Nonlinear Mathematical Physics 1997, V.4, N 1–2, 102–106.

Symmetry Reduction of Poincaré-Invariant Nonlinear Wave Equations

A.F. BARANNYK and Yu.D. MOSKALENKO

Department of Mathematics, Pedagogical University,
Ostrogradsky Street 2, 314003, Poltava, Ukraïna

Abstract

Reduction of multidimensional Poincaré-invariant equations to ordinary differential equations and 2-dimensional equations is considered.

Let us consider the nonlinear wave equation

$$\Phi(\Box u, (\nabla u)^2, u) = 0, \tag{1}$$

where $u = u(x)$ is a scalar function of the variable x, $x = (x_0, x_1, \ldots, x_n) \in R_{1,n}$,

$$\Box u = \frac{\partial^2 u}{\partial x_0^2} - \frac{\partial^2 u}{\partial x_1^2} - \cdots - \frac{\partial^2 u}{\partial x_n^2}, \quad (\nabla u)^2 = \left(\frac{\partial u}{\partial x_0}\right)^2 - \left(\frac{\partial u}{\partial x_1}\right)^2 - \cdots - \left(\frac{\partial u}{\partial x_n}\right)^2.$$

The equation (1) is invariant under the Poincaré algebra $AP(1,n)$. The basis of this algebra is formed by the following vector fields

$$P_\mu = \partial_\mu, \quad J_{0a} = x_0\partial_a + x_a\partial_0, \ J_{ab} = x_b\partial_a - x_a\partial_b, \tag{2}$$

$\mu = 0, 1, \ldots, n;\ a, b, = 1, 2, \ldots, n.$

A great number of well-known equations are particular cases of the equation (1).

Let's take for instance the d'Alembert equation

$$\Box u + \lambda u^k = 0; \tag{3}$$

the Liouville equation

$$\Box u + \lambda \exp u = 0; \tag{4}$$

the sine-Gordon equation

$$\Box u + \lambda \sin u = 0; \tag{5}$$

the eikonal equation

$$(\nabla u)^2 = 1. \tag{6}$$

The reduction problem of the equation (1) is very important. The main point of reduction consists in introduction of new variables $\omega_1(x), \ldots, \omega_k(x)$ $(1 \le k \le n)$ being functions of x and having property that the ansatz $u = \varphi(\omega_1, \ldots, \omega_k)$ reduces the equation

(1) to one with a smaller number of variables $\omega_1, \dots, \omega_k$. The construction of all ansatzes for the equation (1) is a very difficult problem.

The problem is more easy if variables $\omega_1, \dots, \omega_k$ are invariants of some subalgebra of the algebra $AP(1,n)$. These variables are called invariant variables. It is easy to find invariant variables if the optimal system of subalgebras of the algebra $AP(1,n)$ is known. It is impossible in practice to construct the optimal system of subalgebras of the algera $AP(1,n)$ in a general case (for an arbitrary n). But the situation isn't hopeless, because for symmetry reduction it is enough to know subalgebras having essentially different systems of invariants.

Definition. Two subalgebras $L_1, L_2 \subset AP(1,n)$ are called equivalent if there exists a group transformation φ transforming the system of invariants of the subalgebra L_1 into that L_2.

This relation is a more strong relation on the set of all subalgebras of the algebra $AP(1,n)$ than the relation of conjugation.

Among all subalgebras of the algebra $AP(1,n)$ having the same invariants, there exists the algebra containing all subalgebras having this property. This subalgebra is called I-maximal. Two I-maximal subalgebras are equivalent iff they are conjugate. I-maximal subalgebras differ advantageously from the rest subalgebras of the algebra $AP(1,n)$. They are determined uniquely and have a more easy structure. Thus it is enough to construct the system of I-maximal subalgebras of the algebra $AP(1,n)$ instead of the optimal system of subalgebras.

Grundland A.M., Harnad J., Winternitz P. [1] classified I-maximal subalgebras of rank n of the algebra $AP(1,n)$. It allowed to construct seven ansatzes reducing the equation (1) to ordinary differential equations.

The problem of classification of I-maximal subalgebras of the algebra $AP(1,n)$ was solved in works [2, 3] . These results, in particular, imply that there exist 14 types of ansatzes reducing the equation (1) to 2-dimensional equations. Let us adduce main types of these ansatzes

$$u = \varphi(\omega_1, \omega_2):$$

1) $\omega_1 = x_0,\ \omega_2 = x_n;$

2) $\omega_1 = x_0,\ \ \omega_2 = x_1^2 + \ldots + x_m^2, m = 1, 2, \ldots, n;$

3) $\omega_1 = x_0 - x_n,\ \ \omega_2 = x_0^2 - x_1^2 - \ldots - x_m^2 - x_n^2,\ \ m = 1, 2, \ldots, n-1;$

4) $\omega_1 = x_1^2 + \ldots + x_m^2,\ \ \omega_2 = x_0^2 - x_n^2, m = 1, 2, \ldots, n-1;$

5) $\omega_1 = x_0^2 - x_1^2 - \ldots - x_m^2 - x_n^2,\ \ \omega_2 = x_{m+1},\ \ m = 2, 3, \ldots, n-2;$

6) $\omega_1 = x_0^2 - x_1^2 - \ldots - x_m^2 - x_n^2,\ \ \omega_2 = x_{m+1}^2 + \ldots + x_q^2,$

$m = 2, 3, \ldots, n-2;\quad q = m+1, \ldots, n;$

7) $\omega_1 = (x_0 - x_n)^2 - 4x_1,\ \ \omega_2 = (x_0 - x_n)^3 - 6x_1(x_0 - x_n) + 6(x_0 + x_n);$

8) $\omega_2 = x_0 - x_n,\ \ \omega_1 = \left(x_0^2 - \sum_{i=1}^{n} \frac{x_0 - x_n}{x_0 - x_n - \gamma_i}(x_{d_{i-1}+1}^2 + \cdots + x_{d_i}^2) - x_n^2 \right)^{\frac{1}{2}},$

$$d_0 = 0,\ d_1, d_2, \ldots, d_t \in R,\ d_0 < d_1 < \ldots < d_t = m,\ m \leq n,\ \gamma_i \in R;$$

9) $\omega_1 = x_0^2 - x_n^2, \quad \omega_2 = \alpha \ln(x_0 + x_n) - x_1, \quad \alpha > 0;$

10) $\omega_1 = x_0^2 - x_1^2 - \ldots - x_m^2 - x_n^2, \quad \omega_2 = \alpha \ln(x_0 - x_n) + x_{m+1},$

$m = 1, 2, \ldots, n-2, \quad \alpha > 0;$

11) $\omega_1 = x_1^2 + x_2^2, \quad \omega_2 = x_0 + \arctan \dfrac{x_2}{x_1}.$

Each of these ansatzes may be written in a more general form using transformations of the group $P(1,n)$.

Let us consider the nonlinear d'Alembert equation

$$\Box u + \lambda u^k = 0 \tag{7}$$

in the Minkowski space $R_{1,n}$. The equation (7) has been investigated in works [4, 5].

The equation (7) is invariant under the extended Poincaré algebra $A\tilde{P}(1,n)$ being obtained from the algebra $AP(1,n)$ by adding the dilatation operator

$$D = -x_0\partial_0 - \ldots - x_n\partial_n + \frac{2}{k-1} u\partial_u.$$

There exists the simple algorithm which allows to classify I-maximal subalgebras of the algebra $AP(1,n)$ if the classification of I-maximal subalgebras of the algebra $AP(1,n)$ is known. It allows to construct all symmetry ansatzes reducing the equation (7) to ordinary differential equations. The following ansatzes are obtained:

1) $u = x_0^{\frac{2}{1-k}} \varphi(\omega), \quad \omega = \dfrac{x_1^2 + x_2^2 + \ldots + x_m^2}{x_0^2}, \quad m = 1, 2, \ldots, n;$

2) $u = (x_0 - x_n)^{\frac{2}{1-k}} \varphi(\omega), \quad \omega = \dfrac{(x_0^2 - x_1^2 - \ldots - x_m^2 - x_n^2)^{\frac{1}{2}}}{x_0 - x_n},$

$m = 1, 2, \ldots, n-1;$

3) $u = (x_0^2 - x_1^2 - \ldots - x_m^2 - x_n^2)^{\frac{1}{1-k}} \varphi(\omega),$

$\omega = \delta \ln(x_0^2 - x_1^2 - \ldots - x_m^2 - x_n^2) - \ln(x_0 - x_n), \quad m = 1, 2, \ldots, n-1;$

4) $u = (x_0 - x_n)^{\frac{1}{1-k}} \varphi(\omega), \quad \omega = \dfrac{x_0^2 - x_1^2 - \ldots - x_m^2 - x_n^2}{x_0 - x_n} + \ln(x_0 - x_n),$

$m = 1, 2, \ldots, n-1;$

5) $u = (x_1^2 + \ldots + x_m^2)^{\frac{1}{1-k}} \varphi(\omega), \quad \omega = \dfrac{x_1^2 + x_2^2 + \ldots + x_m^2}{x_0^2 - x_n^2}, \ m = 1, 2, \ldots, n-1;$

6) $u = x_{m+1}^{\frac{2}{1-k}} \varphi(\omega), \quad \omega = \dfrac{x_0^2 - x_1^2 - \ldots - x_m^2 - x_n^2}{x_{m+1}^2}, \ m = 1, 2, \ldots, n-2.$

Let us consider the multidimensional eikonal equation

$$\Big(\frac{\partial u}{\partial x_0}\Big)^2 - \Big(\frac{\partial u}{\partial x_1}\Big)^2 - \ldots - \Big(\frac{\partial^2 u}{\partial x_{n-1}}\Big)^2 = 1, \tag{8}$$

where $u = u(x)$ is a scalar function of the variable $x = (x_0, x_1, \ldots, x_{n-1})$, $n \geq 2$. The equation (8) is invariant under the conformal algebra $AC(1,n)$. The algebra $AC(1,n)$ contains the extended Poincaré algebra $AP(1,n)$ being generated by the vector fields:

$$P_\alpha = \partial_\alpha,\ J_{0a} = x_0\partial_a + x_a\partial_0,\ J_{ab} = x_b\partial_a - x_a\partial_b,\ D = -x^\alpha\partial_\alpha,\ x_n = u$$

$$(\alpha = 0, 1, \ldots, n; \quad a, b = 1, 2, ..., n).$$

Let us use maximal subalgebras of rank $n-1$ of the algebra $A\tilde{P}(1,n)$ to find ansatzes reducing the equation (8) to ordinary differential equations. As consequence we obtain 18 types of the following ansatzes [6]. All these ansatzes are split into three classes. Below we adduce the examples of ansatzes for each of the classes.

I. <u>Ansatzes of the type $u = f(x)\varphi(\omega) + g(x),\ \omega = \omega(x)$</u>

1) $u = \varphi(\omega),\ \omega = x_0;$
2) $u = \varphi(\omega) + \ln(x_o + x_{m+1}), \quad \omega = x_0^2 - x_1^2 - \ldots - x_m^2 - x_{m+1}^2,$ $m = 0, 1, \ldots, n-1;\ n \geq 3.$

II. <u>Ansatzes of the type $u^2 = f(x)\varphi(\omega) + g(x),\ \omega = \omega(x)$</u>

1) $u^2 = \varphi(\omega) - x_1^2 - \ldots - x_m^2,\ \omega = x_0 - x_n, \quad m = 1, 2, \ldots, n-1;$
2) $u^2 = \varphi(\omega) + x_0^2 - x_1^2 - \ldots - x_m^2,\ \omega = x_0 - x_m, \quad m = 1, 2, \ldots, n-1;$
3) $u^2 = (x_{n-2}^2 + x_{n-1}^2)\varphi(\omega) + x_0^2 - x_1^2 - \ldots - x_m^2,\ \omega = 2\ln(x_0 - x_m) - (1+\alpha)\ln(x_{n-2}^2 + x_{n-1}^2) - 2C\arctan\frac{x_{n-1}}{x_{n-2}}, \quad m = 1, 2, \ldots, n-3;\ n \geq 4, C > 0, \quad \alpha \geq 0.$

III. <u>Ansatzes of the type $h(u,x) = f(x)\varphi(\omega) + g(x),\ \omega = \omega(u,x)$</u>

1) $u = \frac{1}{4}\varphi(\omega) + \frac{1}{4}(x_0 - x_1)^2,\ \omega = (x_0 - x_1)^3 - 6u(x_0 - x_1) + 6(x_0 + x_1);$
2) $u^2 = \varphi(\omega) - x_1^2,\ \omega = x_0 + \arctan\frac{u}{x_1};$
3) $u^2 = (x_0 - x_2)\varphi(\omega) - x_1^2,\ \omega = x_0 + x_2 + \ln(x_0 - x_2) + 2\alpha\arctan\frac{u}{x_1},\ \alpha \geq 0.$

The search for additional symmetries of differential equations is an important problem of investigations concerning partial differential equations. One of the possible ways to solve this problem is a study of the symmetry of the 2-dimensional reduced equations.

Let us consider, for instance, the symmetry ansatz

$$u = u(\omega_1, \omega_2), \tag{9}$$

where $\omega_1 = x_0 - x_m,\ \omega_2 = x_0^2 - x_1^2 - \ldots - x_m^2 \ (m = 2, 3, \ldots, n)$. The ansatz (9) reduces the d'Alembert equation (7) to the 2-dimensional equation

$$4\omega_1 u_{12} + 4\omega_2 u_{22} + 2(m+1)u_2 + \lambda u^k = 0, \tag{10}$$

where $u_{12} = \dfrac{\partial^2 u}{\partial\omega_1\partial\omega_2},\ u_{22} = \dfrac{\partial^2 u}{\partial\omega_2^2},\ u_2 = \dfrac{\partial u}{\partial\omega_2}.$

Theorem 1 *The maximal algebra of invariance of equation (10) in the case of* $k \neq 0, \dfrac{m+1}{m-1}$ *and* $m > 1$ *in the Lie sense is the 4-dimensional Lie algebra* $A(4)$ *which is generated by such operators:*

$$X_1 = \omega_1 \frac{\partial}{\partial \omega_1} + \omega_2 \frac{\partial}{\partial \omega_2} - \frac{1}{k-1} u \frac{\partial}{\partial u}, \quad X_2 = \omega_2 \frac{\partial}{\partial \omega_2} - \frac{1}{k-1} u \frac{\partial}{\partial u},$$

$$X_3 = \omega_1 \frac{\partial}{\partial \omega_2}, \quad M = \omega_1^l (\omega_1 \frac{\partial}{\partial \omega_1} + \omega_2 \frac{\partial}{\partial \omega_2} - \frac{m-1}{2} u \frac{\partial}{\partial u}),$$

where $l = \dfrac{(m-1)(k-1)}{2} - 1$.

Theorem 2 *The maximal algebra of invariance of equation (10) in the case of* $k = \dfrac{m+1}{m-1}$ *and* $m > 1$ *in the sense of Lie is the 4-dimensional Lie algebra* $B(4)$ *which is generated by such operators:*

$$S = \omega_1 \ln \omega_1 \frac{\partial}{\partial \omega_1} + \omega_2 \ln \omega_1 \frac{\partial}{\partial \omega_2} - \frac{m-1}{2} \ln(\omega_1 + 1) u \frac{\partial}{\partial u},$$

$$Z_1 = \omega_1 \frac{\partial}{\partial \omega_1} + \omega_2 \frac{\partial}{\partial \omega_2} - \frac{m-1}{2} u \frac{\partial}{\partial u}, \quad Z_2 = \omega_2 \frac{\partial}{\partial \omega_2} - \frac{m-1}{2} u \frac{\partial}{\partial u}, \quad Z_3 = \omega_1 \frac{\partial}{\partial \omega_2}.$$

Let us note that X_1, X_2, X_3 (Z_1, Z_2, Z_3) are operators of the algebra of invariance of the d'Alembert equation. But these operators are written in new variables. The operator M isn't a symmetry operator of the equation (7). Also the operator S isn't a symmetry operator of the equation (7).

The operators M and S allow to construct new ansatzes reducing the d'Alembert equation to ordinary differential equations. Let us adduce some types of such ansatzes:

$$1)\ u = \left[(x_0 - x_m)^{\frac{(m-1)(k-1)}{2} - 1} (x_0^2 - x_1^2 - \ldots - x_m^2) \right]^{\frac{1}{1-k}} \varphi(\omega),$$

$$\omega = \frac{\alpha}{l} (x_0 - x_m)^{-l} + \ln \frac{x_0^2 - x_1^2 - \ldots - x_m^2}{x_0 - x_m};$$

$$2)\ u = (x_0 - x_m)^{\frac{1-m}{2}} \varphi(\omega), \qquad \omega = \frac{x_0^2 - x_1^2 - \ldots - x_m^2}{x_0 - x_m} + \frac{\varepsilon}{l} (x_0 - x_m)^{-l}.$$

References

[1] Grundland A.M., Harnad J., Winternitz P. Symmetry reduction for nonlinear relativistically invariant equations, *J. Math. Phys.*, 1984, V.25, N 4, 791.

[2] Fushchych W.I., Barannyk A.F., Maximal subalgebras of the rank $n-1$ of the algebra $AP(1,n)$ and reduction of nonlinear wave equations, *Ukr. Math. J.,* 1990, V.42, N 11, 1250–1256; N 12, 1693–1700.

[3] Barannyk A.F., Barannyk L.F., Fushchych W.I., Reduction of multidimensional Poincaré-invariant nonlinear equation to 2-dimensional equations, *Ukr. Math. J.,* 1991, V.43, N 10, 1311–1323.

[4] Fushchych W.I., Serov N.I., The symmetry and some exact solutions of the nonlinear many-dimensional Liouville, d'Alembert and eikonal equations, *J. Phys. A,* 1983, V.16, 3645–3658.

[5] Barannyk A.F., Barannyk L.F., Fushchych W.I., Reduction of the multi-dimensional d'Alembert equation to 2-dimensional equations, *Ukr. Math. J.,* 1994, V.46, N 6, 651–662.

[6] Barannyk A.F., Fushchych W.I., Ansatzes for the eikonal equations, *Dop. NAN Ukr.,* 1993, N 12, 41–43.