A Nonrelativistic Chiral Soliton
in One Dimension

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Dedicated to   W. Fushchych on the occasion of his sixtieth birthday

Abstract

I analyze the one-dimensional, cubic Schrödinger equation with a nonlinearity constructed from the current density rather than, as is usual, from the charge density. A soliton solution is found, where the soliton moves only in one direction. Relation to the higher-dimensional Chern–Simons theory is indicated. The theory is quantized and results for the two-body quantum problem agree at weak coupling with those coming from a semiclassical quantization of the soliton.

1   Introduction

The Schrödinger equation in one spatial dimension with the cubic nonlinearity

$$i\hbar\frac{\partial}{\partial t}\psi(t,x) = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(t,x) - g\rho(t,x)\psi(t,x), \qquad g > 0, \quad \rho = \psi^*\psi \ . \tag{1}$$

plays a cycle of interrelated roles in mathematical physics. (Here * signifies complex conjugation.)

Viewed as a nonlinear partial differential equation for the function ψ, it possesses the famous soliton solution

$$\psi_s(t,x) = \pm e^{i\frac{mv}{\hbar}(x-ut)}\frac{\hbar}{\sqrt{gm}}\frac{\alpha}{\cosh\alpha(x-vt)}, \qquad \alpha^2 = (m^2v^2/\hbar^2)(1-2u/v) \ . \tag{2}$$

Moreover, Eq. (1) is completely integrable and multisoliton solutions can also be explicitly constructed. Note that the soliton described by Eq. (2) moves with (group) velocity v, which is unrestricted as to sign and magnitude. (The phase velocity u must be less than $v/2$.) In particular, the soliton can be brought to rest: the theory is Galileo-invariant, so that any solution to (1) can be mapped into another solution by the Galileo boost

$$x \longrightarrow x - Vt, \qquad \psi(t,x) \longrightarrow e^{\frac{i}{\hbar}mV(x-\frac{1}{2}Vt)}\psi(t, x - Vt) \ . \tag{3}$$

[Galileo transformations for the dynamical system (1) are accompanied by a phase change of ψ – a so-called "1-cocycle".] Applying transformation (3) to (2) with $V = -v$ results in

$$\psi_s(t,x) \longrightarrow \pm e^{i\frac{\hbar\alpha^2}{2m}t}\frac{\hbar}{\sqrt{gm}}\frac{\alpha}{\cosh\alpha x} \tag{4}$$

which also solves (1) and describes a soliton at rest.

Alternatively, one may view (1) as a quantal Heisenberg equation of motion for the quantum field operator $\psi(t,x)$, which is taken to satisfy the commutation relation

$$[\psi(t,x_1),\psi^*(t,x_2)] = \delta(x_1 - x_2) \ . \tag{5}$$

(Now * denotes Hermitian conjugation.) Since dynamics (1) conserves the number operator $\int dx\,\rho$, the quantum Hilbert space can be decomposed according to the (integer) eigenvalues N of $\int dx\,\rho$, and one finds that the N-body wave function

$$\psi_N(t;x_1,\ldots,x_N) \equiv \frac{1}{\sqrt{N!}}\langle 0|\psi(t,x_1)\cdots\psi(t,x_N)|N\rangle \tag{6}$$

satisfies an N-body Schrödinger equation with two-body, pairwise attractive δ-function interactions. The bound-state spectrum can be found explicitly and the energy eigenvalue is determined as

$$E_N = \frac{-g^2m}{6\hbar^2}(N^3 - N) \ . \tag{7}$$

The cycle of formulas closes: the static solution (4) can be quantized semiclassically and the bound energy spectrum obtained in this way coincides with (7) [1]. Specifically, the classical energy of (4) is $\frac{-g^2m}{6\hbar^2}N^3$, where N is the value of $\int dx\,\rho$ evaluated on (4). The semiclassical (first quantum) correction diminishes N^3 by N (so that the bound state energy vanishes at $N = 1$). (Also, the semiclassical phase shift of the scattering problem reproduces the exact quantal formula for weak coupling [2].)

In this article, I shall review some properties of another nonlinear Schrödinger equation, which (in a transformed formulation) has recently arisen in discussions of one-dimensional condensed matter systems (quantum wires, Hall edge states) [3] and which possesses a remarkable soliton solution, in that the soliton is chiral, *i.e.*, it can move only in one direction [4].

2   The Equation and Its Soliton Solution

The equation that we consider differs from (1) in that the nonlinearity involves the current density j

$$j = \frac{\hbar}{m}\mathrm{Im}\left(\psi^*\frac{\partial}{\partial x}\psi\right) \tag{8}$$

rather than charge density ρ:

$$i\hbar\frac{\partial}{\partial t}\psi(t,x) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(t,x) - \hbar\lambda j(t,x)\psi(t,x) \ . \tag{9}$$

Note that j can be substituted for ρ only in one spatial dimension, where the "vector" nature of j is absent. The current and charge densities are linked by the continuity equation, which is valid both for (1) and (9):

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}j = 0 . \tag{10}$$

When ψ is decomposed into modulus and phase

$$\psi = \sqrt{\rho}e^{i\theta} \tag{11}$$

one sees that

$$j = \frac{\hbar}{m}\rho\frac{\partial}{\partial x}\theta \tag{12}$$

and (9) takes the form of the usual nonlinear Schrödinger equation (1),

$$i\hbar\frac{\partial}{\partial t}\psi(t,x) = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(t,x) - g(t,x)\rho(t,x)\psi(t,x) \tag{13}$$

but with the coupling strength

$$g(t,x) = \hbar^2\frac{\lambda}{m}\frac{\partial}{\partial x}\theta(t,x) \tag{14}$$

modulated by the spatial variation of the phase. There is no *a priori* restriction on the sign of λ, but for definiteness we shall take it to be positive, $\lambda > 0$.

To find the one-soliton solution ψ_s, we take the phase to be as in a propagating wave

$$\psi_s = e^{-i(\omega t - kx)}\sqrt{\rho} \tag{15}$$

so that

$$j = v\rho, \qquad v \equiv \frac{\hbar k}{m}, \tag{16}$$

and our equation (9) becomes identical to (1) with

$$g = \hbar\lambda v . \tag{17}$$

The soliton solution exists provided $g > 0$; this requires

$$v > 0 . \tag{18}$$

The soliton can only move to the right; it is chiral. (Had we taken $\lambda < 0$, then the soliton could move only to the left.) The profile that solves (9) is found from (2) and (17) to be

$$\psi_s(t,x) = \pm e^{i\frac{mv}{\hbar}(x-ut)}\sqrt{\frac{\hbar}{\lambda mv}}\frac{\alpha}{\cosh\alpha(x-vt)} \tag{19}$$

where $u \equiv \omega/k$. The presence of $1/\sqrt{v}$ in the amplitude reinforces the statement that the velocity must be positive; in particular, the soliton cannot be brought to rest. Evidently, dynamics (9) is *not* Galileo-invariant; the velocity of the soliton cannot be arbitrarily reduced. This fact will be seen clearly in the next section, where the action, Lagrangian, and other dynamical quantities relevant to (9) are discussed.

3   Action Principle, Symmetries, and Constants of Motion

Eq. (9) may be obtained from an action principle, which also is useful for identifying the Hamiltonian (energy), momentum, and other constants of motion.

Consider the Lagrange density

$$\mathcal{L} = i\hbar \Psi^* \frac{\partial}{\partial t}\Psi - \frac{\hbar^2}{2m}\left|\left(\frac{\partial}{\partial x} - i\frac{\lambda}{2}\rho\right)\Psi\right|^2 . \tag{20}$$

The Euler–Lagrange equation reads

$$i\hbar \frac{\partial}{\partial t}\Psi = -\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x} - i\frac{\lambda}{2}\rho\right)^2 \Psi - \frac{\hbar\lambda}{2}j\Psi \tag{21}$$

where

$$\rho = \Psi^*\Psi, \tag{22}$$

$$j = \frac{\hbar}{m}\,\mathrm{Im}\left(\Psi^*\left(\frac{\partial}{\partial x} - i\frac{\lambda}{2}\rho\right)\Psi\right) \tag{23}$$

and both are linked by the continuity equation (10). Next, we redefine the Ψ field by

$$\Psi(t,x) = e^{i\frac{\lambda}{2}\int^x \mathrm{d}y\,\rho(t,y)}\psi(t,x) \tag{24}$$

(the lower limit on the integral is immaterial – it affects only the phase of ψ) so that

$$i\hbar \frac{\partial}{\partial t}\psi(t,x) - \frac{\hbar\lambda}{2}\int^x \mathrm{d}y\,\frac{\partial}{\partial t}\rho(t,y)\psi(t,x) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(t,x) - \frac{\hbar\lambda}{2}j(t,x)\psi(t,x) . \tag{25}$$

But the integral may be evaluated with the help of (10) and transferred to the right side. The resulting equation is just (9).

Energy is conserved as a consequence of time-translation invariance, and its form can be deduced with the help of Noether's theorem, or by inspection from (20). Evidently the Hamiltonian (energy) density is

$$\mathcal{E} = \frac{\hbar^2}{2m}\left|\left(\frac{\partial}{\partial x} - i\frac{\lambda}{2}\rho\right)\Psi\right|^2 = \frac{\hbar^2}{2m}\left|\frac{\partial}{\partial x}\psi\right|^2 . \tag{26}$$

Similarly, space translation invariance ensures momentum conservation, and the momentum density reads

$$\mathcal{P} = \hbar\,\mathrm{Im}\left(\Psi^*\frac{\partial}{\partial x}\Psi\right) = mj + \frac{\hbar\lambda}{2}\rho^2 . \tag{27}$$

These quantities obey continuity equations. With the energy density $\mathcal{E}$, there is associated the energy flux T^{0x}

$$T^{0x} = \frac{-\hbar^2}{m}\,\mathrm{Re}\left(\frac{\partial}{\partial t}\Psi^*\left[\frac{\partial}{\partial x} - \frac{i\lambda}{2}\rho\right]\Psi\right) = \frac{-\hbar^2}{m}\,\mathrm{Re}\left(\frac{\partial}{\partial t}\psi^*\frac{\partial}{\partial x}\psi\right) + \frac{\hbar\lambda}{2}j^2 \tag{28}$$

and together they satisfy

$$\frac{\partial}{\partial t}\mathcal{E} + \frac{\partial}{\partial x}T^{0x} = 0 . \tag{29}$$

Similarly, with the momentum density $\mathcal{P}$ supplemented by the momentum flux

$$T^{xx} = \frac{\hbar^2}{m}\left|\left(\frac{\partial}{\partial x} - \frac{i\lambda}{2}\rho\right)\Psi\right|^2 - \frac{\hbar^2}{4m}\frac{\partial^2}{\partial x^2}\rho = \frac{\hbar^2}{m}\left|\frac{\partial}{\partial x}\psi\right|^2 - \frac{\hbar^2}{4m}\frac{\partial^2}{\partial x^2}\rho, \tag{30}$$

their continuity equation is verified as

$$\frac{\partial}{\partial t}\mathcal{P} + \frac{\partial}{\partial x}T^{xx} = 0 \ . \tag{31}$$

Therefore, when fields decrease rapidly at spatial infinity, the energy and momentum are time-independent

$$E = \int \mathrm{d}x\,\mathcal{E} \qquad \frac{\mathrm{d}}{\mathrm{d}t}E = 0 \tag{32}$$

$$P = \int \mathrm{d}x\,\mathcal{P} \qquad \frac{\mathrm{d}}{\mathrm{d}t}P = 0 \ . \tag{33}$$

Note that the momentum density $\mathcal{P}$ is not equal to the energy flux T^{0x} because the theory is not Lorentz-invariant. Also it is not Galileo-invariant. This is seen in the present context by constructing the usual Galileo boost generator

$$G = tP - m\int \mathrm{d}x\, x\,\rho \ . \tag{34}$$

From (10) and (33) we find

$$\frac{\mathrm{d}G}{\mathrm{d}t} = P + m\int \mathrm{d}x\, x\frac{\partial}{\partial x}j = P - m\int \mathrm{d}x\, j \ . \tag{35}$$

But Eq. (27) shows that P possesses the dynamical contribution $\frac{\hbar\lambda}{2}\int \mathrm{d}x\,\rho^2$ in addition to the usual kinematical term $m\int \mathrm{d}x\, j$; therefore, it follows that G is not conserved,

$$\frac{\mathrm{d}}{\mathrm{d}t}G = \frac{\hbar\lambda}{2}\int \mathrm{d}x\,\rho^2, \tag{36}$$

but always increases in time.

There does exist, however, another symmetry, beyond time and space translation invariance, and another conserved quantity beyond H and P. The additional symmetry is dilation invariance corresponding to the possibility of rescaling space and time:

$$t \longrightarrow at, \qquad x \longrightarrow \sqrt{a}x, \qquad \psi(t,x) \longrightarrow a^{1/4}\psi(at,\sqrt{a}x) \ . \tag{37}$$

Applied to the soliton solution (19), transformation (37) yields another soliton solution with group and phase velocities v, u rescaled by $\sqrt{a}$. In order to construct the constant of motion associated with dilation invariance, it is useful to "improve" the energy density and energy flux by the addition of terms ("superpotentials") that do not affect the continuity equation nor the spatial integral defining the energy. Instead of (26) and (28), we use

$$\mathcal{E}_{\mathrm{improved}} = \mathcal{E} - \frac{\hbar^2}{8m}\frac{\partial^2}{\partial x^2}\rho \tag{38}$$

$$T^{0x}_{\mathrm{improved}} = T^{0x} - \frac{\hbar^2}{8m}\frac{\partial^2}{\partial x^2}j \ . \tag{39}$$

Owing to (10), we still have

$$\frac{\partial}{\partial t}\mathcal{E}_{\text{improved}} + \frac{\partial}{\partial x}T^{0x}_{\text{improved}} = 0 \tag{40}$$

and for fields that decrease at large distances

$$E = \int \mathrm{d}x\, \mathcal{E} = \int \mathrm{d}x\, \mathcal{E}_{\text{improved}} . \tag{41}$$

The improved energy density satisfies

$$\mathcal{E}_{\text{improved}} = \tfrac{1}{2}T^{xx} \tag{42}$$

which is the nonrelativistic criterion of scale invariance in a field theory [5]. It follows from (42) that the dilation change

$$D = tE - \tfrac{1}{2}\int \mathrm{d}x\, x\, \mathcal{P} \tag{43}$$

is time-independent:

$$\frac{\mathrm{d}D}{\mathrm{d}t} = E - \tfrac{1}{2}\int \mathrm{d}x\, x\, \frac{\partial}{\partial t}\mathcal{P} = E + \tfrac{1}{2}\int \mathrm{d}x\, x\, \frac{\partial}{\partial x}T^{xx} = \int \mathrm{d}x\left(\mathcal{E}_{\text{improved}} - \tfrac{1}{2}T^{xx}\right) = 0 . \tag{44}$$

[Condition (42) also ensures conformal invariance in a Galileo-invariant theory [5]. Here the absence of Galileo invariance requires the absence of conformal invariance, as is seen from the Lie algebra of the corresponding generators: the bracket of P with the conformal generator closes on the Galileo boost generator.]

The momentum and energy of soliton (19) are obtained by evaluating P and E on profile (19):

$$P_{\text{soliton}} = Mv \tag{45}$$

$$E_{\text{soliton}} = \tfrac{1}{2}Mv^2 = \tfrac{1}{2M}P^2_{\text{soliton}} \tag{46}$$

$$M = mN\left(1 + \frac{\lambda^2}{12}N^2\right) \tag{47}$$

with

$$N = \int \mathrm{d}x\, \rho = \frac{2\hbar\alpha}{\lambda mv} . \tag{48}$$

The soliton's dynamical characteristics are those of a nonrelativistic particle of mass M moving with the velocity v. The consequent connection between energy and momentum exhibited by (45) and (46),

$$E_{\text{soliton}} = \tfrac{1}{2}vP_{\text{soliton}} \tag{49}$$

in fact follows dilation invariance. For soliton (19), $\mathcal{P}$ is a function $x - vt$, hence, the time-independent dilation change (43) reads

$$D = tE - \tfrac{1}{2}\int \mathrm{d}x\, x\, \mathcal{P}(x - vt) = t\left(E - \tfrac{v}{2}P\right) - \tfrac{1}{2}\int \mathrm{d}x\, x\, \mathcal{P}(x) \tag{50}$$

where the second equality follows from the first by a shift of the integration variable. Since both D and the last term in (49) are time-independent, the coefficient of t must vanish, and this implies (49). Since for our solution $\mathcal{P}(x)$ is an even function of x, the last integral also vanishes and

$$D_{\text{soliton}} = 0 . \tag{51}$$

4   Other Solutions

Thus far we have considered only the attractive interaction, which binds a soliton: $g > 0$ in (1) and $\lambda v > 0$ in (17, 18). With a repulsive interaction, there also exist classical solutions, variously known as "kinks" or "dark solitons" [6]. They carry infinite energy, and we shall not discuss them here.

Unlike the conventional nonlinear Schrödinger equation (1), our chiral equation (9) does not appear to be completely integrable [7], and analytic expressions for multisoliton solutions to (9) are not available. It has been remarked, however, that if the theory is modified by adding to (20) the potential $V(\rho) = -\frac{\hbar^2\lambda^2}{8m}\rho^3$, then Eq. (21) acquires the extra term $-\frac{3\hbar^2\lambda^2}{8m}\rho^2\Psi$ and becomes an integrable nonlinear, derivative Schrödinger equation with the nonlinearity $i\frac{\hbar^2\lambda^2}{2m}\rho\frac{\partial}{\partial x}\Psi$. However, the solitons are no longer chiral [8].

5   Relation to the (2+1)-dimensional Chern–Simons Theory

Our dynamical, chiral model with the Lagrange density (20) is partially related, through dimensional reduction, to a $(2+1)$-dimensional model of nonrelativistic fields interacting with a $U(1)$ gauge potential, whose kinetic term is the Chern–Simons expression.

Consider the $(2+1)$-dimensional Lagrange density

$$\mathcal{L}_{(2+1)} = \frac{1}{2\bar{\kappa}}\epsilon^{\alpha\beta\gamma}A_\alpha\partial_\beta A_\gamma + i\hbar\Psi^*\left(\frac{\partial}{\partial t} + iA_0\right)\Psi - \frac{\hbar^2}{2m}\sum_{i=1}^{2}\left|\left(\frac{\partial}{\partial r^i} + iA_i\right)\Psi\right|^2. \quad (52)$$

When dependence on the second spatial coordinate is suppressed, and A_2 is renamed $\frac{mc}{\hbar^2}B$, $\mathcal{L}_{(2+1)}$ leads to a B–F gauge theory:

$$\mathcal{L}_{(1+1)} = \frac{1}{2\kappa}B\epsilon^{\mu\nu}F_{\mu\nu} + i\hbar\Psi^*\left(\frac{\partial}{\partial t} + iA_0\right)\Psi - \frac{\hbar^2}{2m}\left|\left(\frac{\partial}{\partial x} + iA_x\right)\Psi\right|^2 - \frac{mc^2}{2\hbar^2}B^2\rho. \quad (53)$$

Here $F_{\mu\nu} = \frac{\partial}{\partial x^\mu}A_\nu - \frac{\partial}{\partial x^\nu}A_\mu$, $x^\mu = \{t, x\}$, $A_\mu = (A_0, A_x)$, and $\kappa = \frac{\hbar^2}{mc}\bar{\kappa}$, where c is the velocity of light, which plays no role in the following. This is not yet our theory (20), because when the nonpropagating B and A_μ fields are eliminated, they decouple completely in the sense that the phase of Ψ may be adjusted so that all interactions disappear:

$$\mathcal{L}_{(1+1)} \longrightarrow i\hbar\Psi^*\frac{\partial}{\partial t}\Psi - \frac{\hbar^2}{2m}\left|\frac{\partial}{\partial x}\Psi\right|^2. \quad (54)$$

In order to make the vector potential A_μ and the B field dynamically active, thereby allowing the Ψ particles to interact, we include a kinetic term for B, which could be taken in the Klein–Gordon form. However, we prefer a simpler expression that describes "chiral" Bose fields propagating only in one direction, whose Lagrangian density is proportional to $\pm\frac{\partial B}{\partial t}\frac{\partial B}{\partial x} + v\frac{\partial B}{\partial x}\frac{\partial B}{\partial x}$ [9]. Here v is a velocity and the consequent equation of motion arising from this kinetic term (without further interaction) is solved by $B = B(x \mp vt)$ (with suitable boundary conditions at spatial infinity), describing propagation in one direction, with velocity $\pm v$. Note that $\frac{\partial B}{\partial t}\frac{\partial B}{\partial x}$ is not invariant against a Galileo transformation, which is a symmetry of $\mathcal{L}_{(1+1)}$ and of $\frac{\partial B}{\partial x}\frac{\partial B}{\partial x}$: performing a Galileo boost on $\frac{\partial B}{\partial t}\frac{\partial B}{\partial x}$ with velocity $\tilde{v}$ gives rise to $\tilde{v}\frac{\partial B}{\partial x}\frac{\partial B}{\partial x}$, effectively boosting the v parameter by $\tilde{v}$. Consequently one can

drop the $v \frac{\partial B}{\partial x} \frac{\partial B}{\partial x}$ contribution to the kinetic B Lagrangian, thereby selecting to work in a global "rest frame". Boosting a solution in this rest frame then produces a solution to the theory with a $\frac{\partial B}{\partial x} \frac{\partial B}{\partial x}$ term.

In view of the above, we choose the B-kinetic Lagrange density to be

$$\mathcal{L}_B = \pm \frac{1}{\hbar} \frac{\partial B}{\partial t} \frac{\partial B}{\partial x} \tag{55}$$

and the total Lagrange density is $\mathcal{L}_B + \mathcal{L}_{(1+1)}$. It is possible to remove the A_μ and B fields by a Hamiltonian reduction, as described in Ref. [10], and by a phase redefinition of Ψ. Once this is done, one is left with the Lagrange density (20)

$$\mathcal{L}_B + \mathcal{L}_{(1+1)} \longrightarrow \mathcal{L} \tag{56}$$

with $\pm \kappa^2$ entering as $-\frac{\lambda}{2}$. The "$\pm$" sign reflects the sign arbitrariness of the $\frac{\partial B}{\partial t} \frac{\partial B}{\partial x}$ kinetic term, and without loss of generality can be chosen so that $\lambda > 0$. Also, the origin of Galileo noninvariance is now recognized as arising from $\mathcal{L}_B$ [11].

6   Quantum Theory

The theory described by the Lagrange density (20) may be quantized. For the quantum Hamiltonian, we take the normal ordered expression

$$H = \frac{\hbar^2}{2m} \int \mathrm{d}x : \left| \left(\frac{\partial}{\partial x} - i \frac{\lambda}{2} \rho \right) \Psi \right|^2 : \tag{57}$$

and posit the canonical commutation relations (5). It follows that the two-body wave function, defined as in (6), satisfies

$$i\hbar \frac{\partial}{\partial t} \psi_2(t; x_1, x_2) = -\frac{\hbar^2}{2m} \left[\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - i\lambda \delta(x_1 - x_2) \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right) \right] \psi_2(t; x_1, x_2). \tag{58}$$

Owing to time- and space-translation invariance, one can separate the time and center-of-mass coordinates

$$\psi_2(t; x_1, x_2) = e^{-i(Et/\hbar)} e^{i(P/\hbar)(x_1 + x_2)/2} u(x_1 - x_2) . \tag{59}$$

The wave function for relative motion satisfies

$$\left(-\frac{\hbar^2}{m} \frac{\partial^2}{\partial x^2} + \frac{P^2}{4m} - \hbar \frac{P}{2m} \lambda \delta(x) \right) u(x) = E u(x) . \tag{60}$$

The presence of the total momentum P in the δ-function potential for relative motion vividly demonstrates the absence of Galileo invariance. Provided

$$\frac{P}{m} > 0, \tag{61}$$

Eq. (60) possesses a bound state solution with the energy

$$E = \frac{P^2}{4m} \left(1 - \frac{\lambda^2}{4} \right) . \tag{62}$$

Since P/m may be identified with (a multiple of) the classical velocity v, we recognize condition (61) as the same as (18), which guarantees soliton binding. So we expect that there is a relation between the classical soliton and quantum bound states.

Justification for this can be see from the following argument. If we write (62) as

$$E = \frac{P^2}{2M_{\text{quantum}}},$$

(63)

we find that M_{quantum} is given by

$$M_{\text{quantum}} = \frac{2m}{1 - \frac{\lambda^2}{4}}.$$

(64)

Next we conjecture that the classical formula for M, Eq. (47), should be modified in semiclassical quantization in the same way as in the conventional nonlinear Schrödinger equation: *viz.* as in (7) N^3 is diminished by N [1]. Thus, we replace (47) by

$$M_{\text{semiclassical}} = mN + \frac{m\lambda^2}{12}(N^3 - N).$$

(65)

For $N = 2$, this gives

$$M_{\text{semiclassical}} = 2m\left(1 + \frac{\lambda^2}{4}\right).$$

(66)

which agrees with (64) at weak coupling, *i.e.*, small λ^2. Although explicit solutions to the N-body quantum Schrödinger equation are not known for $N > 2$, one may establish perturbatively in λ that (65) is consistent with quantum bound states, at weak coupling.

The two-body problem inherits from the full N-body field theory, the latter's symmetries and constants of motion. In particular, the total momentum

$$P_2 \equiv p_1 + p_2$$

(67)

and the total energy

$$\begin{aligned} H_2 &= \frac{1}{2m}(p_1^2 + p_2^2) - \frac{\hbar}{2m}(p_1 + p_2)\lambda\delta(x_1 - x_2) \\ &= \frac{1}{2m}p^2 + \frac{1}{4m}P_2^2 - \frac{\hbar}{2m}P_2\lambda\delta(x) \qquad \left(p \equiv \tfrac{1}{2}(p_1 - p_2), \quad x \equiv x_1 - x_2\right) \end{aligned}$$

(68)

together with the dilaton charge

$$\begin{aligned} D_2 &= tH_2 - \tfrac{1}{4}(x_1p_1 + p_1x_1 + x_2p_2 + p_2x_2) \\ &= tH_2 - \tfrac{1}{4}(XP_2 + P_2X + xp + px) \qquad \left(X \equiv \tfrac{1}{2}(x_1 + x_2)\right) \end{aligned}$$

(69)

are time-independent and close on the algebra [5]

$$[H_2, P_2] = 0,$$

(70)

$$[H_2, D_2] = i\hbar H_2,$$

(71)

$$[P_2, D_2] = i\hbar\tfrac{1}{2}P_2.$$

(72)

It is interesting that in the present problem scale invariance does not prevent the formation of a bound state. That is because the absence of the Galileo invariance allows the bound state energy to depend on the total momentum, which is not quantized, and, as a consequence the bound state energy (62) lies in the continuum, as is required by scale invariance.

7 Conclusion

While our chiral, nonlinear Schrödinger equation apparently cannot be completely integrated, it should be possible, at least numerically, to study the N-soliton solution. From this, one could extract the nature of the solitons' mutual scattering and possible dissipation. It should then be most interesting to examine the same processes in the N-body quantum Schrödinger equation. Also the procedure for quantizing the classical solutions could be further developed. While the analytic intractability may be a daunting obstacle for arbitrary N, one should at least unravel the $N = 2$ case.

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