Representations of the $Q$-deformed Euclidean Algebra $U_q(iso_3)$ and Spectra of their Operators

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Abstract

Representations of the $q$-deformed Euclidean algebra $U_q(iso_3)$, which at $q \to 1$ gives the universal enveloping algebra $U(iso_3)$ of the Lie algebra $iso_3$ of the Euclidean Lie group $ISO(3)$, are studied. Explicit formulas for operators of irreducible $\star$-representations defined by two parameters $\rho \in \mathbb{R}$ and $s \in \frac{1}{2} \mathbb{Z}$ are given. At $q \to 1$, these representations exhaust all irreducible infinite-dimensional $\star$-representations of $U(iso_3)$. The spectrum of the operator $T_{\rho,s}(I_3)$ corresponding to a $q$-analogue of the infinitesimal operator of shifts along the third axis is given. Contrary to the case of the classical Euclidean algebra $iso_3$, this spectrum is discrete and has one point of accumulation.

1. Introduction

The aim of this paper is to study irreducible representations of the $q$-deformed Euclidean algebra $U_q(iso_3)$ defined on the base of the algebra $U_q(so_3)$ given in [1] (see also [2]). The classical Lie algebras $so_3$ and $sl_2$ are isomorphic. The algebra $U_q(so_3)$ from [1] differs from the quantum algebra $U_q(sl_2)$ defined by Drinfeld [3] and Jimbo [4]. Namely, $U_q(sl_2)$ is defined by means of the Cartan subalgebra and root subspaces of the Lie algebra $sl_2$. The $q$-deformed algebra $U_q(iso_3)$ is a $q$-deformation of the defining relations $[J_1, J_2] = J_3,$ $[J_2, J_3] = J_1,$ $[J_3, J_1] = J_2$.

Adding to $U_q(so_3)$ the generator $I_3$ corresponding to infinitesimal shifts along the third axis and postulating commutation relations of $I_3$ with other generators, we obtain the $q$-deformed algebra $U_q(iso_3)$. This algebra is a $q$-deformation of the Lie algebra $iso_3$ of the Euclidean group $ISO(3)$ which is the semidirect product of the rotation group $SO(3)$ and the translation group of a 3-dimensional Euclidean space. There are difficulties with definition of the Hopf algebra structure in this algebra. We do not consider this problem here and only note that our algebra $U_q(iso_3)$ can be embedded into the quantum algebra $U_q(iu_3)$ (the $q$-deformation of the Lie algebra of the inhomogeneous unitary group). The last quantum algebra is equipped with the structure of a Hopf algebra.

We construct infinite-dimensional irreducible representations of the algebra $U_q(iso_3)$. They are given by two numbers $\rho \in \mathbb{R}$ and $s \in \frac{1}{2} \mathbb{Z}$. Unfortunately, we cannot state that they exhaust all irreducible $\star$-representations of $U_q(iso_3)$. But at $q \to 1$, they give all irreducible $\star$-representations of the Lie algebra $iso_3$. Thus, we can state that we constructed...
\( q \)-deformations of all irreducible \( \ast \)-representations of \( \text{iso}_3 \). Remark that irreducible \( \ast \)-representations of \( U_q(\text{iso}_3) \) of class 1 with respect to the subalgebra \( U_q(\text{so}_3) \) (that is, in the spaces of these representations, there exist vectors invariant with respect to this subalgebra) were constructed in [5].

We find the spectrum and spectral measure of the representation operator corresponding to the generator \( I_3 \) of \( U_q(\text{iso}_3) \). This operator is bounded and has a discrete spectrum. It is interesting that in the classical case (i.e., when \( q = 1 \)) this operator is bounded and has a continuous spectrum. We could find this spectrum and the spectral measure by means of involving into consideration the theory of \( q \)-orthogonal polynomials [6, 7]. The operators \( T_{p,q}(I_3) \) are representable by Jacobi matrices. Thus, we can employ the theory of such operators [8] and this leads to the theory of \( q \)-orthogonal polynomials.

Everywhere below we assume that the deformation parameter \( q \) lies in the finite interval \( 0 < q < 1 \).

### 2. The \( q \)-deformed algebra \( U_q(\text{so}_3) \) and its representations

The algebra \( U_q(\text{so}_3) \) is a \( q \)-deformation of the universal enveloping algebra of the Lie algebra \( \text{so}(3) \) of the rotation group \( SO(3) \). It is generated by three elements \( I_{21}, I_{32} \) and \( I_{31} \) satisfying the relations

\[
[I_{21}, I_{32}]_{q^{1/4}} \equiv q^{1/4} I_{21} I_{32} - q^{-1/4} I_{32} I_{21} = I_{31}, \tag{1}
\]

\[
[I_{32}, I_{31}]_{q^{1/4}} = I_{21}, \quad [I_{31}, I_{21}]_{q^{1/4}} = I_{32}. \tag{2}
\]

Unfortunately, a Hopf algebra structure is not known on \( U_q(\text{so}_3) \). Nevertheless, it is shown [9] that we can consider tensor products of irreducible finite dimensional representations which are \( q \)-deformations of irreducible representations of the Lie algebra \( \text{so}_3 \).

Let us remark that according to (1), the element \( I_{31} \) is determined by \( I_{21} \) and \( I_{32} \). Thus, the algebra \( U_q(\text{so}_3) \) can be defined by \( I_{21} \) and \( I_{32} \), but now, instead of the quadratic relations (1) and (2), we must take the cubic relations [10]

\[
I_{21} I_{32}^2 - (q^{1/2} + q^{-1/2}) I_{32} I_{21} I_{32} + I_{32}^3 I_{21} = -I_{21},
\]

\[
I_{31} I_{21}^2 - (q^{1/2} + q^{-1/2}) I_{21} I_{31} I_{21} + I_{31}^3 I_{21} = -I_{31}
\]

which can be written down in the form

\[
[[I_{21}, I_{32}]_{q^{1/4}}, I_{32}]_{q^{-1/4}} = -I_{21}, \quad [[I_{32}, I_{21}]_{q^{1/4}}, I_{21}]_{q^{-1/4}} = -I_{32}.
\]

The formulas \( I_{21}^q = -I_{21} \) and \( I_{32}^q = -I_{32} \) determine the \( \ast \)-algebra structure on \( U_q(\text{so}_3) \). The formulas \( I_{31}^q = -I_{31} \) and \( I_{21}^q = I_{21} \) determine on \( U_q(\text{so}_3) \) the \( \ast \)-structure defining the \( q \)-deformed algebra \( U_q(\text{so}_{2,1}) \).

We need below only those irreducible representations of \( U_q(\text{so}_3) \) which are \( q \)-deformations of irreducible representations of the Lie algebra \( \text{so}_3 \). These representations are given by nonnegative integers or half-integers \( l \). The representation \( T_l \), labeled by a number \( l \), acts on the linear space \( V_l \) with the orthonormal basis

\[
|m\rangle, \quad m = -l, -l + 1, \ldots, l, \tag{3}
\]
and is given in terms of \( q \)-numbers \([a] = (q^{a/2} - q^{-a/2})/(q^{1/2} - q^{-1/2})\) by the formulas

\[
T_l(I_1m) = i[m][m],
\]

\[
T_l(I_{21}m) = d(m)(l-m)[l+m+1]^{1/2}|m+1\rangle - d(m-1)(l-m+1)[l+m]^{1/2}|m-1\rangle, \tag{5}
\]

\[
T_l(I_{31}m) = i^{1/2}\{q^md(m)(l-m)[l+m+1]^{1/2}|m+1\rangle + q^{-m}d(m-1)(l-m+1)[l+m]^{1/2}|m-1\rangle\},
\]

where \( i = \sqrt{-1} \) and

\[
d(m) = \left(\frac{[m][m+1]}{[2m][2m+2]}\right)^{1/2} = \left(\frac{1}{(q^m + q^{-m})(q^{m+1} + q^{-m-1})}\right)^{1/2}.
\]

Let us note that the operators \( T_l(I_{21}) \) and \( T_l(I_{32}) \) are anti-Hermitian. The operator \( T_l(I_{31}) \) is not anti-Hermitian. Relations (1) and (2) do not allow us to make both operators \( T_l(I_{21}) \) and \( T_l(I_{31}) \) anti-Hermitian since the element \( I_{31} \) is not invariant under the \( * \)-operation.

Note that \( T_l, l = 0, \frac{1}{2}, 1, \ldots \), do not exhaust all irreducible representations of \( U_q(\text{so}_3) \). The classification of irreducible \( * \)-representations of \( U_q(\text{so}_3) \) is given in [11].

3. The \( q \)-deformed algebra \( U_q(\text{iso}_3) \)

In order to construct the \( q \)-deformed algebra \( U_q(\text{iso}_3) \), we add to the generating elements \( I_{21} \) and \( I_{32} \) of the algebra \( U_q(\text{so}_3) \) the element \( I_3 \) which satisfies the relations

\[
[I_3, I_{21}] \equiv I_3I_{21} - I_{21}I_3 = 0, \tag{7}
\]

\[
I_{32}^2I_3 - (q^{1/2} + q^{-1/2})I_{32}I_3I_{32} + I_3I_{32}^2 = -I_3, \tag{8}
\]

\[
I_3^2 - (q^{1/2} + q^{-1/2})I_3I_3 + I_{32}I_{32}^2 = 0. \tag{9}
\]

The associative algebra generated by the elements \( I_{i,i-1}, i = 2, 3, \) and \( I_3 \) obeying relations (1), (2) and (7)–(9) is denoted by \( U_q(\text{iso}(3, \mathbb{C})) \). Introducing the involution (antilinear antiautomorphism)

\[
I_{i,i-1}^* = -I_{i,i-1}, \quad I_3^* = -I_3 \tag{10}
\]

into \( U_q(\text{iso}(3, \mathbb{C})) \), we obtain the algebra \( U_q(\text{so}_3) \). The involution

\[
I_{21}^* = -I_{21}, \quad I_{32}^* = I_{32}, \quad I_3^* = -I_3 \tag{11}
\]
determines the algebra \( U_q(\text{so}_{2,1}) \).

If \( q \to 1 \), then the algebra \( U_q(\text{iso}_3) \) tends to the universal enveloping algebra of the Lie algebra of the Euclidean group \( ISO(3) \) (the group of motions of a 3-dimensional Euclidean space).

The element \( I_3 \) is a \( q \)-analogue of the infinitesimal operator for shifts along to the third axis. We may determine the elements in \( U_q(\text{so}_3) \) which are \( q \)-analouges of infinitesimal operators for shifts along the first and second axes. They are given by

\[
I_2 = q^{1/4}I_{32}I_3 - q^{-1/4}I_3I_{32}, \quad I_1 = q^{1/4}I_{31}I_3 - q^{-1/4}I_3I_{31} \equiv q^{1/4}I_{21}I_2 - q^{-1/4}I_2I_{21}.
\]
However, the elements \( I_1 \) and \( I_2 \) are not invariant under the *-operation.

The algebra \( U_q(\text{iso}_3) \) can be obtained by means of the contraction from the algebra \( U_q(\text{so}_4) \). The last algebra is a generalization of the algebra \( U_q(\text{so}_3) \) and is generated by the elements \( I_{i,i-1}, i = 1, 2, 3 \), satisfying the defining relations

\[
[I_{43}, I_{21}] = I_{43}I_{21} - I_{21}I_{43} = 0,
\]

\[
I_{i,i-1}^2 I_{i+1,i} - (q^{1/2} + q^{-1/2})I_{i,i-1}I_{i+1,i}I_{i,i-1} + I_{i+1,i}I_{i,i-1}^2 = -I_{i+1,i},
\]

\[
I_{i+1,i}^2 I_{i,i-1} - (q^{1/2} + q^{-1/2})I_{i+1,i}I_{i,i-1}I_{i+1,i} + I_{i,i-1}I_{i+1,i}^2 = -I_{i,i-1},
\]

where \( i = 2, 3 \). Replacing \( I_{43} \) by \( RI_3 \) in the last two relations taken for \( i = 3 \) and tending \( R \) to infinity, we obtain the defining relations for the algebra \( U_q(\text{iso}_3) \).

4. Representations of \( U_q(\text{iso}_3) \)

We describe those irreducible infinite-dimensional *-representations of \( U_q(\text{iso}_3) \) which are \( q \)-deformation of irreducible infinite-dimensional *-representations of the Lie algebra \( \text{iso}_3 \) (they are infinitesimal forms of irreducible unitary representations of the group ISO(3)).

The last representations \( T_{\rho,s}^q \) of \( \text{iso}_3 \) are given by two numbers \( \rho \in \mathbb{R} \) and \( s \in \frac{1}{2}\mathbb{Z} \). They act in the Hilbert space \( V_s \) with the orthonormal basis

\[
|l, m\rangle, \quad l = |s|, |s| + 1, |s| + 2, \ldots, \quad m = -l, -l + 1, \ldots, l.
\]

In fact, this basis is the set of bases \( |l, m\rangle, m = -l, -l + 1, \ldots, l \), of irreducible representations of the subalgebra \( \text{so}_3(3) \) and the restriction of \( T_{\rho,s}^q \) onto the subalgebra \( \text{so}_3 \) decomposes into the sum of the irreducible representations \( T_l \) of this subalgebra, for which \( l = |s|, |s| + 1, \ldots, \infty \).

The corresponding irreducible representations of \( U_q(\text{iso}_3) \) are denoted by \( T_{\rho,s} \), where \( \rho \) and \( s \) take the same values. The representation \( T_{\rho,s} \) acts in the space \( V_s \) described above and is given in the basis \( \{|l, m\rangle\} \) by (4) and (5) for the operators \( T_{\rho,s}(I_{21}) \) and \( T_{\rho,s}(I_{32}) \) and by the formula

\[
T_{\rho,s}(I_3)|l, m\rangle = i\rho \left[ \frac{|s|m}{|l|l+1} \right]|l, m\rangle - \rho \left( \frac{|l+s||l-s||l+m||l-m|}{|l||l+1||l+2||l+3|} \right)^{1/2} |l-1, m\rangle
\]

\[
+ \rho \left( \frac{|l+s+1||l-s+1||l+m+1||l-m+1|}{|l+1||l+2||l+3||l+4|} \right)^{1/2} |l+1, m\rangle,
\]

where numbers in the square brackets are \( q \)-numbers.

It is proved by direct (but awkward) calculation that the operators \( T_{\rho,s}(I_{21}), T_{\rho,s}(I_{32}) \) and \( T_{\rho,s}(I_3) \) satisfy the defining relations of the algebra \( U_q(\text{iso}_3) \). We omit this calculation.

**Theorem 1** The representations \( T_{\rho,s}, \rho \neq 0 \), are *-representations for \( U_q(\text{iso}_3) \). They are irreducible and pairwise nonequivalent.

**Proof.** It is checked by direct calculation that the operators \( T_{\rho,s}(I_{21}), T_{\rho,s}(I_{32}) \) and \( T_{\rho,s}(I_3) \) satisfy the conditions defining *-representations of \( U_q(\text{iso}_3) \). Irreducibility of \( T_{\rho,s} \), \( \rho \neq 0 \), is proved by the standard method. Pairwise nonequivalence of these representations follows from the fact that the operator \( T_{\rho,s}(I_3) \) has different spectra for different values of the pair \( (\rho, s) \). The spectrum of \( T_{\rho,s}(I_3) \) will be found in the next section.
5. Spectrum of the operator $T_{\rho,s}(I_3)$

Let us find the spectrum of the operator $L_\rho = iT_{\rho,s}(I_3)$, $i = \sqrt{-1}$. The carrier space $V_\rho$ of the representation $T_{\rho,s}$ can be represented as the direct sum $V_\rho = \sum_{m=0}^{\infty} \otimes V_{s,m}$, where

$$V_{s,m} = \sum_{l=\max\{s,m\}}^{\infty} \otimes C[l,m].$$

The subspaces $V_{s,m}$ are invariant with respect to the operator $L_\rho$. We shall find spectra of $L_\rho$ on each of these subspaces. The spectrum of $L_\rho$ on $V_\rho$ is obtained by uniting these spectra.

Further we consider the vectors $(-i)^{-|l,m|}$ instead of the vectors $|l,m\rangle$. In this case, the third summand in (12) must be multiplied by $-i$ and the second one by $i$.

If $|x,m\rangle$ is an eigenvector of the operator $L_\rho$: $L_\rho|x,m\rangle = x|x,m\rangle$, then

$$|x,m\rangle = \sum_{l=k}^{\infty} P_{l-k}(x)|l,m\rangle, \quad k = \max(|m|, |s|).$$

Formula (12) is symmetric with respect to permutation of $s$ and $m$ and to change of signs at $m$ and $s$. Therefore, we may assume, without loss of generality, that $s$ and $m$ are positive and that $s \geq m$.

Substituting expression (13) for $|x,m\rangle$ into the relation $L_\rho|x,m\rangle = x|x,m\rangle$ and acting by $L_\rho$ upon $|l,m\rangle$, we easily find that the vector $|x,m\rangle$ is an eigenvector of $L_\rho$ with the eigenvalue $x$ if $P_{l-k}$ satisfy the recurrence relation

$$\left(\frac{[n+2s+1][n+1][n+s+m+1][n+s-m+1]}{[n+s+1]^2[2n+2s+1][2n+2s+3]}ight)^{1/2} P_{n+1}(x) + \left(\frac{[n+2s][n+s+m][n+s-m]}{[n+s]^2[2n+2s-1][2n+2s+1]}\right)^{1/2} P_{n-1}(x) - \left[\frac{s}{m}\right] \frac{n+s}{n+s+1} P_n(x) = \frac{x}{\rho} P_n(x)$$

(14)

(here $n = l - k$) and the initial conditions $P_0(x) = 1$, $P_{-1}(x) = 0$.

Making in (14) the substitution

$$P_n(x) = q^{-n(n+2s+1)/4} \left[\frac{n+2s}{n}!\left[\frac{n+s+m}{n+s-m}!\right][2n+2s+1]!\right]^{1/2} P'_n(x),$$

where $[n]! = [n][n-1]...[1]$, we reduce (22) to the recurrence relation

$$\left(1 - q^{n+2s+1}\right)\left(1 - q^{n+s+m+1}\right) \frac{P_{n+1}(x)}{\left(1 - q^{2n+2s+1}\right)\left(1 - q^{2n+2s+2}\right)} + \frac{q^{n+2s+m+1}\left(1 - q^n\right)\left(1 + q^{n+s}\right)}{\left(1 - q^{2n+2s+1}\right)\left(1 - q^{2n+2s}\right)} \frac{P_{n-1}(x)}{\left(1 - q^{n+s}\right)\left(1 - q^{n+s+m}\right)}$$

$$- \frac{q^{n+1}\left(1 - q^n\right)}{\left(1 - q^{n+s}\right)} P_n(x) = \frac{q^{(s+m+1)/2} x}{\rho} P_n(x).$$

(15)
To solve this recurrence relation, we use the following recurrence relation
\[ A_n p_{n+1}(y) - C_n p_{n-1}(y) - (A_n - C_n - 1)p_n(y) = yp_n(y) \] (16)
for big $q$-Jacobi polynomials [6]
\[ p_n(y) = p_n(y; a, b, c|q) = \varphi_2\left( q^{-n}, \frac{abq^{n+1}}{aq}, \frac{cq}{q};q\right), \]
where $\varphi_2$ is the $q$-hypergeometric function and
\[ A_n = \frac{(1 - aq^{n+1})(1 - cq^{n+1})(1 - acq^{n+1})}{(1 - abq^{2n+1})(1 - abq^{2n+2})}, \quad C_n = \frac{(1 - q^n)(1 - bq^n)(1 - abc^{-1}q^n)acq^{n+1}}{(1 - abq^{2n})(1 - abq^{2n+1})}. \]

Setting into (16)
\[ a = q^{s+m}, \quad b = q^{s-m}, \quad c = -q^s, \quad y = \frac{x}{\rho} q^{(m+s+1)/2}, \] (17)
after some calculation, we reduce (16) to (15). This means that the solution of the recurrence relations (14), normed by the condition $P_0(x) = 1$, is the polynomial
\[ P_n(x) = N_n^{1/2} p_n(y; q^{s+m}, q^{s-m}, -q^s | q), \] (18)
where
\[ N_n = q^{-n(n+2s+1)/2} \frac{[n + 2s]![n + s + m]![s - m]![2n + 2s + 1]}{[n]![n + s - m]![s + m]![2s + 1]}. \] (19)

Note that the same result is obtained by setting $a = b = -q^s, c = q^{s+m}$ in (16) and retaining $y$ from (17).

The big $q$-Jacobi polynomials in a general case satisfy the orthogonality relation, which can be given by formulas (7.3.12-14) from [6] (the formula (7.3.13) is corrected):
\[ \int_{cq}^{aq} p_n(y; a, b, c|q)p_m(y; a, b, c|q)\mu(y)dy = \delta_{nm}/h_{n+1} \]
where
\[ \mu(y) = \frac{(aq; q)_\infty (bg; q)_\infty (cq; q)_\infty (abq/c; q)_\infty}{aq(1 - q)(c/a; q)_\infty (aq/c; q)_\infty (q; q)_\infty (abq^2; q)_\infty} \]
and
\[ h_n = \frac{(1 - abq^{n+1})(abq/c)_n (aq; q)_n (cq; q)_n (-ac)^{-n}q^{-n(n+3)/2}}{(1 - abq)(q; q)_n (aq/c; q)_n (bg; q)_n}. \]

Here $(a; q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n), (a; q)_n = (a; q)_\infty / (aq^n; q)_\infty$. The integral on the left-hand side of (19) is understood as a $q$-integral, see [6]:
\[ \int_{a}^{b} f(t)d_q t = \int_{0}^{b} f(t)d_q t - \int_{a}^{0} f(t)d_q t, \quad \int_{0}^{a} f(t)d_q t = a(1 - q) \sum_{n=0}^{\infty} f(aq^n)q^n. \]
Let us express \( \mu(y) \), \( h_\infty \), and \( h_n \) through the parameters \( a, b, c, \) and \( y \) from (17). Taking into account that \( (aq^{-n}; q)_n = (-a)^n q^{-n(n+1)/2}(qa^{-1}; q)_n \), we have
\[
\mu(y) = (yq^{-s-m}; q)_{s+m} (-yq^{-s}; q)_s / (-yq^{-m}; q)_m.
\]

With formulas
\[
-q^{n+1}; q)_\infty / (-q^{n+1}; q)_\infty = q^{-\alpha} [n]! [2m]! / [m]! [2n]!!, \quad (-q^{n+1}; q)_\infty / (-q^{-m}; q)_\infty = q^{-\alpha} [n]! [m]! / [2m]! [2n]!!
\]
and
\[
(-q; q)_n = q^{n/2} [2n] (-1; q)_n / [2n] = q^{n(n+1)/4} [2n]! / [n]!.
\]

where \( \alpha = (n - m)(n + m + 1)/4 \), \( \mu(q) = q^{n(n-1)/4}(1 - q)^n [n]! \), \( \mu^2 = q^{n^2/2}(1 - q)^n [2n]!! \), [n]! = [n][n-2] \cdots [1] or [2], the expression for \( h_\infty \) can be transformed into the following:
\[
h_\infty = \frac{1}{2} q^{-(s+m+2)/2} \left( \frac{[s]!}{[2s]!!} \right)^2 \frac{[2s+1]!}{[s-m]![s+m]!}.
\]

For \( a = b = q^{-s}, c = q^{s+m} \), the quantity \( h_\infty \) is negative.

The expression for \( h_n \) coincides with that for \( N_n \) in (18), as it should be. This can be easily verified with the help of
\[
(q^{m+1}; q)_n = q^{n(n+2m-1)/4}(1 - q)^n [m+n]! / [m]!.
\]

(21)

The facts mentioned above imply that the polynomials \( P_n(x) \) in (18) satisfy the orthogonality relation
\[
\int_{-q^{s+1}}^{q^{s+1}} P_n(x) P_m(x) w(y) d_q(y) = \delta_{mn},
\]

where
\[
w(y) = \frac{1}{2} q^{-(s+m+2)/2} \left( \frac{[s]!}{[2s]!!} \right)^2 \frac{[2s+1]!}{[s-m]![s+m]!} \frac{(yq^{-s}; q)(yq^{-s-m}; q)_{s+m}}{(-yq^{-m}; q)_m},
\]

\( y = xq^{(s+m+1)/2} / \rho \). In more explicit form,
\[
\sum_{k=0}^{\infty} P_n(z_k) P_m(z_k) W(r_k) + \sum_{k=0}^{\infty} P_n(z'_k) P_m(z'_k) W'(r_k) = \delta_{mn},
\]

(23)

where \( z_k = \rho q^{k+(s+m+1)/2}, z'_k = -q^{-m} z_k, r_k = q^{k+s+m+1} \) and \( W(r_k) = (1 - q)r_k w(r_k), W'(r_k) = -W(-q^{-m} r_k) \). Straightforward calculation with (20) and
\[
(-q^{m+1}; q)_n = q^{n(n+2m+1)/4} [m]! [2m+2n]! / [m+n]! [2m]!!
\]

leads to
\[
W(r_k) = \frac{1}{2} q^{s(s+m+1)/2+k(s+1)} (1 - q)^{s+m+1} \left( \frac{[s]!}{[2s]!!} \right)^2 \frac{[2s+1]!}{[s-m]![s+m]!} \frac{[k+s+m]! [k+m]! [2k+2s]!!}{[k]![k+m]! [2k+2m]!!},
\]
and $W'(r_k) = W(r_k)m \rightarrow m$.

Formula (22) demonstrates that the spectrum of the operator $L_\rho$ on the subspace $V_{sm}$ is a discrete set of points $-\rho q^k(s-m+1)/2$ and $\rho q^{k+(s+m+1)/2}$, $k = 0, 1, 2, \ldots$. Since $0 < q < 1$, the accumulation point of the spectrum is zero. Joining spectra for all subspaces $V_{sm}$, we obtain the spectrum of the operator $T_\rho s(I_3)$ on $V_s$.

6. Case $q = 1$

Let $|x, m \rangle$ at $q = 1$ be an eigenvector of the operator $L_\rho$ with an eigenvalue $x$, $L_\rho |x, m \rangle = x |x, m \rangle$ and

$$|x, m \rangle = \sum_{l=s}^\infty \tilde{P}_{-s}(x)|x, m \rangle.$$  \hspace{1cm} (24)

The formula of action of the operator $L_\rho$ upon the basis vectors $(-i)^{-l}|l, m \rangle$ at $q = 1$ is obtained from (12) by the substitution $[r] \rightarrow r$ for any $c$-number $r$. Then, repeating the procedure of the preceding section, we find that the functions $\tilde{P}_n(x)$ in (23) take the form

$$\tilde{P}_n(x) = \left( \frac{(s-m)!(s+m)!n!(n+2s)!(2n+2s+1)}{(2s+1)!(n+s-m)!(n+s+m)!} \right)^{1/2} P_n^{s+m,s-m}(x/\rho),$$  \hspace{1cm} (25)

where $P_n^{(\alpha,\beta)}(x)$ is an ordinary Jacobi polynomial. With use of the orthogonality relation for these polynomials (see, e.g., [12]), we obtain

$$\int_{-1}^1 \tilde{P}_n(x)\tilde{P}_m(x)\tilde{w}(y)dy = \delta_{mn}.$$  \hspace{1cm} (26)

Here $y = x/\rho$ and

$$\tilde{w}(y) = 2^{-2(2s+1)} \frac{(2s+1)!}{(s-m)!(s+m)!}(1-y)^{s+m}(1+y)^{s-m}.$$

We see that the spectrum of the operator $L_\rho$ on the subspace $V_{sm}$ is continuous at $q = 1$ and consists of points in the interval $[-\rho, \rho]$.

Remark that big $q$-Jacobi polynomials have the property

$$\lim_{q \rightarrow 1} P_n(x; q^\alpha, q^\beta, -q^\gamma|q) = P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(1).$$

In this limit, $P_n(x) \rightarrow \tilde{P}_n(x)$ (see formulas (18) and (24)) and $\tilde{w}(y) \rightarrow \tilde{w}(y)$ (see (21) and (25)). If, in additions, to use the property

$$\lim_{q \rightarrow 1} \int_0^a f(t) d_q(t) = \int_0^a f(t) d(t),$$

where $f$ is a continuous function on $[0, a]$, one easily verifies that the orthogonality relation (21) for the polynomials $P_n(x)$ at $q \rightarrow 1$ goes to the orthogonality relation (25) for the polynomials $\tilde{P}_n(x)$. 

References