Symmetry Reduction of Poincaré-Invariant Nonlinear Wave Equations

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Abstract
Reduction of multidimensional Poincaré-invariant equations to ordinary differential equations and 2-dimensional equations is considered.

Let us consider the nonlinear wave equation
\[ \Phi(\Box u, (\nabla u)^2, u) = 0, \]  
where \( u = u(x) \) is a scalar function of the variable \( x, \quad x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{1,n} \),
\[ \Box u = \frac{\partial^2 u}{\partial x_0^2} - \frac{\partial^2 u}{\partial x_1^2} - \cdots - \frac{\partial^2 u}{\partial x_n^2}, \quad (\nabla u)^2 = \left( \frac{\partial u}{\partial x_0} \right)^2 - \left( \frac{\partial u}{\partial x_1} \right)^2 - \cdots - \left( \frac{\partial u}{\partial x_n} \right)^2. \]

The equation (1) is invariant under the Poincaré algebra \( AP(1,n) \). The basis of this algebra is formed by the following vector fields
\[ P_\mu = \partial_\mu, \quad J_{0a} = x_0 \partial_a + x_a \partial_0, \quad J_{ab} = x_b \partial_a - x_a \partial_b, \]  
\( \mu = 0, 1, \ldots, n; \quad a, b, = 1, 2, \ldots, n. \)

A great number of well-known equations are particular cases of the equation (1).
Let’s take for instance the d’Alembert equation
\[ \Box u + \lambda u^k = 0; \]  
the Liouville equation
\[ \Box u + \lambda \exp u = 0; \]  
the sine-Gordon equation
\[ \Box u + \lambda \sin u = 0; \]  
the eikonal equation
\[ (\nabla u)^2 = 1. \]

The reduction problem of the equation (1) is very important. The main point of reduction consists in introduction of new variables \( \omega_1(x), \ldots, \omega_k(x) (1 \leq k \leq n) \) being functions of \( x \) and having property that the ansatz \( u = \varphi(\omega_1, \ldots, \omega_k) \) reduces the equation
The problem is more easy if variables $\omega_1, \ldots, \omega_k$ are invariants of some subalgebra of the algebra $AP(1,n)$. These variables are called invariant variables. It is easy to find invariant variables if the optimal system of subalgebras of the algebra $AP(1,n)$ is known. It is impossible in practice to construct the optimal system of subalgebras of the algebra $AP(1,n)$ in a general case (for an arbitrary $n$). But the situation isn’t hopeless, because for symmetry reduction it is enough to know subalgebras having essentially different systems of invariants.

**Definition.** Two subalgebras $L_1, L_2 \subset AP(1,n)$ are called equivalent if there exists a group transformation $\varphi$ transforming the system of invariants of the subalgebra $L_1$ into that $L_2$.

This relation is a more strong relation on the set of all subalgebras of the algebra $AP(1,n)$ than the relation of conjugation.

Among all subalgebras of the algebra $AP(1,n)$ having the same invariants, there exists the algebra containing all subalgebras having this property. This subalgebra is called $I$-maximal. Two $I$-maximal subalgebras are equivalent iff they are conjugate. $I$-maximal subalgebras differ advantageously from the rest subalgebras of the algebra $AP(1,n)$. They are determined uniquely and have a more easy structure. Thus it is enough to construct the system of $I$-maximal subalgebras of the algebra $AP(1,n)$ instead of the optimal system of subalgebras.

Grundland A.M., Harnad J., Winternitz P. [1] classified $I$-maximal subalgebras of rank $n$ of the algebra $AP(1,n)$. It allowed to construct seven ansatzes reducing the equation (1) to ordinary differential equations.

The problem of classification of $I$-maximal subalgebras of the algebra $AP(1,n)$ was solved in works [2, 3]. These results, in particular, imply that there exist 14 types of ansatzes reducing the equation (1) to 2-dimensional equations. Let us adduce main types of these ansatzes

\[ u = \varphi(\omega_1, \omega_2) : \]

1) $\omega_1 = x_0$, $\omega_2 = x_n$;
2) $\omega_1 = x_0$, $\omega_2 = x_1^2 + \ldots + x_m^2$, $m = 1, 2, \ldots, n$;
3) $\omega_1 = x_0 - x_n$, $\omega_2 = x_0^2 - x_1^2 - \ldots - x_m^2 - x_n^2$, $m = 1, 2, \ldots, n - 1$;
4) $\omega_1 = x_1^2 + \ldots + x_m^2$, $\omega_2 = x_0^2 - x_n^2$, $m = 1, 2, \ldots, n - 1$;
5) $\omega_1 = x_0^2 - x_1^2 - \ldots - x_m^2 - x_n^2$, $\omega_2 = x_{m+1}$, $m = 2, 3, \ldots, n - 2$;
6) $\omega_1 = x_0^2 - x_1^2 - \ldots - x_m^2 - x_n^2$, $\omega_2 = x_{m+1}^2 + \ldots + x_q^2$,
   \[ m = 2, 3, \ldots, n - 2; \quad q = m + 1, \ldots, n; \]
7) $\omega_1 = (x_0 - x_n)^2 - 4x_1$, $\omega_2 = (x_0 - x_n)^3 - 6x_1(x_0 - x_n) + 6(x_0 + x_n)$;
8) $\omega_2 = x_0 - x_n$, $\omega_1 = \left( x_0^2 - \sum_{i=1}^{n} \frac{x_0 - x_n}{x_0 - x_n - \gamma_i (x_{d_i-1+1}^2 + \cdots + x_{d_i}^2) - x_n^2} \right)^{\frac{1}{2}}$,
d_0 = 0, d_1, d_2, \ldots, d_t \in R, \ d_0 < d_1 < \ldots < d_t = m, \ m \leq n, \ \gamma_i \in R;

9) \omega_1 = x_0^2 - x_n^2, \ \omega_2 = \alpha \ln(x_0 + x_n) - x_1, \ \alpha > 0;

10) \omega_1 = x_0^2 - x_1^2 - \ldots - x_m^2 - x_n^2, \ \omega_2 = \alpha \ln(x_0 - x_n) + x_{m+1},
    m = 1, 2, \ldots, n - 2, \ \alpha > 0;

11) \omega_1 = x_1^2 + x_2^2, \ \omega_2 = x_0 + \arctan \frac{x_2}{x_1}

Each of these ansatzes may be written in a more general form using transformations of the group \( P(1, n) \).

Let us consider the nonlinear d’Alembert equation

\[ \Box u + \lambda u^k = 0 \tag{7} \]

in the Minkowski space \( R_{1,n} \). The equation (7) has been investigated in works [4, 5].

The equation (7) is invariant under the extended Poincaré algebra \( AP(1, n) \) being obtained from the algebra \( AP(1, n) \) by adding the dilatation operator

\[ D = -x_0 \partial_0 - \ldots - x_n \partial_n + \frac{2}{k-1} u \partial_u. \]

There exists the simple algorithm which allows to classify \( I \)-maximal subalgebras of the algebra \( AP(1, n) \) if the classification of \( I \)-maximal subalgebras of the algebra \( AP(1, n) \) is known. It allows to construct all symmetry ansatzes reducing the equation (7) to ordinary differential equations. The following ansatzes are obtained:

1) \( u = x_0^2 \varphi(\omega), \quad \omega = \frac{x_1^2 + x_2^2 + \ldots + x_m^2}{x_0}, \quad m = 1, 2, \ldots, n; \)

2) \( u = (x_0 - x_n)^{\frac{2}{k-1}} \varphi(\omega), \quad \omega = \frac{(x_0^2 - x_1^2 - \ldots - x_m^2 - x_n^2)^{\frac{1}{2}}}{x_0 - x_n}, \quad m = 1, 2, \ldots, n - 1; \)

3) \( u = (x_0^2 - x_1^2 - \ldots - x_m^2 - x_n^2)^{\frac{1}{k-1}} \varphi(\omega), \quad \omega = \delta \ln(x_0^2 - x_1^2 - \ldots - x_m^2 - x_n^2) - \ln(x_0 - x_n), \quad m = 1, 2, \ldots, n - 1; \)

4) \( u = (x_0 - x_n)^{\frac{1}{k-1}} \varphi(\omega), \quad \omega = \frac{x_0^2 - x_1^2 - \ldots - x_m^2 - x_n^2}{x_0 - x_n} + \ln(x_0 - x_n), \quad m = 1, 2, \ldots, n - 1; \)

5) \( u = (x_1^2 + \ldots + x_m^2)^{\frac{1}{k-1}} \varphi(\omega), \quad \omega = \frac{x_1^2 + x_2^2 + \ldots + x_m^2}{x_1^2 - x_m^2}, \quad m = 1, 2, \ldots, n - 1; \)

6) \( u = x_{m+1}^2 \varphi(\omega), \quad \omega = \frac{x_0^2 - x_1^2 - \ldots - x_m^2 - x_n^2}{x_{m+1}^2}, \quad m = 1, 2, \ldots, n - 2. \)
Let us consider the multidimensional eikonal equation
\[
\left( \frac{\partial u}{\partial x_0} \right)^2 - \left( \frac{\partial u}{\partial x_1} \right)^2 - \ldots - \left( \frac{\partial^2 u}{\partial x_{n-1}} \right)^2 = 1, \tag{8}
\]
where \( u = u(x) \) is a scalar function of the variable \( x = (x_0, x_1, \ldots, x_{n-1}) \), \( n \geq 2 \). The equation (8) is invariant under the conformal algebra \( AC(1,n) \). The algebra \( AC(1,n) \)
contains the extended Poincaré algebra \( AP(1,n) \) being generated by the vector fields:
\[
P_\alpha = \partial_\alpha, \quad J_{0a} = x_0 \partial_a + x_a \partial_0, \quad J_{ab} = x_b \partial_a - x_a \partial_b, \quad D = -x^a \partial_\alpha, \quad x_n = u
\]
\[(\alpha = 0, 1, \ldots, n; \quad a, b = 1, 2, \ldots, n).\]

Let us use maximal subalgebras of rank \( n - 1 \) of the algebra \( \tilde{P}(1,n) \) to find ansatzes reducing the equation (8) to ordinary differential equations. As consequence we obtain 18 types of the following ansatzes [6]. All these ansatzes are split into three classes. Below we adduce the examples of ansatzes for each of the classes.

I. Ansatzes of the type \( u = f(x)\varphi(\omega) + g(x), \omega = \omega(x) \)
1) \( u = \varphi(\omega), \omega = x_0; \)
2) \( u = \varphi(\omega) + \ln(x_0 + x_{m+1}), \quad \omega = x^2_0 - x^2_1 - \ldots - x^2_m - x^2_{m+1}; \quad m = 0, 1, \ldots, n - 1; \quad n \geq 3. \)

II. Ansatzes of the type \( u^2 = f(x)\varphi(\omega) + g(x), \omega = \omega(x) \)
1) \( u^2 = \varphi(\omega) - x^2_1 - \ldots - x^2_m, \omega = x_0 - x_n, \quad m = 1, 2, \ldots, n - 1; \)
2) \( u^2 = \varphi(\omega) + x^2_0 - x^2_1 - \ldots - x^2_m, \omega = x_0 - x_m, \quad m = 1, 2, \ldots, n - 1; \)
3) \( u^2 = (x^2_{n-2} + x^2_{n-1})\varphi(\omega) + x^2_0 - x^2_1 - \ldots - x^2_m, \omega = 2 \ln(x_0 - x_m) - (1 + \alpha) \ln(x^2_{n-2} + x^2_{n-1}) - 2C \arctan\frac{x_{n-1}}{x_{n-2}}, \quad m = 1, 2, \ldots, n - 3; \quad n \geq 4, C > 0, \quad \alpha \geq 0. \)

III. Ansatzes of the type \( h(u, x) = f(x)\varphi(\omega) + g(x), \omega = \omega(u, x) \)
1) \( u = \frac{1}{4}\varphi(\omega) + \frac{1}{4}(x_0 - x_1)^2, \omega = (x_0 - x_1)^3 - 6u(x_0 - x_1) + 6(x_0 + x_1); \)
2) \( u^2 = \varphi(\omega) - x^2_1, \omega = x_0 + \arctan\frac{u}{x_1}; \)
3) \( u^2 = (x_0 - x_2)\varphi(\omega) - x^2_1, \omega = x_0 + x_2 + \ln(x_0 - x_2) + 2\alpha \arctan\frac{u}{x_1}, \alpha \geq 0. \)

The search for additional symmetries of differential equations is an important problem of investigations concerning partial differential equations. One of the possible ways to solve this problem is a study of the symmetry of the 2-dimensional reduced equations.

Let us consider, for instance, the symmetry ansatz
\[
u = u(\omega_1, \omega_2), \tag{9}\]
where \( \omega_1 = x_0 - x_m, \omega_2 = x^2_0 - x^2_1 - \ldots - x^2_m (m = 2, 3, \ldots, n) \). The ansatz (9) reduces
the d’Alembert equation (7) to the 2-dimensional equation
\[
4\omega_1 u_{12} + 4\omega_2 u_{22} + 2(m + 1)u_2 + \lambda u^k = 0, \tag{10}\]
where \( u_{12} = \frac{\partial^2 u}{\partial \omega_1 \partial \omega_2}, u_{22} = \frac{\partial^2 u}{\partial \omega_2^2}, u_2 = \frac{\partial u}{\partial \omega_2}. \)
The maximal algebra of invariance of equation (10) in the case of \( k \neq 0, \frac{m+1}{m-1} \) and \( m > 1 \) in the Lie sense is the 4-dimensional Lie algebra \( A(4) \) which is generated by such operators:

\[
X_1 = \omega_1 \frac{\partial}{\partial \omega_1} + \omega_2 \frac{\partial}{\partial \omega_2} - \frac{1}{k-1} u \frac{\partial}{\partial u}, \quad X_2 = \omega_2 \frac{\partial}{\partial \omega_2} - \frac{1}{k-1} u \frac{\partial}{\partial u},
\]

\[
X_3 = \omega_1 \frac{\partial}{\partial \omega_2}, \quad M = \omega_1 (\omega_1 \frac{\partial}{\partial \omega_1} + \omega_2 \frac{\partial}{\partial \omega_2} - \frac{m-1}{2} u \frac{\partial}{\partial u}),
\]

where \( \ell = \frac{(m-1)(k-1)}{2} - 1 \).

**Theorem 2** The maximal algebra of invariance of equation (10) in the case of \( k = \frac{m+1}{m-1} \) and \( m > 1 \) in the sense of Lie is the 4-dimensional Lie algebra \( B(4) \) which is generated by such operators:

\[
S = \omega_1 \ln \omega_1 \frac{\partial}{\partial \omega_1} + \omega_2 \ln \omega_1 \frac{\partial}{\partial \omega_2} - \frac{m-1}{2} \ln(\omega_1 + 1) u \frac{\partial}{\partial u},
\]

\[
Z_1 = \omega_1 \frac{\partial}{\partial \omega_1} + \omega_2 \frac{\partial}{\partial \omega_2} - \frac{m-1}{2} u \frac{\partial}{\partial u}, \quad Z_2 = \omega_2 \frac{\partial}{\partial \omega_2} - \frac{m-1}{2} u \frac{\partial}{\partial u}, \quad Z_3 = \omega_1 \frac{\partial}{\partial \omega_2}.
\]

Let us note that \( X_1, X_2, X_3, \) \( Z_1, Z_2, Z_3 \) are operators of the algebra of invariance of the d’Alembert equation. But these operators are written in new variables. The operator \( M \) isn’t a symmetry operator of the equation (7). Also the operator \( S \) isn’t a symmetry operator of the equation (7).

The operators \( M \) and \( S \) allow to construct new ansätze reducing the d’Alembert equation to ordinary differential equations. Let us adduce some types of such ansätze:

1) \( u = \left[(x_0 - x_m)^{(m-1)(k-1)} - 1 (x_0^2 - x_1^2 - \ldots - x_m^2)^{\frac{1}{2k}} \right]^{\frac{1}{k-1}} \varphi(\omega), \quad \omega = \frac{\alpha}{l} (x_0 - x_m)^{-l} + \ln \frac{x_0^2 - x_1^2 - \ldots - x_m^2}{x_0 - x_m};
\]

2) \( u = (x_0 - x_m)^{\frac{m}{2}} \varphi(\omega), \quad \omega = \frac{x_0^2 - x_1^2 - \ldots - x_m^2}{x_0 - x_m} + \frac{\varepsilon}{l} (x_0 - x_m)^{-l}. \)

**References**


