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On Subalgebras of the Conformal Algebra $AC(2,2)$

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Abstract

Subalgebras of the Lie algebra $AC(2,2)$ of the group $C(2,2)$, which is the group of conformal transformations of the pseudo-Euclidean space $R_{2,2}$, are studied. All subalgebras of the algebra $AC(2,2)$ are splitted into three classes, each of those is characterized by the isotropic rank $0, 1$, or 3. We present the complete classification of the class 0 subalgebras and also of the class 3 subalgebras which satisfy an additional condition. The results obtained are applied to the reduction problem for the d'Alembert equation $\Box u + \lambda u^3 = 0$ in the space $R_{2,2}$.

1. Introduction. A number of equations of the theoretical and mathematical physics are invariant with respect to the group $C(2,2)$ of conformal transformations of the pseudo-Euclidean space $R_{2,2}$ [1]. Therefore subalgebras of the Lie algebra $AC(2,2)$ of the group $C(2,2)$ may be used search for invariant and partially invariant solutions of such equations. It is well-known (see, e.g., [2]) that the problem of subalgebra classification of the algebra $AC(2,2)$ up to $C(2,2)$-conjugacy is equivalent to the problem of classification of subalgebras for the algebra $AO(3,3)$ up to $O(3,3)$-conjugacy. The present paper is devoted to solution of the last problem. Following [3], we split all subalgebra of this algebra into three classes, characterizing each of them by the isotropic rank. In Paragraph 3, we adduce the classification of the class 0 subalgebras for the algebra $AO(3,3)$. In Paragraph 4, the problem of classification of the class 3 subalgebras for the algebra $AO(3,3)$ is reduced to the problem of classification of subalgebras for the algebra $AIGL(3,R)$, which is the Lie algebra for the group $IGL(3,R)$, the group of nonuniform transformations of the three-dimensional real space. We have carried out the complete classification up to $O(3,3)$-conjugacy of the class 3 subalgebras L of the algebra $AO(3,3)$ which do not possess a one-dimensional completely isotropic subspace, invariant with respect to L. In Paragraph 5, we consider the conformal algebra $AC(2,2)$, which is the maximal invariance algebra of the d'Alembert equation $\Box u + \lambda u^3 = 0$ in the space $R_{2,2}$. We give the complete description of the rank 3 and 4 subalgebras of the algebra $AC(2,2)$ which are not conjugated to subalgebras of the algebra $A_{(1)}$, defined as the normalizer in $AO(3,3)$ of the one-dimensional completely isotropic subspace. We have found invariants of these maximal subalgebras and carried out the reduction of the equation for each of these subalgebras. In solution of the above-mentioned problems, we used some general principles of classification of subalgebras for an arbitrary Lie algebra, adduced in [4].

2. Pseudoorthogonal algebra $AO(3,3)$**.** Let R be the field of real numbers, $V = R_{3,3}$ be the pseudo-Euclidean space of the signature (3,3), $\{Q_1, \dots, Q_6\}$ is the orthonormal basis of the space V. We call a group, which preserves the form $x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4 - x_5y_5 - x_6y_6$, a pseudoorthogonal form of the space $O(3,3)$. If a matrix f is equal to S in basis $\{Q_1, \dots, Q_6\}$ of the space V, then only in the case where $S^T J_{3,3} S = J_{3,3}$,

$$J_{3,3} = \begin{pmatrix} E_3 & 0 \\ 0 & -E_3 \end{pmatrix},$$

E_3 is the unit matrix of order 3, S^T is the transposed matrix S. Thus, we can define the group $O(3,3)$ as the group of all square matrices Δ of order 6 over the field of real numbers R, satisfying the matrix equation $\Delta^T J_{3,3} \Delta = J_{3,3}$. Hence, the Lie algebra $AO(3,3)$ of the group $O(3,3)$ is composed of all real matrices X which satisfy the relation $XJ_{3,3} + J_{3,3}X^T = 0$. Let E_{ik} be the matrix of order 6 which has 1 at the crossing of the i-th line and k-th column, and zeros at all other places $(i, k = 1, \dots, 6)$. A basis of the algebra $AO(3,3)$ is formed by the matrices $J_{ab} = E_{ab} - E_{ba}$ $(a < b; a, b = 1, 2, 3)$, $J_{ai} = -E_{ai} - E_{ia}$ $(a = 1, 2, 3; i = 4, 5, 6)$, $J_{cd} = -E_{cd} + E_{cd}$ $(c < d; c, d = 4, 5, 6)$.

Every internal automorphism $\Delta \to C\Delta C^{-1}$ of the group $O(3,3)$ induces an automorphism $\varphi_c : X \to CXC^{-1}$ of the Lie algebra $AO(3,3)$. We shall call this automorphism an $O(3,3)$ automorphism of the algebra $AO(3,3)$, corresponding to the matrix C. We shall call subalgebras L_1 and L_2 $O(3,3)$-conjugated, if $CL_1C^{-1} = L_2$.

A subalgebra $L \subset AO(3,3)$ is called a class 0 subalgebra, if V does not contain a totally isotropic subspace invariant with respect to L. We shall say that a subalgebra $L \subset AO(3,3)$ belongs to the class $r > 0$ or has an isotropic rank r, if the rank of the maximal totally isotropic subspace, invariant with respect to L, is equal to r. It is evident that every subalgebra L of the algebra $AO(3,3)$ has the isotropic rank 0, 1, or 3.

Let $L \subset AO(3,3)$ be an arbitrary subalgebra for which there exists a totally isotropic subspace $V_{(1)}$ of rank 1, invariant with respect to L. We can assume according to the Witt theorem that $V_{(1)} = \langle Q_1 + Q_6 \rangle$. The maximal subalgebra, which leaves $V_{(1)}$ invariant, coincides with the algebra $A_{(1)} = \langle G_2, G_3, G_4, G_5 \rangle \supsetneq (\langle J_{16} \rangle \oplus \langle J_{23}, J_{24}, J_{25}, J_{34}, J_{35}, J_{45} \rangle)$, where $G_a = J_{1a} - J_{a6}$ $(a = 2, 3, 4, 5)$. The subalgebra $\langle J_{23}, J_{24}, J_{25}, J_{34}, J_{35}, J_{45} \rangle$ is the Lie algebra of the pseudoorthogonal group $O(2,2)$, and the subalgebra $\langle G_2, G_3, G_4, G_5 \rangle \supsetneq \langle J_{23}, J_{24}, J_{25}, J_{34}, J_{35}, J_{45} \rangle$ is the Lie algebra of the Poincaré group $P(2,2)$. Taking into account that the element J_{16} plays the role of dilation, we get that $A_{(1)}$ is the Lie algebra of the extended Poincaré group $\tilde{P}(2,2)$, and $A_{(1)} = A\tilde{P}(2,2)$. As the problem of description of subalgebras of the algebra $A\tilde{P}(2,2)$ is solved in [5], it is sufficient to consider only those subalgebras of the algebra $AO(3,3)$ which are not conjugated to subalgebras of the algebra $A_{(1)}$. These subalgebras are all subalgebras of class 0 of the algebra $AO(3,3)$ and subalgebras L of class 3, which do not have a one-dimensional totally isotropic subspace, invariant with respect to L. Now we shall go to description of such subalgebras.

Subalgebras of class 3 of the algebra $AO(3,3)$**.** In this paragraph, we shall study subalgebras of class 0 of the algebra $AO(3,3)$ up to $O(3,3)$-conjugacy. Let L be one of such subalgebras. Then the space V is decomposed into the direct orthogonal sum of irreducible L-spaces $V_1, \dots, V_3$, of those is non-degenerate. If (p_i, q_i) is the signature of the space V_i, then by virtue of the Witt theorem we can assume that V_i has a basis

$$Q_{j_1}, \dots, Q_{j_{p_i}}, Q_{j_{p_i+1}}, \dots, Q_{j_{p_i+q_i}} \tag{1}$$

Here $j_i < \ldots < j_{p_i} \leq 3, 3 < j_{p_i+1} < \ldots < j_{p_i+q_i} \leq 6$. If $J \in L$, we can consider adJ as a linear transformation $\hat{J}_i$ of the space V_i. The matrix $\pi_i(J)$ of the transformation $\hat{J}_i$ in basis (1) of the space V_i is contained in $AO(p_i, q_i)$. The transformation $\pi_i : L \to AO(p_i, q_i)$ is a homomorphism, and $\pi_i(L)$ is an irreducible subalgebra of the algebra $AO(p_i, q_i)$. As the mapping $J \to (\pi_1(J), \ldots, \pi_s(J))$ is an isomorphism of L into the algebra $\pi_1(L) \times \cdots \times \pi_s(L)$, we shall say that L is decomposed with respect to the basis $\{Q_1, \ldots, Q_6\}$ into the subdirect product of algebras $\pi_1(L), \ldots, \pi_s(L)$ and write this in the following way:

$$L = \pi_1(L) \times \cdots \times \pi_s(L).$$

Let $L' \subset AO(3,3)$ be the maximal subalgebra having the mentioned decomposition $V = V_1 \oplus \cdots \oplus V_s$ of the space V into the direct sum of L'-subspaces $V_1, \ldots, V_s$. Then L' is decomposed into the direct product of the algebras $\pi_1(L), \ldots, \pi_s(L)$. Let us put $L_i = \{J \in L' \mid \pi_j(J) = 0 \text{ for every } j \neq i\}$. It is easy to see that L_i is a subalgebra of the algebra L' and we have the decomposition $L = L_1 + \cdots + L_s$ of the algebra L into the subdirect sum of the algebras $L_1, \ldots, L_s$. Let us adopt a convention of considering L_i the same as $\pi_i(L)$. In this sense, we shall say that L_i is an irreducible subalgebra of the algebra $AO(p_i, q_i)$. Thus, a class 0 subalgebra L of the algebra $AO(3,3)$ is either irreducible or can be decomposed into a subdirect sum of irreducible algebras.

Theorem 1 *The algebra $AO(3,3)$ contains up to $O(3,3)$-conjugacy only one proper irreducible subalgebra which is conjugated to the algebra $\langle J_{12} - J_{45}, J_{13} - J_{46}, J_{23} - J_{56}, J_{15} - J_{24}, J_{26} - J_{35}, J_{16} - J_{34} \rangle$.*

The proof of Theorem 1 is adduced in [6].

Let us find all maximal class 0 subalgebras of the algebra $AO(3,3)$, using the type of decomposition of the space V into a direct orthogonal sum of irreducible subspaces. Let, e.g., L is a maximal class 0 subalgebra of the algebra $AO(3,3)$ and $V = V_1 \oplus V_2$ is a direct orthogonal sum of two L-irreducible subspaces $V_1 = \langle Q_1, Q_2, Q_4 \rangle$ and $V_2 = \langle Q_3, Q_5, Q_6 \rangle$. We shall say that the decomposition of the space V is of the type $(+ + -)(+ - -)$. It is evident that the subalgebra L coincides with the algebra $\langle J_{12}, J_{14}, J_{24} \rangle \oplus \langle J_{35}, J_{36}, J_{56} \rangle$. All maximal class 0 subalgebras of the algebra $AO(3,3)$ are adduced in Table 1.

Now it is not difficult to get a description of all class 0 subalgebras of the algebra $AO(3,3)$ up to $O(3,3)$-conjugacy. Let, e.g., a decomposition of the space V is of the type $(+ + +)(- - -)$. A subalgebra L, for which such a decomposition of the space V exists, is either the direct sum $\langle J_{12}, J_{13}, J_{23} \rangle \oplus \langle J_{45}, J_{46}, J_{56} \rangle$ or can be decomposed into a subdirect sum $L_1 + L_2$ of two irreducible subalgebras L_1 and L_2. Thus, $L_1 = \langle J_{12}, J_{13}, J_{23} \rangle, L_2 = \langle J_{45}, J_{46}, J_{56} \rangle$. It is easy to verify that, up to $O(3,3)$-conjugacy, the subalgebra $L_1 + L_2$ is conjugated to the algebra $L' = \langle J_{12} + J_{45}, J_{13} - J_{46}, J_{23} + J_{56} \rangle$. However, the isotropic rank of the algebra L' is equal to 3 as V contains a three-dimensional totally isotropic subspace $\langle Q_1 + Q_4, Q_2 - Q_5, Q_3 + Q_6 \rangle$ invariant with respect to L'. This fact proves that if a decomposition of the space V is of the type $(+ + +)(- - -)$, then there is the only subalgebra (up to $O(3,3)$-conjugacy) corresponding to this decomposition: $\langle J_{12}, J_{13}, J_{23} \rangle \oplus \langle J_{45}, J_{46}, J_{56} \rangle$. Other cases are considered similarly. Finally, we come to the following result.

Statement 1 *Let L be a class 0 subalgebra of the algebra $AO(3,3)$ which is not maximal. Then L is $O(3,3)$-conjugated to one of the following algebras:*

1) $F_1 = \langle J_{12} - J_{45}, J_{13} - J_{46}, J_{23} - J_{56}, J_{15} - J_{24}, J_{26} - J_{35}, J_{16} - J_{34}\rangle$;

2) $F_2 = \langle -2J_{12} + J_{45}, J_{14} + J_{25} + \sqrt{3}J_{35}, -J_{15} + J_{24} - \sqrt{3}J_{34}\rangle$;

3) $F_3 = \langle -2J_{56} + J_{23}, J_{36} + J_{25} + \sqrt{3}J_{24}, -J_{26} + J_{35} - \sqrt{3}J_{34}\rangle$.

Table 1. Maximal class 0 subalgebras of the algebra $AO(3,3)$

No	Type of decomposition of the space V	Maximal class 0 subalgebras
1	$(+++---)$	$AO(3,3)$
2	$(+++--)(-)$	$AO(3,2) = \langle J_{ab} \mid a,b = 1,\dots,5\rangle$
3	$(+)(++---)$	$AO(2,3) = \langle J_{ab} \mid a,b = 2,\dots,6\rangle$
4	$(++)(+---)$	$\langle J_{12}\rangle \oplus \langle J_{ab} \mid a,b = 3,\dots,6\rangle$
5	$(+++-)(--)$	$\langle J_{ab} a, b = 1,\dots,4\rangle \oplus \langle J_{56}\rangle$
6	$(+++)(---)$	$\langle J_{12}, J_{13}, J_{23}\rangle \oplus \langle J_{45}, J_{46}, J_{56}\rangle$
7	$(++-)(+--)$	$\langle J_{12}, J_{14}, J_{24}\rangle \oplus \langle J_{35}, J_{36}, J_{56}\rangle$
8	$(+)(+)(+---)$	$AO(1,3) = \langle J_{ab} \mid a,b = 3,\dots 6\rangle$
9	$(+++-)(-)(-)$	$AO(3,1) = \langle J_{ab} a, b = 1,\dots,4\rangle$
10	$(+)(++)(---)$	$\langle J_{23}\rangle \oplus \langle J_{45}, J_{46}, J_{56}\rangle$
11	$(+)(++-)(--)$	$\langle J_{23}, J_{24}, J_{34}\rangle \oplus \langle J_{56}\rangle$
12	$(++)(+--)(-)$	$\langle J_{12}\rangle \oplus \langle J_{34}, J_{35}, J_{45}\rangle$
13	$(+++)(--)(-)$	$\langle J_{12}, J_{13}, J_{23}\rangle \oplus \langle J_{45}$
14	$(+)(+)(+)(---)$	$AO(3) = \langle J_{45}, J_{46}, J_{56}\rangle$
15	$(-)(-)(-)(+++)$	$AO(3) = \langle J_{12}, J_{13}, J_{23}\rangle$

Class 3 subalgebras of the algebra $AO(3,3)$. In the present paragraph, the problem of classification of class 3 subalgebras $L \subset AO(3,3)$ is reduced to the problem of classification of subalgebras of the algebra $AIG(3,R)$, which is the Lie algebra of the group of nonuniform real transformations of the three-dimensional real space.

Let $L \subset AO(3,3)$ be an arbitrary class 3 subalgebra. By virtue of the Witt theorem, we can assume that L leaves a subspace $V_{(3)} = \langle Q_1 + Q_4, Q_2 + q_5, Q_3 + Q_6\rangle$ invariant. All such subalgebras are contained in the maximal class 3 subalgebra $A_{(3)}$ which is a normalizer in $AO(3,3)$ of the totally isotropic space $V_{(3)}$. According to [3], every element J of the algebra $A_{(3)}$ can be uniquely represented in the form

$$J = \begin{pmatrix} J_1 & -J_1 \\ J_1 & -J_1 \end{pmatrix} + \begin{pmatrix} 0 & J_2 \\ J_2 & J_2 - J_2^T \end{pmatrix}, \tag{2}$$

where $J_1 \in AO(3), J_2 \in AGL(3,R)$. That can be written symbolically in the following way: $J = (J_1; J_2)$. According to decomposition (3), we can assert that the algebra $A_{(3)}$, considered as a vector space, can be decomposed into a Cartesian product $AO(3) \times AGL(3,R)$

of the spaces $AO(3)$ and $AGL(3,R)$. Hence it follows that a basis of the algebra $A_{(3)}$ is formed by the matrices $K_{12} = J_{12} - J_{45}$, $K_{13} = J_{13} - J_{46}$, $K_{23} = J_{23} - J_{56}$, $\mathbb{D}_1 = J_{14} - J_{25}$, $\mathbb{D}_2 = J_{14} - J_{36}$, $L_{12} = J_{15} + J_{24}$, $L_{13} = J_{16} + J_{34}$, $L_{23} = J_{26} + J_{35}$, $S = -\frac{1}{2}(J_{14} + J_{25} + J_{36})$, $T_1 = \frac{1}{2}(J_{23} + J_{26} - J_{35} + J_{56})$, $T_2 = \frac{1}{2}(-J_{13} - J_{16} + J_{34} - J_{46})$, $T_3 = \frac{1}{2}(J_{12} + J_{15} - J_{24} + J_{45})$. The matrix algebra $(J_1; 0)$, where J_1 runs over $AO(3)$, forms a commutative ideal V_1 of the algebra $A_{(3)}$, the quotient algebra $A_{(3)}/V_1$ of which is isomorphic to the algebra $AIGL(3,R)$. It not difficult to verify that $A_{(3)}$ is isomorphic to the algebra $AIGL(3,R)$ (see [7]). Further we shall consider the following basis of the algebra $A_{(3)}$:

$$\begin{aligned}
&A_1 = -\mathbb{D}_1,\ A_2 = \frac{1}{2}(K_{12} - L_{12}),\ A_3 = \frac{1}{2}(K_{12} + L_{12}),\\
&\mathbb{D} = -\frac{1}{3}\mathbb{D}_1 + \frac{2}{3}\mathbb{D}_2 + \frac{2}{3}S,\ S,\ P_1 = \frac{1}{2}(K_{13} + L_{13}),\\
&P_2 = \frac{1}{2}(K_{23} + L_{23}),\ K_{13},\ A_2' = \frac{1}{2}(K_{23} - L_{23}),\ T_1,\ T_2,\ T_3.
\end{aligned}$$

For basis elements of the algebra $A_{(3)}$, we use the same notations as for the algebra $AIGL(3,R)$ in [7]. Hence we designate the subalgebra $\langle A_1, A_2, A_3, \mathbb{D}, P_1, P_2, K_{13}, A_2\rangle$, which is isomorphic to the algebra $ASL(3,R)$, as $ASL(3,R)$. Similarly $ASL(3,R) \oplus \langle S\rangle = AGL(3,R)$ and $A_{(3)} = AIGL(3,R)$. This allows us to use automatically the classification of subalgebras of the algebra $AIGL(3,R)$ adduced in [7].

Let F be some subalgebra of the algebra $AGL(3,R)$. The subalgebra F is called irreducible, and the space $W = \langle T_1, T_2, T_3\rangle$ is called F-irreducible if W contains only nontrivial subspaces invariant with respect to F. The subalgebra F is called fully reducible if, for each F-invariant subspace $W_1 \subset W$, there exists such an F-invariant subspace $W_2 \subset W$ that $W = W_1 \oplus W_2$.

Let us consider the following sequences of subspaces:

$$0 \subset \langle T_1, T_2\rangle \subset \langle T_1, T_2, T_3\rangle, \tag{3}$$

$$0 \subset \langle T_1\rangle \subset \langle T_1, T_2, T_3\rangle, \tag{4}$$

$$0 \subset \langle T_1\rangle \subset \langle T_1, T_2\rangle \subset \langle T_1, T_2, T_3\rangle. \tag{5}$$

We shall say a subalgebra $F \subset AGL(3,R)$, which is not fully reducible, belongs to the class $\mathfrak{M}_1$ (correspondingly to $\mathfrak{M}_2$ and $\mathfrak{M}_3$), if series (4) (correspondingly (5) and (6)) is a composition series of the F-module of W. If L is an arbitrary subalgebra of the algebra $A_{(3)}$ and L_1 is its projection on $AGL(3,R)$, then we can always assume that L_1 is either fully reducible or belongs to one of the classes $\mathfrak{M}_1, \mathfrak{M}_2$ and $\mathfrak{M}_3$. We shall assume just that in the following. We shall say that a subalgebra $L \subset AIGL(3,R)$ belongs to the class $\tilde{\mathfrak{M}}_i$ if L is an extension of a subalgebra F which belongs to the class $\mathfrak{M}_i$ of the algebra $AGL(3,R)(i = 1,2,3)$. Note that the maximal subalgebra M_2 from the class $\tilde{\mathfrak{M}}_2$ of the algebra $AGL(3,R)$ has a basis $\{A_1', A_2', A_3', \mathbb{D}', P_1', P_2', T_1, T_2, T_3\}$. Using [3], we get that the group G_1 of $O(3,3)$-automorphisms which leaves invariant a totally isotropic subspace $V_{(3)}$ induces a group of $IGL(3,R)$-automorphisms on $A_{(3)}$. Therefore, we can distinguish the following two steps at classification of subalgebras of the algebra $A_{(3)}$. At the first step, we find all subalgebras of the algebra $A_{(3)}$ nonequivalent up to $IGL(3,R)$-conjugacy.

The set of subalgebras received is designated as $\mathfrak{U}$. Two subalgebras $L_1, L_2 \in \mathfrak{U}$ can be conjugated by means of some $O(3,3)$-automorphism which does not belong to G_1. Thus, at the second step, we have the problem of classification of subalgebras from the set $\mathfrak{U}$ up to $O(3,3)$-conjugacy. To solve this problem, let us consider the following totally isotropic subspaces V:

$$S_1 = \langle Q_1 + Q_4, Q_2 + Q_5, Q_3 + Q_6 \rangle, \quad S_2 = \langle Q_1 + Q_4, Q_2 + Q_5, Q_3 - Q_6 \rangle,$$
$$S_3 = \langle Q_1 + Q_4, Q_2 - Q_5, Q_3 - Q_6 \rangle, \quad S_4 = \langle Q_1 - Q_4, Q_2 - Q_5, Q_3 - Q_6 \rangle,$$
$$S_5 = \langle Q_1 - Q_4, Q_2 + Q_5, Q_3 - Q_6 \rangle, \quad S_6 = \langle Q_1 - Q_4, Q_2 + Q_5, Q_3 + Q_6 \rangle,$$
$$S_7 = \langle Q_1 + Q_4, Q_2 - Q_5, Q_3 + Q_6 \rangle, \quad S_8 = \langle Q_1 - Q_4, Q_2 - Q_5, Q_3 + Q_6 \rangle.$$

Let us designate the following matrices as C_i: $C_1 = \operatorname{diag}\,[1,1,1,1,1,-1]$, $C_2 = \operatorname{diag}\,[1,1,1,1,-1,1]$ $C_3 = \operatorname{diag}\,[1,1,1,-1,1,1]$. Let φ_i $(i = 1,2,3)$ be an $O(3,3)$-automorphism of the algebra $AO(3,3)$ determined by the matrix $C_i, (i = 1,2,3)$. The group $\{\varphi_1, \varphi_2, \varphi_3\}$ generated by automorphisms φ_i is designated as G_2. The order of the group G_2 is equal to 8.

Theorem 2 *If subalgebras $L_1, L_2 \subset A_{(3)}$ are conjugated with respect to the group of $O(3,3)$-automorphisms, they are conjugated also with respect to the group $\{G_1, G_2\}$.*

Proof. Note that there exist only the following totally isotropic subspaces of rank 3: S_1, S_2, S_3, S_4.. Let f be an $O(3,3)$-automorphism, mapping the algebra $L_1 \subset A_{(3)}$ on the algebra $L_2 \subset A_{(3)}$. The subspace $f^{-1}(S_1)$ is totally isotropic and invariant with respect to the subalgebra L. It easy to make sure that there exists some $IGL(3,R)$-automorphism ψ mapping $f^{-1}(S_1)$ on some subspace $S_i (i \in \{1,\dots,7\})$, and $\psi(L_1) = L_1$. The automorphism $f\psi$ maps L_1 on L_2, and S_1 on S_2. So we can assume that $f(L_1) = L_2$ and $f(S_i) = S_1$. Then $f = f_1\varphi$ for some $IGL(3,R)$-automorphism $f \in G_1$ and $\varphi \in G_2$. The theorem is proved.

It follows from Theorem 2 that, at the second step, the classification of subalgebras of the set $\mathfrak{U}$ should be done up to automorphisms of the form $f\varphi$ where $f \in G_1$ and $\varphi \in G_2$. Let us apply Theorem 2 to the problem of classification up to $O(3,3)$-conjugacy of class 3 subalgebras of the algebra $AO(3,3)$, which are not conjugate to subalgebras of the algebra $A_{(1)}$. Let L be one of such subalgebras. It is an extension in $A_{(3)} = AIGL(3,R)$ of some subalgebra $F \subset AG(3,R)$. Taking into account that V does not contain a one-dimensional totally isotropic subspace invariant with respect to L, we get that F is either irreducible or belongs to the class M_2. Using the description of these classes adduced in [7] and additionally studying them up to conjugacy with respect to automorphisms of the form $f\varphi$, where $f \in G_1, \varphi \in G_2$, we get the complete classification up to $O(3,3)$-conjugacy of class 3 subalgebras of the algebra $AO(3,3)$, which are not conjugate to subalgebras of the algebra $A_{(1)}$. Further we use the following notations:

$$A_1' = \frac{1}{2}A_1 - \frac{3}{2}\mathbb{D} + S,\ A_3' = P_2,\ P_1' = P_1,\ P_2' = -A_3,\ \mathbb{D}' = -\frac{1}{2}A_1 + \frac{1}{2}\mathbb{D} - S.$$

Class 3 subalgebras of the algebra $AO(3,3)$, which are not conjugate to subalgebras of the algebra $A_{(1)}$:

$$1)\ \langle J_{23} + J_{45}, J_{13} + J_{46}, J_{12} + J_{56} \rangle \oplus \langle s_1 S \rangle \oplus \langle s_2 T_1, s_2 T_2, s_2 T_3 \rangle \qquad (s_1, s_2 = 0; 1);$$

2) $\langle J_{24}+J_{35}, J_{12}+J_{56}, J_{14}+J_{36}\rangle \oplus \langle s_1 S\rangle \oplus \langle s_2T_1, s_2T_2, s_2T_3\rangle \qquad (s_1, s_2 = 0;1)$;

3) $ASL(3,R) \oplus \langle s_1 S\rangle \oplus \langle s_2T_1, s_2T_2, s_2T_3\rangle \qquad (s_1, s_2 = 0;1)$,

where $ASL(3,R) = \langle J_{23}+J_{45}, J_{13}+J_{46}, A_1', A_2', A_3, P_1', P_2', \mathbb{D}'\rangle S$;

4) $\langle A_2'+A_3'+\alpha\mathbb{D}'+\beta S, P_1', P_2', T_1, T_2, T_3\rangle \qquad (\alpha \geq 0, 2\alpha-\beta \geq 0)$;

5) $\langle A_2'+A_3'+\alpha S, \mathbb{D}'+\beta S, P_1', P_2', T_1, T_2, T_3\rangle \qquad (\alpha \geq 0, 0 \leq \beta \leq 2)$;

6) $\langle A_2'+A_3', \mathbb{D}', S, P_1', P_2', T_1, T_2, T_3\rangle$;

7) $\langle A_1', A_2', A_3', P_1', P_2', T_1, T_2, T_3\rangle$;

8) $\langle A_1', A_2', A_3', \mathbb{D}'+\alpha S, P_1', P_2', T_1, T_2, T_3\rangle \qquad (0 \leq \alpha \leq 2)$;

9) $\langle A_1', A_2', A_3', \mathbb{D}', S, P_1', P_2', T_1, T_2, T_3\rangle$;

10) $\langle A_2'+A_3', +\alpha(\mathbb{D}'+2S), P_1'+T_3, P_2'-T_2\rangle \qquad (\alpha \geq 0)$;

11) $\langle A_2'+A_3'+T_1, P_1'+T_3, P_2'-T_2\rangle$

12) $\langle A_2'+A_3'+\alpha(\mathbb{D}'+2S), P_2'+T_2+\beta T_3, P_2'-\beta T_2+T_3, T_1\rangle \qquad (\alpha \leq 0)$;

13) $\langle A_2'+A_3', P_1'+T_2+\beta T_3, P_2'-\beta T_2+T_3, T_1\rangle \qquad (\beta \leq 0)$;

14) $\langle A_2'+A_3', \mathbb{D}'+2S, P_1'+T_3, P_2'-T_2\rangle \ (\alpha \leq 0)$;

15) $\langle A_2'+A_3', \mathbb{D}+2S, P_1'+T_3, P_2'-T_2\rangle$;

16) $\langle A_2'+A_3', \mathbb{D}+2S, P_1'+T_2+\beta T_3, P_2'-\beta T_2+T_3, T_1\rangle \qquad (\beta \geq 0)$;

17) $\langle A_2'+A_3', \mathbb{D}'+2S, P_1'+T_3, P_2'-T_2, T_1\rangle$;

18) $\langle A_1', A_2', A_3', P_1'+T_2+T_3, T_1\rangle$;

19) $\langle A_1', A_2', A_3', \mathbb{D}'+2S, P_1'+T_2, P_2'+T_3, T_1\rangle$.

On reduction of the d'Alembert equation in the pseudo-Euclidean space $\mathbf{R_{2,2}}$. We consider a nonlinear wave equation

$$\Box u + \lambda u^3 = 0$$

in the pseudo-Euclidean space $R_{2,2}$, where

$$\Box u = u_{11}+u_{22}-u_{33}-u_{44},\ u_{\alpha\beta} = \frac{\partial^2 u}{\partial x_\alpha \partial x_\beta}, u \equiv u(x),\ x = (x_1, x_2, x_3, x_4); \alpha, \beta = 1,2,3,4.$$

The maximal invariance algebra of this equation is the conformal algebra $AC(2,2)$ [1]. It is isomorphic to the algebra $AO(3,3)$ and is realized by the following operators:

$$\begin{aligned}
&\Omega_{\alpha\beta} = J_{\alpha+1,\beta+1} = g_{\alpha\alpha}x_\alpha\frac{\partial}{\partial x_\beta} - g_{\beta\beta}x_\beta\frac{\partial}{\partial x_\alpha},\\
&P^\alpha = J_{1,\alpha+1} - J_{\alpha+1,6} = \frac{\partial}{\partial x_\alpha},\\
&\mathbb{D} = -J_{16} = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3} + x_4\frac{\partial}{\partial x_4} + u\frac{\partial}{\partial u},\\
&K^\alpha = J_{1,\alpha+1} + J_{\alpha+1,6} = -2g_{\alpha\alpha}x_\alpha\mathbb{D} - x^2\frac{\partial}{\partial x_\alpha},
\end{aligned} \tag{6}$$

where $x^2 = x_1^2 + x_2^2 - x_3^2 - x_4^2, g_{11} = g_{22} = -g_{33} = -g_{44} = 1; \alpha, \beta = 1, 2, 3, 4$. We use subalgebras of the algebra $AC(2,2)$ to look for invariant solutions of the d'Alembert equation. For this purpose, we will describe maximal subalgebras of ranks 3 and 4 of the algebra $AO(3,3)$, which are not conjugate to subalgebras of the algebra $A_{(1)}$. As shown earlier, such subalgebras belong to the class 0 or are contained in the algebra $A_{(3)}$. For convenience, we go from the algebra $A_{(3)}$ to the algebra $\varphi_c(A_3)$, where φ_c is an $O(3,3)$-automorphism determined by the matrix $C = \operatorname{diag}[E_3, J]$. Here E_3 is the unit matrix of order 3, $J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. We will designate elements of the algebras $A_{(3)}$ and $\varphi_c(A_3)$ corresponding to each other by the automorphism φ_c by the same symbols. Hence, the maximal subalgebra M_2 from the class $\mathfrak{M}_2$ of the algebra $\varphi_c(A_3)$ is realized by the following operators:

$$\begin{aligned} &\mathbb{D}' = y_3\partial_3 - y_4\partial_4,\ A_2' = (A_2' = (y_1y_2 + y_3y_4)\partial_1 + \mathbb{D}, \\ &A_3 = -\partial_2, \quad A_1' = 2y\partial_2 + y_3\partial_3 + y_4\partial_4 - u\partial_u, \\ &P_1' - \partial_4, \quad P_2' = -y_3\partial_1 + y_2\partial_4, \quad T_1 = -\partial_1, \\ &T_2 = \partial_3, \quad T_3 = y_4\partial_1 - y_2\partial_3, \quad S = -y_1\partial_1 - y_3\partial_3 + \tfrac{1}{2}u\partial_u, \end{aligned} \tag{7}$$

where $y_1 = x_1 + x_4, y_2 = x_1 - x_4, y_3 = x_2 + x_3, y_4 = x_2 - x_3$, and $\mathbb{D} = y_1\partial_1 + y_2\partial_2 + y_3\partial_3 + y_4\partial_4 + u\partial_u,\ \partial_\alpha = \dfrac{\partial}{\partial y_\alpha} (\alpha = 1, 2, 3, 4)$. Using the classification of subalgebras explained in Paragraphs 3 and 4, and formulae (7) and (8), we come to the following results.

Theorem 3 *Let L be a maximal rank 4 subalgebra of the algebra $AC(2,2)$ not conjugated to a subalgebra of the algebra $A_{(1)}$. Then it is $C(2,2)$-conjugated to one of the following algebras:*

1) $F_1 = \langle J_{12}, J_{13}, J_{23}, J_{45}, J_{46}, J_{56}\rangle$;
2) $F_2 = \langle J_{23} + J_{45}, J_{13} + J_{46}, J_{12} + J_{56}, S\rangle$;
3) $F_3 = \langle J_{24} + J_{35}, J_{12} + J_{56}, J_{14} + J_{36}, S\rangle$;
4) $F_4 = \langle A_2' + A_3', \mathbb{D}' + 2S, P_1' + T_3, P_2' - T_2\rangle$.

Theorem 4 *Let L be a maximal rank 3 subalgebra of the algebra $AC(2,2)$ not conjugated to a subalgebra of the algebra $A_{(1)}$. Then it is $C(2,2)$-conjugated to one of the following algebras:*

1) $L_1 = \langle J_{12}, J_{13}, J_{23}, J_{45}\rangle$;
2) $L_2 = \langle J_{23}, J_{45}, J_{46}, J_{56}\rangle$
3) $L_3 = \langle 2_{16} + J_{34}, J_{14} + J_{36} + \sqrt{3}J_{35}, J_{13} + J_{46} - \sqrt{3}J_{45}\rangle$;
4) $L_4 = \langle 2_{16} + J_{34}, J_{14} + J_{36} + \sqrt{3}J_{24}, J_{13} + J_{46} - \sqrt{3}J_{23}\rangle$;
5) $L_5 = \langle J_{23} + J_{45}, J_{13} + J_{46}, J_{12} + J_{56}\rangle$;
6) $L_6 = \langle J_{24} + J_{35}, J_{12} + J_{56}, J_{14} + J_{36}\rangle$;
7) $L_7 = \langle A_2' + A_3' + \alpha(\mathbb{D}' + 2S), P_1' + T_3, P_2' - T_2\rangle\ (\alpha \geq 0)$;
8) $L_8 = \langle A_2' + A_3' + T_1, P_1' + T_3, P_2' - T_2\rangle$;

Let us write down the complete systems of invariants of the subalgebras represented in Theorems 3 and 4.

$$F_1 : u^2[1 + (y_1y_2 + y_3y_4)^2 + y_1^2 + y_2^2 + y_3^2 + y_4^2].$$

$$F_2 : u^4(1 + y_2^2 + y_4^2)[(y_1y_2 + y_3y_4)^2 + y_1^2 + y_3^2].$$

$$F_3 : u^4(1 + y_2^2 - y_4^2)[(y_1y_2 + y_3y_4)^2 + y_1^2 + y_3^2].$$

$$F_4 : u^4(1 + y_2^2)[2y_1 + 2y_2(y_1y_2 + y_3y_4) - y_3^2 + y_4^2].$$

$$L_1 : \omega' = u(1 + y_1y_2 + y_3y_4),\ \omega = \frac{(1 + y_1y_2 + y_3y_4)^2}{(y_1 - y_2)^2 + (y_3 - y_4)^2}.$$

$$L_2 : \omega' = u(1 - y_1y_2 - y_3y_4),\ \omega = \frac{(1 - y_1y_2 - y_3y_4)^2}{(y_1 + y_2)^2 + (y_3 + y_4)^2}.$$

$$L_3 : \omega' = u\omega_1,\ \omega = \frac{\omega_2(3\omega_1 - \omega_2)^2 - 3\sqrt{3}\omega_3^2}{\omega_1^3},$$

where $\omega_1 = y_1 + y_2,\ \omega_2 = 4y_1 + \sqrt{3}y_4^2,\ \omega_3 = \sqrt{3}(y_1 - y_2)y_4 + 2y_3 + y_4^3$.

$$L_4 : \omega' = u\omega_1,\ \omega = \frac{\omega_2(3\omega_1 - \omega_2)^2 - 3\sqrt{3}\omega_3^2}{\omega_1^3}$$

where $\omega_1 = -y_1 + y_2,\ \omega_2 = -4y_1 + \sqrt{3}y_4^2,\ \omega_3 = -\sqrt{3}(y_1 + y_2), -2y_3 + y_4^3$;

$$L_5 : \omega' = u^2(1 + y_2^2 + y_4^2),\ \omega = \frac{(y_1y_2 + y_3y_4)^2 + y_1^2 + y_3^2}{1 + y_2^2 + y_4^2}.$$

$$L_6 : \omega' = u^2(1 + y_2^2 + y_4^2),\ \omega = \frac{(y_1y_2 + y_3y_4)^2 + y_1^2 + y_3^2}{1 + y_2^2 - y_4^2}.$$

$$L_7 : \omega' = u(1 + y_2^2)^{1/2}\exp(\alpha \arctan y_2),\ \omega = \frac{2y_1 - 2y_3^2 + y_4^2 + 2y_2(y_1y_2 + y_3y_4)}{1 + y_2^2 - y_4^2}.$$

$$L_8 : \omega' = u(1 + y_2^2)^{1/2},\ \omega = \frac{2y_1 - y_3^2 + y_4^2 + 2y_2(y_1y_2 + y_3y_4)}{1 + y_2^2} - 2\arctan y_2.$$

We can check directly that the d'Alembert equation have no solutions invariant under respect to subalgebras $F_i (i = 1, ..., 4)$, L_5, and L_6. Considering all the remaining subalgebras of the rank 3 and using the ansatz $\omega' = \varphi(\omega)$, we reduce the d'Alembert equation to ordinary differential equations with an unknown function φ :

$$L_1 : -4(4 + \exp\omega)\ddot{\varphi} + 2\varphi\dot{\varphi} - 8\varphi + \lambda\varphi^3 = 0.$$

$$L_2 : 4(4 + \exp\omega)\ddot{\varphi} - 2\varphi\dot{\varphi} - 8\varphi + \lambda\varphi^3 = 0.$$

$$L_3 : 9\omega(\omega - 4)\ddot{\varphi} + 18(\omega - 2)\dot{\varphi} + 2\varphi + \frac{\lambda}{4}\varphi^3 = 0.$$

$$L_4 : 9\omega(\omega-4)\ddot{\varphi} + 18(\omega-2)\dot{\varphi} + 2\varphi - \frac{\lambda}{4}\varphi^3 = 0.$$

$$L_7 : -16\alpha\exp(-\omega)\ddot{\varphi} - 8\alpha\exp(-\omega)\dot{\varphi} + \lambda\varphi^3 = 0.$$

$$L_8 : -16\ddot{\varphi} + \lambda\varphi^3 = 0.$$

A general integral of the equation $-16\ddot{+}\lambda\varphi^3 = 0$ has the form

$$\omega + C = \int \left(\frac{\lambda\varphi^4}{32} + C_1\right)^{-1/2} d\varphi.$$

When $C_1 = 0$, we get $\varphi = \dfrac{4\sqrt{2}}{\sqrt{\lambda}(\omega + C)}$. Thus, we have the following exact solution of the d'Alember equation:

$$u = \frac{4\sqrt{2}}{[\lambda(1+y_2^2)]^{1/2}\left(\dfrac{2y_1 - y_3^2 + y_4^2 + 2y_2(y_1y_2 + y_3y_4)}{1+y_2^2} - 2\arctan y_2 + C\right)}.$$

References

[1] Fushchych W., Shtelen W. and Serov N., Symmetry Analysis and Exact Solutions of Equations of Nonlinear Mathematical Physics, Kluwer Academic Publishers, 1993, 436p. (Russian version, Naukova Dumka, Kiev, 1989).

[2] Fushchych W., Barannyk L. and Barannyk A., Subgroup Analysis of the Galilei and Poincaré Groups and Reductions of Nonlinear Equations, Naukova Dumka, Kiev, 1991, 300p.

[3] Fushchych W., Barannyk L. and Barannyk A., On continuous subgroups of pseudoorthogonal and pseudounitary subgroups, Preprint N 86.87, Institute of Math. of Ukrainian Acad. of Sci., 1986.

[4] Patera J., Winternitz P. and Zassenhaus H., Continuous subgroups of the fundamental groups of physics, *J. Math. Phys.* 1975, V. 16, N8, 1597–1624.

[5] Fushchych W.I., Barannyk A.F. and Moskalenko Yu.D., On exact solutions of the nonlinear d'Alembert and Liouville equations in space $R_{2,2}$. I. *Ukr. Math. J.*, 1990, V.42, N 8, 1142–1148.

[6] Barannyk A.F., Moskalenko Yu.D. and Fushchych W.I., Subalgebras of pseudoorthogonal algebra $AO(3,3)$, Preprint N 89.67, Institute of Math. of Ukrainian Acad. of Sci., 1989.

[7] Barannyk A.F., Moskalenko Yu.D. and Fushchych W.I., Subalgebras of affine algebra $AIGL(3,R)$, Preprint N 89.65, Institute of Math. of Ukrainian Acad. of Sci., 1989.