1\left\{2\beta x_2\left[\ln\left|\frac{x_2+\beta x_0-\alpha\beta x_1}{x_2-\beta x_0+\alpha\beta x_1}\right| + C_2\right]^{-1} + \beta^2(x_0-\alpha x_1)\right\}^{-1},$$

for $\beta^2 - \mu C_1 = 0$;

$$u = C_1\left\{\gamma x_2\tan\left\{\frac{\gamma}{2\beta}\ln\left|\frac{x_2+\beta x_0-\alpha\beta x_1}{x_2-\beta x_0+\alpha\beta x_1}\right| + C_2\right\} + \beta^2(x_0-\alpha x_1)\right\}^{-1},$$

for $\beta^2 - \mu C_1 = -\gamma^2 < 0$.

If $\alpha \neq 0$ and $\frac{1}{1-\alpha^2} = -\beta^2$, then it is possible to represent equation (3.3) in the form

$$\omega(\omega^2+\beta^2)\dot{\varphi} + (\omega^2-\beta^2)\varphi - \mu\omega^2\varphi^2 = C_1, \ \mu = \frac{\lambda}{2(1-\alpha^2)}. \tag{3.6}$$

A solution of equation (3.6) is looked for in the form $\varphi = \frac{C_1}{\omega\psi(\omega) - \beta^2}$. The function ψ is defined by the equation

$$\frac{d\psi}{\psi^2+\beta^2+\mu C_1} = \frac{d\omega}{-(\omega^2+\beta^2)}. \tag{3.7}$$

If $\beta^2+\mu C_1=\gamma^2>0$, then a general solution of equation (3.7) is:

$$\psi=\gamma\tan\left\{-\frac{\gamma}{\beta}\arctan\frac{\omega}{\beta}+C_2\right\},$$

and the corresponding solution of equation (3.6) is

$$\varphi=-C_1\left[\gamma\omega\tan\left\{\frac{\gamma}{\beta}\arctan\frac{\omega}{\beta}+C_2\right\}+\beta^2\right]^{-1}.$$

If $\beta^2+\mu C_1=0$, then

$$\psi=\frac{\beta}{\arctan\frac{\omega}{\beta}+C_2}\quad\text{and}\quad\varphi=C_1\left[\frac{\beta\omega}{\arctan\frac{\omega}{\beta}+C_2}-\beta^2\right]^{-1}.$$

Provided $\beta^2+\mu C_1=-\gamma^2$, then

$$\psi=\gamma\frac{C_2\exp\left\{\frac{2\gamma}{\beta}\arctan\frac{\omega}{\beta}\right\}+1}{C_2\exp\left\{\frac{2\gamma}{\beta}\arctan\frac{\omega}{\beta}\right\}-1}\ \text{and}\ \varphi=C_1\left\{\gamma\omega\frac{C_2\exp\left\{\frac{2\gamma}{\beta}\arctan\frac{\omega}{\beta}\right\}+1}{C_2\exp\left\{\frac{2\gamma}{\beta}\arctan\frac{\omega}{\beta}\right\}-1}-\beta^2\right\}^{-1}.$$

For the obtained values φ we derive the following solutions of equation (1.1)

$$u=-C_1\left\{\gamma x_2\tan\left\{\frac{\gamma}{\beta}\arctan\frac{x_2}{\beta x_0-\alpha\beta x_1}+C_2\right\}+\beta^2(x_0-\alpha x_1)\right\}^{-1},$$

for $\beta^2+\mu C_1=\gamma^2>0$;

$$u=C_1\left\{\frac{\beta x_2}{\arctan\frac{x_2}{\beta x_0-\alpha\beta x_1}+C_2}-\beta^2(x_0-\alpha x_1)\right\}^{-1}\quad\text{for }\beta^2+\mu C_1=0;$$

$$u=C_1\left\{\gamma x_2\frac{C_2\exp\left\{\frac{2\gamma}{\beta}\arctan\frac{x_2}{\beta x_0-\alpha\beta x_1}\right\}+1}{C_2\exp\left\{\frac{2\gamma}{\beta}\arctan\frac{x_2}{\beta x_0-\alpha\beta x_1}\right\}-1}-\beta^2(x_0-\alpha x_1)\right\}^{-1},$$

for $\beta^2+\mu C_1=-\gamma^2>0$.

3.3. $<\alpha P_0+P_3,J_{12},D>(\alpha\geq 0):u=\frac{1}{x_0-\alpha x_3}\varphi(\omega),\omega=\frac{x_1^2+x_2^2}{(x_0-\alpha x_3)^2}$,

$$4\omega[(1-\alpha^2)\omega-1]\ddot{\varphi}+[10(1-\alpha^2)\omega-4]\dot{\varphi}+2(1-\alpha^2)\varphi-\lambda\varphi^2-2\lambda\omega\varphi\dot{\varphi}=0.$$

The reduced equation is equivalent to one:

$$4\omega[(1-\alpha^2)\omega-1]\dot{\varphi}+2(1-\alpha^2)\omega\varphi-\lambda\omega\varphi^2=C_1, \tag{3.8}$$

where C_1 is an arbitrary constant.

Let $C_1 = 0$. Equation (3.8) is transformed into a separable differential equation. In this case we obtain that

$$u = \frac{4(x_0 - x_3)}{\lambda(x_1^2 + x_2^2) + C(x_0 - x_3)^2}, \text{ for } \alpha = 1,$$

and for $\alpha \neq 1$

$$u = \frac{2(1-\alpha^2)}{\lambda(x_0 - \alpha x_3) + C\sqrt{(1-\alpha^2)(x_1^2 + x_2^2) - (x_0 - \alpha x_3)^2}}.$$

3.4. $AO(3) \bigoplus < D >$: $u = \frac{1}{x_0}\varphi(\omega), \omega = \frac{x_1^2 + x_2^2 + x_3^2}{x_0^2}$,

$$4\omega(\omega - 1)\ddot{\varphi} + (10\omega - 6)\dot{\varphi} - 2\lambda\omega\varphi\dot{\varphi} + 2\varphi - \lambda\varphi^2 = 0.$$

By integrating this equation, we arrive at the Riccati equation:

$$4\omega(\omega - 1)\dot{\varphi} + 2(\omega - 1)\varphi - \lambda\omega\varphi^2 = C_1. \tag{3.9}$$

The substitution $\varphi(\omega) = \frac{1}{t}\psi(t),\ t = \sqrt{\omega}$ reduces equation (3.9) to

$$\frac{d\psi}{\lambda(\psi^2 + \lambda^{-1}C_1)} = \frac{dt}{2(t^2 - 1)}.$$

The general solution of equation (3.9) has the form

$$\varphi = \frac{a}{\sqrt{\omega}} \tan\left\{\frac{\lambda a}{4} \ln\left|C_2 \frac{\sqrt{\omega} - 1}{\sqrt{\omega} + 1}\right|\right\}, \text{ for } C_1 = \lambda a^2, a > 0,$$

$$\varphi = \frac{1}{\sqrt{\omega}}\left(\frac{\lambda}{4} \ln\left|\frac{\sqrt{\omega} + 1}{\sqrt{\omega} - 1}\right| + C_2\right)^{-1}, \text{ for } C_1 = 0,$$

$$\varphi = \frac{a}{\sqrt{\omega}} \frac{\sqrt{|\sqrt{\omega} + 1|^{\lambda a}} + C_2\sqrt{|\sqrt{\omega} - 1|^{\lambda a}}}{\sqrt{|\sqrt{\omega} + 1|^{\lambda a}} - C_2\sqrt{|\sqrt{\omega} - 1|^{\lambda a}}}, \text{ for } C_1 = -\lambda a^2, a > 0.$$

The corresponding solution of equation (1.1) has the form

$$u(x) = \frac{a}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \tan\left\{\frac{\lambda a}{4} \ln\left|C_2 \frac{\sqrt{x_1^2 + x_2^2 + x_3^2} - x_0}{\sqrt{x_1^2 + x_2^2 + x_3^2} + x_0}\right|\right\}, \text{ for } C_1 = \lambda a^2, a > 0;$$

$$u(x) = \frac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \left\{\frac{\lambda}{4} \ln\left|\frac{\sqrt{x_1^2 + x_2^2 + x_3^2} + x_0}{\sqrt{x_1^2 + x_2^2 + x_3^2} - x_0}\right| + C_2\right\}^{-1}, \text{ for } C_1 = 0;$$

$$u(x) = \frac{a}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \frac{\sqrt{|\sqrt{x_1^2 + x_2^2 + x_3^2} + x_0|^{\lambda a}} + C_2\sqrt{|\sqrt{x_1^2 + x_2^2 + x_3^2} - x_0|^{\lambda a}}}{\sqrt{|\sqrt{x_1^2 + x_2^2 + x_3^2} + x_0|^{\lambda a}} - C_2\sqrt{\sqrt{|x_1^2 + x_2^2 + x_3^2} - x_0|^{\lambda a}}}$$

for $C_1 = -\lambda a^2, \quad a > 0$.

4 Reduction of equation (1.2) to ordinary differential equations

4.1. $<\alpha P_0 + P_1, P_3, D>$ $(\alpha \geq 0)$: $u = \varphi(\omega) - \ln\{x_0 - \alpha x_1\}, \omega = \dfrac{x_2}{x_0 - \alpha x_1}$,

$$((1-\alpha^2)\omega^2 - 1)\ddot{\varphi} + 2(1-\alpha^2)\omega\dot{\varphi} - \lambda\omega\exp(\varphi)\dot{\varphi} - \lambda\exp(\varphi) + 1 - \alpha^2 = 0. \qquad (4.1)$$

If we integrate equation (4.1), we obtain

$$((1-\alpha^2)\omega^2 - 1)\dot{\varphi} - \lambda\omega\exp(\varphi) + (1-\alpha^2)\omega = C_1. \qquad (4.2)$$

The substitution $\varphi = \ln\psi$ transforms equation (4.2) into the Bernoulli equation

$$((1-\alpha^2)\omega^2 - 1)\dot{\psi} - \lambda\omega\psi^2 + ((1-\alpha^2)\omega - C_1)\psi = 0. \qquad (4.3)$$

If $\alpha = 1, C_1 \neq 0$, the general solution of equation (4.3) has the form

$$\psi = \frac{C_1^2}{\lambda C_1\omega - \lambda + C_1^2 C_2\exp(-C_1\omega)}.$$

Then

$$\varphi = \ln\frac{C_1^2}{\lambda C_1\omega - \lambda + C_1^2 C_2\exp(-C_1\omega)},$$

and therefore

$$u = \ln\frac{C_1^2}{\lambda C_1 x_2 - \lambda(x_0 - x_1) + C_1^2 C_2(x_0 - x_1)\exp\left\{-\dfrac{C_1 x_2}{x_0 - x_1}\right\}}.$$

For $\alpha = 1, C_1 = 0$, we find that $\psi = \dfrac{2}{\lambda\omega^2 + C}$, and therefore the corresponding solution of equation (1.2) has the form $u = \ln\dfrac{2(x_0 - x_1)}{\lambda x_2^2 + C(x_0 - x_1)^2}$.

If $\alpha \neq 1$ and $\dfrac{1}{1-\alpha^2} = \beta^2 > 0$, equation (4.3) has such a solution depending on $C_1\beta$:

$$\frac{1}{\psi} = \left\{\frac{\lambda\beta|\omega+\beta|}{2(C_1\beta+1)} + \frac{\lambda\beta|\omega-\beta|}{2(C_1\beta-1)} + C_2|\omega-\beta|^{\frac{1+C_1\beta}{2}}|\omega+\beta|^{\frac{1-C_1\beta}{2}}\right\}, \text{ for } C_1\beta \neq 1;$$

$$\frac{1}{\psi} = \frac{\lambda\beta}{4}\left\{(\omega+\beta) + (\omega-\beta)\ln\left|\frac{\omega+\beta}{\omega-\beta}\right| + C_2(\omega-\beta)\right\}, \text{ for } C_1\beta = 1;$$

$$\frac{1}{\psi} = \frac{\lambda\beta}{4}\left\{-(\omega-\beta) + (\omega+\beta)\ln\left|\frac{\omega+\beta}{\omega-\beta}\right| + C_2(\omega+\beta)\right\}, \text{ for } C_1\beta = -1.$$

Corresponding solutions of equation (1.2) have the form

$$u = -\ln\Big\{\frac{\lambda\beta|x_2 + \beta x_0 - \beta\alpha x_1|}{2(C_1\beta+1)} + \frac{\lambda\beta|x_2 - \beta x_0 + \beta\alpha x_1|}{2(C_1\beta-1)} +$$

$$C_2|x_2 - \beta x_0 + \beta\alpha x_1|^{\frac{1+C_1\beta}{2}}|x_2 + \beta x_0 - \beta\alpha x_1|^{\frac{1-C_1\beta}{2}}\Big\}, \text{ for } C_1\beta \neq \pm 1;$$

$$u = -\ln\Big\{\frac{\lambda\beta}{4}(x_2+\beta x_0-\beta\alpha x_1)+\frac{\lambda\beta}{4}(x_2-\beta x_0+\beta\alpha x_1)\ln\left|\frac{x_2+\beta x_0-\alpha\beta x_1}{x_2-\beta x_0+\alpha\beta x_1}\right|+$$

$$C_2(x_2-\beta x_0+\alpha\beta x_1)\Big\},\ \text{for } C_1\beta=1;$$

$$u = -\ln\Big\{-\frac{\lambda\beta}{4}(x_2-\beta x_0+\beta\alpha x_1)+\frac{\lambda\beta}{4}(x_2+\beta x_0-\beta\alpha x_1)\ln\left|\frac{x_2+\beta x_0-\alpha\beta x_1}{x_2-\beta x_0+\alpha\beta x_1}\right|+$$

$$C_2(x_2+\beta x_0-\alpha\beta x_1)\Big\},\ \text{for } C_1\beta=-1.$$

If $\alpha \neq 1$ and $\dfrac{1}{1-\alpha^2} = -\beta^2 < 0$, equation (4.3) has the solution

$$\frac{1}{\psi} = \frac{\lambda\beta^2(C_1\omega-1)}{1+\beta^2C_1^2} + C_2\sqrt{\omega^2+\beta^2}\exp\left\{-\beta C_1\arctan\frac{\omega}{\beta}\right\}.$$

The corresponding solution of equation (1.2) is

$$u = -\ln\Big\{\frac{\lambda\beta^2(C_1x_2-x_0+\alpha x_1)}{1+\beta^2C_1^2}+$$

$$C_2\sqrt{x_2^2+\beta^2(x_0-\alpha x_1)^2}\exp\left\{-\beta C_1\arctan\frac{x_2}{\beta(x_0-\alpha x_1)}\right\}\Big\}.$$

4.2. $<\alpha P_0+P_1,P_2,P_3,J_{23}>$ $(\alpha\geq 0): u=\varphi(\omega),\ \omega = x_0-\alpha x_1,$

$$(1-\alpha^2)\ddot\varphi+\lambda\dot\varphi\exp(\varphi)=0. \qquad (4.4)$$

If $\alpha = 1$, then $\varphi = C$. If $\alpha\neq 1$, then the expression

$$\int\frac{d\varphi}{\lambda\exp(\varphi)+C_1} = \frac{\omega}{\alpha^2-1}+C_2$$

is a general solution of equation (4.4). Hence it appears that

$$\varphi = \ln\left\{\frac{1-\alpha^2}{\lambda(\omega+C_2)}\right\}\ \text{for } C_1=0$$

and

$$\varphi = \ln\left\{\frac{C_1C_2\exp\left\{\frac{C_1\omega}{\alpha^2-1}\right\}}{1-\lambda C_2\exp\left\{\frac{C_1\omega}{\alpha^2-1}\right\}}\right\}\ \text{for } C_1\neq 0.$$

The functions

$$u = \ln\left\{\frac{1-\alpha^2}{\lambda(x_0-\alpha x_1+C)}\right\}\ \text{and}\ u = \ln\left\{\frac{C_1C_2\exp\left\{\frac{C_1}{\alpha^2-1}(x_0-\alpha x_1)\right\}}{1-\lambda C_2\exp\left\{\frac{C_1}{\alpha^2-1}(x_0-\alpha x_1)\right\}}\right\}$$

are corresponding solutions of equation (1.2).

4.3. $<\alpha P_0+P_3,J_{12},D>$ $(\alpha\geq 0): u=\varphi(\omega)-\ln\{x_0-\alpha x_3\}, \omega = \dfrac{x_1^2+x_2^2}{(x_0-\alpha x_3)^2},$

$$4\omega((1-\alpha^2)\omega-1)\ddot\varphi+(6(1-\alpha^2)\omega-4)\dot\varphi-\lambda(2\omega\dot\varphi+1)\exp(\varphi)+1-\alpha^2=0.$$

4.4. $AO(3)\bigoplus <D>$: $u=\varphi(\omega)-\ln x_0, \omega=\dfrac{x_1^2+x_2^2+x_3^2}{x_0^2},$

$$4\omega(\omega-1)\ddot\varphi+6(\omega-1)\dot\varphi-\lambda(2\omega\dot\varphi+1)\exp(\varphi)+1=0.$$

References

[1] Fushchych W.I. and Serova M.M., On some exact solutions of the nonlinear equations which are invariant under Euclidean and Galilei groups, In: Algebraic-theoretical methods in mathematical physics problems, Kiev, Institute of Mathematics, 1983, 24–54.

[2] Rosen G., Solutions of certain nonlinear wave equations, *J. Math. Phys.*, 1966, V.45, N 3–4, 48–56.

[3] Fushchych W.I., Shtelen V.M., Serov N.I., Symmetry Analysis and Exact Solutions of Equations of Nonlinear Mathematical Physics, Kluwer Academic Publishers, 1993, 400p.

[4] Ovsyannikov L.V., Group Analysis of Differential Equations, New York, Academic Press, 1982, 400p.

[5] Olver P. Applications of Lie Groups to Differential Equations, New York, Springer-Verlag, 1986, 400p.

[6] Patera J., Winternitz P. and Zassenhaus H., *J. Math. Phys.*, 1975, V.16, N 8, 1597–1624.

[7] Fushchych W.I., Barannik L.F., Barannik A.F., Subgroups Analysis of the Galilei, Poincaré Groups and Reduction of Nonlinear Equation, Kiev, Naukova Dumka, 1991, 304p.

[8] Lang S., Algebra, New York, Addison-Wesly Publishing Company, Reading, Mass., 1965, 564p.

[9] Beckers J., Patera J., Perroud M. and Winternitz P., Subgroups of the Euclidean group and symmetry breaking in nonrelativistic quantum mechanics, *J. Math. Phys.*, 1977, V.18, N 1, 72–83.