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The Cohomology of the Variational Bicomplex Invariant under the Symmetry Algebra of the Potential Kadomtsev-Petviashvili Equation

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Abstract

The variational bicomplex of forms invariant under the symmetry algebra of the potential Kadomtsev-Petviashvili equation is described and the cohomology of the associated Euler-Lagrange complex is computed. The results are applied to a characterization problem of the Kadomtsev-Petviashvili equation by its symmetry algebra originally posed by David, Levi, and Winternitz.

§1. Introduction. Let $E = (t, x, y, u) \to (t, x, y)$ be the bundle of the independent and dependent variables for the potential Kadomtsev-Petviashvili (PKP) equation

$$u_{tx} + \frac{3}{2}u_x u_{xx} + u_{xxxx} + \frac{3}{4}s^2 u_{yy} = 0,$$

where $s^2 = \pm 1$. In this paper, we compute the cohomology of the variational bicomplex and the associated Euler-Lagrange complex on E invariant under the Lie algebra Γ_{PKP} of point symmetries of the PKP equation. The algebra Γ_{PKP} is an infinite-dimensional infinitesimal Lie pseudo-group spanned by the vector fields

$$\begin{aligned}
X_f &= f\,\frac{\partial}{\partial t} + \frac{2}{3}yf'\frac{\partial}{\partial y} + \left(\frac{1}{3}xf' - \frac{2}{9}s^2y^2f''\right)\frac{\partial}{\partial x} + \\
&\qquad \left(-\frac{1}{3}uf' + \frac{1}{9}x^2f'' - \frac{4}{27}s^2xy^2f''' + \frac{4}{243}y^4f''''\right)\frac{\partial}{\partial u}, \\
Y_g &= g\,\frac{\partial}{\partial y} - \frac{2}{3}s^2yg'\frac{\partial}{\partial x} + \left(-\frac{4}{9}s^2xyg'' + \frac{8}{81}y^3g'''\right)\frac{\partial}{\partial u}, \\
Z_h &= h\,\frac{\partial}{\partial x} + \left(\frac{2}{3}xh' - \frac{4}{9}s^2y^2h''\right)\frac{\partial}{\partial u}, \\
W_k &= yk\,\frac{\partial}{\partial u}, \qquad \text{and} \qquad U_l = l\,\frac{\partial}{\partial u}\,,
\end{aligned} \tag{1.1}$$

where $f = f(t)$, $g = g(t)$, $h = h(t)$, $k = k(t)$ and $l = l(t)$ are arbitrary smooth functions of t. See [5].

Our work is motivated by the following two considerations.

Firstly, the cohomology of variational bicomplexes and the associated Euler-Lagrange complexes invariant under some transformation group naturally arise in a host of problems in differential geometry and mathematical physics; see, for example, [1], [12]. It is therefore important to develop effective methods for computing the cohomology spaces of these complexes and to find new and useful applications for the cohomology classes.

In [2], Anderson and Pohjanpelto find that, under some mild regularity conditions the local cohomology of the Euler-Lagrange complex invariant under a finite-dimensional transformation group Γ can be computed from the Lie algebra cohomology of Γ. In [11], it is discovered that this relationship continues to hold in a variety of examples involving infinite-dimensional transformation groups Γ provided that the Lie algebra cohomology is replaced by the Gelfand-Fuks cohomology of Γ relative to some subalgebra of Γ. Accordingly, the local cohomology of the Γ_{PKP} invariant Euler-Lagrange complex is isomorphic with the Gelfand-Fuks cohomology of Γ_{PKP}; see [3]. In contrast, as the results of this paper show, the Euler-Lagrange complex of Γ_{PKP} invariant everywhere smooth forms provides an example where this simple correspondence fails to hold. In addition, it is presently not clear what the precise relationship, if any, between the two cohomology spaces is in this case.

Secondly, the symmetry algebras of several other integrable equations in $2+1$ dimensions possess a Kac-Moody-Virasoro structure similar to that of Γ_{PKP}; see, for example, [4], [7], [8], [9]. With these examples at hand, it is reasonable to speculate that the symmetry algebras of integrable equations in 2+1 dimensions characterize the equations in some fashion. In fact, the formulation of virtually all physical models is based on symmetry considerations. In [6], David, Levi, and Winternitz study the characterization problem for the KP equation

$$u_{tx} + \frac{3}{2}uu_{xx} + \frac{3}{2}u_x^2 + \frac{1}{4}u_{xxxx} + \frac{3}{4}s^2u_{yy} = 0$$

in terms of the differential invariants of its symmetry algebra Γ_{KP} – specifically, they classify all fourth-order equations involving only a first or second-order time derivative that are invariant under Γ_{KP}. However, David, Levi, and Winternitz find that the algebra Γ_{KP} admits 10 functionally independent differential invariants of the specified type rendering the sought after characterization unfeasible.

One possible way to shorten the list of equations is to repeat the analysis in [6] for the potential Kadomtsev-Petviashvili equation and only consider those differential invariants of the symmetry algebra Γ_{PKP} that are the Euler-Lagrange expressions of some Lagrangian on E. Surprisingly, the source form

$$\Delta_{PKP} = \left(u_{tx} + \frac{3}{2}u_xu_{xx} + u_{xxxx} + \frac{3}{4}s^2u_{yy}\right) dt \wedge dx \wedge dy \wedge du$$

associated with the PKP equation is invariant under Γ_{PKP}, that is, the algebra Γ_{PKP} consists solely of distinguished symmetries of the PKP equation. However, even though the source form Δ_{PKP} is the Euler-Lagrange expression of the Lagrangian

$$\lambda = \left(-\frac{1}{2}u_tu_x - \frac{1}{4}u_x^3 + \frac{1}{8}u_{xx}^2 - \frac{3}{8}s^2u_y^2\right) dt \wedge dx \wedge dy,$$

one can prove without too much trouble that it is not the Euler-Lagrange expression of any Γ_{PKP} invariant Lagrangian. Thus, the source form Δ_{PKP} generates nontrivial cohomology in the Γ_{PKP} invariant Euler-Lagrange complex. The results of this paper show that the PKP equation is essentially uniquely characterized by this property, that is, as an obstruction to constructing Γ_{PKP} invariant Lagrangians for Γ_{PKP} invariant differential equations satisfying the Helmholtz conditions.

Our main result is the following.

Theorem 1.1

i) The interior rows $(\Omega^{*,s}_{PKP}(J^\infty(E)), d_H)$, $s \geq 1$, of the augmented Γ_{PKP} invariant variational bicomplex are exact.

ii) The dimensions of the vertical cohomology spaces $H^{r,s}(\Omega_{PKP}(J^\infty(E)), d_V)$ of the Γ_{PKP} invariant variational bicomplex are

$s \geq 4$	0	0	0	0
$s = 3$	0	0	0	2
$s = 2$	0	1	4	1
$s = 1$	0	2	1	0
$s = 0$	1	0	0	0
	$r = 0$	$r = 1$	$r = 2$	$r = 3$

iii) The dimensions of the cohomology spaces $H^r(\mathcal{E}_{\mathcal{PKP}}(\mathcal{J}^\infty(\mathcal{E})))$ of the Γ_{PKP} invariant Euler-Lagrange complex are

	$r = 1$	2	3	4	5	6	$r \geq 7$
$\dim H^r(\mathcal{E}_{\mathcal{PKP}})$	0	1	0	2	0	2	0

We will obtain explicit generators for all the cohomology classes in the course of the proof of Theorem 1.1.

Let $\Delta_{u_{xxxx}}$ be the source form

$$\Delta_{u_{xxxx}} = u_{xxxx} dt \wedge dx \wedge dy \wedge du.$$

As a Corollary, Theorem 1.1 yields the following characterization of the PKP equation.

Corollary 1.2. *Let Δ be a Γ_{PKP} invariant source form on E satisfying the Helmholtz conditions. Then there are constants c_1, c_2 and a Γ_{PKP} invariant Lagrangian form λ such that*

$$\Delta = c_1 \Delta_{PKP} + c_2 \Delta_{u_{xxxx}} + \mathrm{E}(\lambda).$$

The rest of the paper is dedicated to the proof of Theorem 1.1. In section 2, we review some basic definitions and results from the theory of variational bicomplexes and transformation groups needed in the proof. In section 3, we first describe a basis for the

Γ_{PKP} invariant variational bicomplex. Then with this description at hand, it is an easy task to complete the proof of Theorem 1.1.

§2. Preliminaries. In this section, we collect together some definitions and results from calculus of variations needed in the sequel. For more details and proofs, we refer to [1], [10], [12], [13].

Let $\pi\ E = (t, x, y, u) \to (t, x, y)$ be the bundle of the independent and dependent variables for the PKP equation. Often we will also write $x^1 = t$, $x^2 = x$, $x^3 = y$. Let $J^\infty(E)$ be the infinite jet bundle of local sections of E with coordinates $(t, x, y, u, u_t, u_x, \ldots)$. Let $\Omega^*(J^\infty(E))$ be the de Rham complex on $J^\infty(E)$, and let

$$\Omega^p(J^\infty(E)) = \bigoplus_{p=r+s} \Omega^{r,s}(J^\infty(E))$$

be the decomposition of the de Rham complex into forms of horizontal degree r and vertical degree s. A type (r, s) form $\omega \in \Omega^{r,s}(J^\infty(E))$ is a sum of terms

$$f(x^i, u^{[k]}) dx^{i_1} \wedge \cdots \wedge dx^{i_r} \wedge \theta_{x^{j_{11}} \cdots x^{j_{1t_1}}} \wedge \cdots \wedge \theta_{x^{j_{s1}} \cdots x^{j_{st_s}}},$$

where $f(x^i, u^{[k]})$ is a smooth function on $J^\infty(E)$ and the forms

$$\theta_{x^{k_1} \cdots x^{k_v}} = du_{x^{k_1} \cdots x^{k_v}} - u_{x^{k_1} \cdots x^{k_v} x^l} dx^l$$

are the contact one forms on $J^\infty(E)$.

We write

$$d = d_H + d_V$$

for the induced decomposition of the exterior derivative into the horizontal and vertical derivatives.

Let

$$\mathrm{I}\ : \Omega^{3,s}(J^\infty(E)) \to \Omega^{3,s}(J^\infty(E)), \qquad s \geq 1,$$

be the interior Euler operator given by

$$\mathrm{I}(\omega) = \frac{1}{s}\, \theta \wedge \left[\left(\frac{\partial}{\partial u} \lrcorner\, \omega \right) - \mathcal{L}_{D_i} \left(\frac{\partial}{\partial u_{x^i}} \lrcorner\, \omega \right) + \mathcal{L}_{D_i} \mathcal{L}_{D_j} \left(\frac{\partial}{\partial u_{x^i x^j}} \lrcorner\, \omega \right) - \cdots \right]. \quad (2.1)$$

Here $\mathcal{L}_{D_i}$ stands for the Lie derivative with respect to the total vector field

$$D_i = \frac{\partial}{\partial x^i} + u_{x^i} \frac{\partial}{\partial u} + u_{x^i x^j} \frac{\partial}{\partial u_{x^j}} + \cdots.$$

Recall that

$$\mathrm{I}\ \circ d_H = 0. \quad (2.2)$$

The spaces $\mathcal{F}^s(J^\infty(E))$, $s \geq 1$, of functional s forms are, by definition, the images of the spaces $\Omega^{3,s}(J^\infty(E))$ under the mapping I. The vertical derivative d_V induces differentials

$$\mathrm{E}\, \Omega^{3,0}(J^\infty(E)) \to \mathcal{F}^1(J^\infty(E)) \quad \text{and} \quad \delta_V\, \mathcal{F}^s(J^\infty(E)) \to \mathcal{F}^{s+1}(J^\infty(E)), \quad s \geq 1,$$

by $\mathrm{E}\ =\ \mathrm{I} \circ d_V$ and $\delta_V\ =\ \mathrm{I} \circ d_V$.

The elements of $\Omega^{3,0}(J^\infty(E))$ can be regarded as Lagrangians for variational problems on E and source forms, the elements of $\mathcal{F}^1(J^\infty(E))$, as partial differential equations on E. The mapping E $\Omega^{3,0}(J^\infty(E)) \to \mathcal{F}^1(J^\infty(E))$ is just a usual Euler-Lagrange operator while the differential $\delta_V\, \mathcal{F}^1(J^\infty(E)) \to \mathcal{F}^2(J^\infty(E))$ agrees with the Helmholtz operator of the inverse problem of calculus of variations.

The infinitesimal transformation group Γ_{PKP} acts projectably on E; consequently, its prolongation $\mathrm{pr}\Gamma_{PKP}$ preserves the spaces $\Omega^{r,s}(J^\infty(E))$ and commutes with the operators d_H, d_V, and I. Recall that the prolongation of a vector field

$$X = \xi^i(x^j)\frac{\partial}{\partial x^i} + \phi(x^j, u)\frac{\partial}{\partial u}$$

on E is given by

$$\mathrm{pr}X = \mathrm{tot}X + \mathrm{pr}X_{\mathrm{ev}}, \tag{2.3}$$

where

$$\mathrm{tot}X = \xi^i D_i \qquad \text{and} \qquad \mathrm{pr}X_{\mathrm{ev}} = \sum_{p\geq 0} \mathrm{D}_{i_1}\cdots D_{i_p}(\phi_{\mathrm{ev}})\frac{\partial}{\partial u_{x^{i_1}\cdots x^{i_p}}},$$

and where

$$\phi_{\mathrm{ev}} = \phi - \xi^i u_i.$$

Also recall the commutation formula

$$[D_i,\ \mathrm{pr}Y] = 0, \qquad i = 1, 2, 3, \tag{2.4}$$

where $Y = \phi(x^j, u)\,\partial/\partial u$ is any evolutionary vector field.

Let $\Omega^{r,s}_{PKP}(J^\infty(E))$ consist of all Γ_{PKP} invariant type (r, s) forms on $J^\infty(E)$, that is,

$$\Omega^{r,s}_{PKP}(J^\infty(E)) = \{\, \omega \in \Omega^{r,s}(J^\infty(E)) \,|\, \mathcal{L}\,\mathrm{pr}_X\omega = 0 \text{ for all } \ X \in \Gamma_{PKP}\,\}.$$

We define the spaces $\mathcal{F}^s_{PKP}(J^\infty(E))$ of Γ_{PKP} invariant functional s forms in a similar fashion. We thus obtain the augmented Γ_{PKP} invariant variational bicomplex $(\Omega^{r,s}_{PKP}(J^\infty(E)), d_H, d_V)$

$$\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & \uparrow d_V & & \uparrow d_V & & \uparrow d_V & & \uparrow d_V & & \uparrow \delta_V \\
 & 0 & \longrightarrow & \Omega^{0,2}_{PKP} & \xrightarrow{d_H} & \Omega^{1,2}_{PKP} & \xrightarrow{d_H} & \Omega^{2,2}_{PKP} & \xrightarrow{d_H} & \Omega^{3,2}_{PKP} & \xrightarrow{I} \mathcal{F}^2_{PKP} \longrightarrow 0 \\
 & & \uparrow d_V & & \uparrow d_V & & \uparrow d_V & & \uparrow d_V & & \uparrow \delta_V \\
 & 0 & \longrightarrow & \Omega^{0,1}_{PKP} & \xrightarrow{d_H} & \Omega^{1,1}_{PKP} & \xrightarrow{d_H} & \Omega^{2,1}_{PKP} & \xrightarrow{d_H} & \Omega^{3,1}_{PKP} & \xrightarrow{I} \mathcal{F}^1_{PKP} \longrightarrow 0 \\
 & & \uparrow d_V & & \uparrow d_V & & \uparrow d_V & & \uparrow d_V & & \\
0 & \longrightarrow & \mathbf{R} & \longrightarrow & \Omega^{0,0}_{PKP} & \xrightarrow{d_H} & \Omega^{1,0}_{PKP} & \xrightarrow{d_H} & \Omega^{2,0}_{PKP} & \xrightarrow{d_H} & \Omega^{3,0}_{PKP}
\end{array}$$

and the Γ_{PKP} invariant Euler-Lagrange complex $\mathcal{E}^*_{PKP}(J^\infty(E))$

$$0 \longrightarrow \mathbf{R} \longrightarrow \Omega^{0,0}_{PKP} \xrightarrow{d_H} \Omega^{1,0}_{PKP} \xrightarrow{d_H}$$
$$\xrightarrow{d_H} \Omega^{2,0}_{PKP} \xrightarrow{d_H} \Omega^{3,0}_{PKP} \xrightarrow{E} \mathcal{F}^1_{PKP} \xrightarrow{\delta_V} \mathcal{F}^2_{PKP} \xrightarrow{\delta_V} \mathcal{F}^3_{PKP} \xrightarrow{\delta_V} \cdots.$$

The vertical cohomology spaces $H^{r,s}(\Omega_{PKP}(J^\infty(E)), d_V)$, $r, s \geq 0$, of the Γ_{PKP} invariant variational bicomplex are, by definition, the quotient spaces

$$H^{r,s}(\Omega_{PKP}(J^\infty(E)), d_V) = \frac{\text{Ker } \{d_V\, \Omega^{r,s}_{PKP}(J^\infty(E)) \to \Omega^{r,s+1}_{PKP}(J^\infty(E))\}}{\text{Im } \{d_V\, \Omega^{r,s-1}_{PKP}(J^\infty(E)) \to \Omega^{r,s}_{PKP}(J^\infty(E))\}}.$$

Here $\Omega^{0,-1}_{PKP} = \{0\}$. The horizontal cohomology spaces $H^{r,s}(\Omega_{PKP}(J^\infty(E)), d_H)$ of the Γ_{PKP} invariant variational bicomplex and the cohomology spaces $H^r(\mathcal{E}_{PKP}(J^\infty(E)))$ of the Γ_{PKP} invariant Euler-Lagrange complex are defined similarly. Note, in particular, that the elements of $H^4(\mathcal{E}_{PKP}(J^\infty(E)))$ characterize those Γ_{PKP} invariant source forms on E which satisfy the Helmholtz conditions and consequently are the Euler-Lagrange expression of some Lagrangian form on E but which are not the Euler-Lagrange expression of any Γ_{PKP} invariant Lagrangian form on E.

§3. Proof of Theorem 1.1 As is easily checked by a direct computation, generators (1.1) of the Lie algebra Γ_{PKP} satisfy the bracket relations

	$X_{\hat f}$	$Y_{\hat g}$	$Z_{\hat h}$	$W_{\hat k}$	$U_{\hat l}$
X_f	$X_{(f\hat f' - f'\hat f)}$	$Y_{(f\hat g' - \frac{2}{3}f'\hat g)}$	$Z_{(f\hat h' - \frac{1}{3}f'\hat h)}$	$W_{(f\hat k' + f'\hat k)}$	$U_{(f\hat l' + \frac{1}{3}f'\hat l)}$
Y_g		$-\frac{2}{3}s^2 Z_{(g\hat g' - g'\hat g)}$	$-\frac{4}{9}s^2 W_{(2g\hat h'' + g'\hat h' - g''\hat h)}$	$U_{g\hat k}$	0
Z_h			$\frac{2}{3}U_{(h\hat h' - h'\hat h)}$	0	0
W_k				0	0
U_l					0

(3.1)

Let $\mathcal{H} \subset \Gamma_{PKP}$ be the subalgebra generated by the vector fields

$$\mathcal{H} = \{X_f \,|\, f = f(t) \text{ is a smooth function}\}.$$

Note that, by virtue of the bracket relations (3.1), a form $\omega \in \Omega^{r,s}(J^\infty(E))$ that is invariant under the algebra $\mathcal{H}$ and under the single vector field $Y_1 = \partial/\partial y$ is necessarily invariant under the full algebra Γ_{PKP}.

Next let R_t, R_x, and R_y be the total differential operators

$$\mathrm{R}_t = D_t + \frac{3}{2}u_x D_x, \quad \mathrm{R}_x = D_x, \qquad \mathrm{R}_y = D_y, \tag{3.2}$$

and let S stand for the operator

$$\mathrm{S} = \mathrm{R}_t\mathrm{R}_x + \frac{3}{4}s^2\mathrm{R}_y\mathrm{R}_y. \tag{3.3}$$

Proposition 3.1. *The total differential operators* R_t, R_x, *and* R_y *and the operator* S *satisfy*

$$\begin{aligned} &[\,\mathrm{pr}X_f, \mathrm{R}_t] = -f'(t)\mathrm{R}_t - \frac{2}{3}yf''(t)\mathrm{R}_y, && [\,\mathrm{pr}X_f, \mathrm{R}_x] = -\frac{1}{3}f'(t)\mathrm{R}_x, \\ &[\,\mathrm{pr}X_f, \mathrm{R}_y] = -\frac{2}{3}f'(t)\mathrm{R}_y + \frac{4}{9}s^2yf''(t)\mathrm{R}_x, && [\,\mathrm{pr}X_f, \mathrm{S}] = -\frac{4}{3}f'(t)\mathrm{S} \end{aligned} \tag{3.4}$$

for all smooth $f = f(t)$.

Proof. By (3.2), we have that

$$[\,\mathrm{pr}X_f, \mathrm{R}_t] = \frac{3}{2}\,\mathrm{pr}X_f(u_x)D_x + [\,\mathrm{pr}X_f, D_t] + \frac{3}{2}u_x[\,\mathrm{pr}X_f, D_x]. \tag{3.5}$$

We compute

$$[\,\mathrm{pr}X_f, D_t] = [\,\mathrm{tot}X_f, D_t] = -f'(t)D_t - \frac{2}{3}yf''(t)D_y - \left(\frac{1}{3}xf''(t) - \frac{2}{9}s^2y^2f'''(t)\right)D_x, \tag{3.6}$$

where we used the prolongation formula (2.3) and the commutation formula (2.4). Also by (1.1) and (2.3),

$$\mathrm{pr}X_f(u_x) = -\frac{2}{3}f'(t)u_x + \frac{2}{9}xf''(t) - \frac{4}{27}s^2y^2f'''(t).$$

Hence,

$$\frac{3}{2}\,\mathrm{pr}X_f(u_x)D_x + [\,\mathrm{pr}X_f, D_t] = -f'(t)(D_t + u_xD_x) - \frac{2}{3}yf''(t)D_y. \tag{3.7}$$

We also have that

$$[\,\mathrm{pr}X_f, D_x] = -\frac{1}{3}f'(t)D_x. \tag{3.8}$$

Now substitute (3.7), (3.8) into (3.5) and simplify to arrive at the first equation in (3.4). The proof of the remaining identities in (3.4) is based on a similar computation and will be omitted. ∎

We let

$$\begin{aligned} &v_{tx} = \mathrm{S}u, && v_{txx} = \mathrm{R}_x v_{tx}, \\ &v_{ty} = \mathrm{R}_t\mathrm{R}_y u, && v_{x^ky^l} = u_{x^ky^l}, \qquad k, l \geq 0. \end{aligned} \tag{3.9}$$

Let $\{\sigma^t, \sigma^x, \sigma^y\}$ be the horizontal frame dual to the total differential operators $\{\mathrm{R}_t, \mathrm{R}_x, \mathrm{R}_y\}$,

$$\sigma^t = dt, \qquad \sigma^x = dx - \frac{3}{2}v_x\,dt, \qquad \sigma^y = dy, \tag{3.10}$$

and define new contact 1 forms by

$$\begin{aligned} &\Theta_t = \mathcal{L}_{\mathrm{R}_t}\theta + \frac{3}{2}v_{xx}\theta, && \Theta_{tx^l} = d_V v_{tx^l}, \quad l = 1, 2, \\ &\Theta_{ty} = d_V v_{ty} + \frac{9}{2}v_{xx}\theta_y, && \Theta_{x^lyy} = \theta_{x^lyy} - 2s^2v_{xx}\theta_{x^{l+1}}, \quad l \geq 0, \end{aligned} \tag{3.11}$$

and

$$\Theta_{x^l y^m} = \theta_{x^l y^m}, \qquad l, m \geq 0;\ m \neq 2.$$

By (1.1) and (2.3), we have that

$$\mathrm{pr}X_f(v_{xx}) = -f'(t)v_{xx} + \frac{2}{9}f''(t). \tag{3.12}$$

Now a straightforward but tedious computation employing the commutation relations (3.4) establishes the following Lie derivative formulas:

$$\begin{aligned} &\mathrm{pr}X_f v_{tx^l} = -\frac{l+4}{3}f'(t)v_{tx^l}, \qquad l = 1, 2, \\ &\mathrm{pr}X_f v_{x^l y^m} = -\frac{l+2m+1}{3}f'(t)v_{x^l y^m} + \frac{4m}{9}s^2 y f''(t)v_{x^{l+1}y^{m-1}}, \\ &m = 0, 1; \qquad l + m \geq 3. \end{aligned} \tag{3.13}$$

Furthermore,

$$\begin{aligned} &\mathcal{L}_{X_f}\sigma^t = f'(t)\sigma^t, \qquad \mathcal{L}_{X_f}\sigma^x = \frac{1}{3}f'(t)\sigma^x - \frac{4}{9}s^2 y f''(t)\sigma^y, \\ &\mathcal{L}_{X_f}\sigma^y = \frac{2}{3}f'(t)\sigma^y + \frac{2}{3}y f''(t)\sigma^t, \end{aligned} \tag{3.14}$$

and, on account of (3.12),

$$\begin{aligned} &\mathcal{L}_{X_f}\Theta_t = -\frac{4}{3}f'(t)\Theta_t - \frac{2}{3}y f''(t)\Theta_y, \quad \mathcal{L}_{X_f}\Theta_{tx^l} = -\frac{l+4}{3}f'(t)\Theta_{tx^l}, l = 1, 2, \\ &\mathcal{L}_{X_f}\Theta_{ty} = -2f'(t)\Theta_{ty} + y f''(t)\left(\frac{4}{9}s^2\Theta_{tx} - \Theta_{yy}\right), \\ &\mathcal{L}_{X_f}\Theta_{x^l y^m} = -\frac{l+2m+1}{3}f'(t)\Theta_{x^l y^m} + \frac{4m}{9}s^2 y f''(t)\Theta_{x^{l+1}y^{m-1}}, \\ &l \geq 0,\ m = 0, 1, 2. \end{aligned} \tag{3.15}$$

Note that all the Lie derivative formulas above only involve the terms $f'(t)$ and $yf''(t)$ and no other terms in $f(t)$ or its derivatives.

We say that a form $\omega \in \Omega^{r,s}(J^\infty(E))$ has a weight w if

$$\mathcal{L}_{X_{3t}}\omega = w \cdot \omega.$$

In this case, write

$$W(\omega) = w.$$

For example, the scalar form $\omega_1 = u_{tx}$ has the weight $W(\omega_1) = -5$ while the type $(1, 1)$ form $\omega_2 = dx \wedge \Theta_t$ has the weight $W(\omega_2) = -3$.

Lemma 3.2. *Let $\omega \in \Omega^{r,s}_{PKP}(J^\infty(E))$ be Γ_{PKP} invariant. Then ω can be expressed in terms of exterior products of the forms σ^t, σ^x, σ^y and $\Theta_{t^k x^l y^m}$ $(3k + l + 2m \leq 5)$ with coefficients which are polynomial in the variables v_{tx^l} $(l = 1, 2)$, v_{x^l} $(l = 3, 4, 5)$, and $v_{x^l y}$ $(l = 2, 3)$.*

Proof. Let $\omega \in \Omega^{r,s}_{PKP}(J^\infty(E))$ be Γ_{PKP} invariant. Our first goal is to show that the components $\omega^{I_1\cdots I_s}_{i_1\cdots i_r}$ of ω in the basis $\{dt, dx, dy, \theta, \theta_t, \ldots\}$ are polynomials in the variables $u_{t^k x^l y^m}$. For this, let $\Phi_\epsilon\, J^\infty(E) \to J^\infty(E)$,

$$\Phi_\epsilon(t, x, y, u_{t^k x^l y^m}) = \left(e^{3\epsilon}t, e^\epsilon x, e^{2\epsilon}y, e^{-(3k+l+2m+1)\epsilon}u_{t^k x^l y^m}\right), \tag{3.16}$$

be the one-parameter group of transformations generated by the vector field $\mathrm{pr}X_{3t}$. Since the action of Φ_ϵ leaves ω invariant, it follows from (3.16) that the components $\omega^{I_1\cdots I_s}_{i_1\cdots i_r}$ satisfy

$$\omega^{I_1\cdots I_s}_{i_1\cdots i_r} = e^{p\epsilon}\omega^{I_1\cdots I_s}_{i_1\cdots i_r} \circ \Phi_\epsilon, \tag{3.17}$$

for some $p \in \mathbf{R}$.

Let k, l, m be some nonnegative integers, and apply a q-fold derivative with respect to the variable $u_{t^k x^l y^m}$ to equation (3.17) to obtain

$$\frac{\partial^q \omega^{I_1\cdots I_s}_{i_1\cdots i_r}}{\partial u_{t^k x^l y^m}{}^q} = e^{[p-q(3k+l+2m+1)]\epsilon}\frac{\partial^q \omega^{I_1\cdots I_s}_{i_1\cdots i_r}}{\partial u_{t^k x^l y^m}{}^q} \circ \Phi_\epsilon. \tag{3.18}$$

Suppose that $q > p$, and let $\epsilon \to \infty$ in (3.18). Recall that by the Γ_{PKP} invariance of ω the component functions $\omega^{I_1\cdots I_s}_{i_1\cdots i_r}$ do not involve the independent variables t, x, y. Thus, we see that

$$\frac{\partial^q \omega^{I_1\cdots I_s}_{i_1\cdots i_r}}{\partial u_{t^k x^l y^m}{}^q} = 0.$$

Hence, any q-fold, $q > p$, derivative with respect to the variable $u_{t^k x^l y^m}$ vanishes identically. Since k, l, m are arbitrary nonnegative integers, it follows that the $\omega^{I_1\cdots I_s}_{i_1\cdots i_r}$ must be polynomial in the derivative variables $u_{t^k x^l y^m}$.

Next write

$$\omega = \omega_{i_1\cdots i_r} dx^{i_1} \wedge \cdots \wedge dx^{i_r},$$

where $\omega_{i_1\cdots i_r} \in \Omega^{0,s}(J^\infty(E))$. Each monomial $dx^{i_1} \wedge \cdots \wedge dx^{i_r}$ has a weight

$$0 \le W(dx^{i_1} \wedge \cdots \wedge dx^{i_r}) \le 6.$$

Hence, we have that

$$-6 \le W(\omega_{i_1\cdots i_r}) \le 0 \qquad \text{for all } \ i_1, \ldots, i_r.$$

By (3.16), the weight of the derivative variable $u_{t^k x^l y^m}$ is

$$W(u_{t^k x^l y^m}) = -3k - l - 2m - 1.$$

Note, moreover, that the invariance of the form ω under the vector fields $\mathrm{pr}W_k$ and $\mathrm{pr}U_l$ simply means that the components of ω do not depend on the variables u_{t^k} or $u_{t^k y}$, where $k \ge 0$. Therefore, by (3.9)–(3.11), we are able to express ω as a combination of the exterior products of the adapted horizontal forms σ^t, σ^x, σ^y, the adapted contact forms $\Theta_{t^k x^l y^m}$ $(3k + l + 2m \le 5)$, the variables v_{tx^l} $(l = 1, 2)$, the variables $v_{x^{l+1}y^m}$ $(l + 2m \le 4)$, and the variable v_{yy}.

We still need to show that the coefficients $\hat{\omega}^{I_1\cdots I_s}_{i_1\cdots i_r}$ of ω in the adapted frame σ^t, σ^x, σ^y and $\Theta_{t^k x^l y^m}$ do not involve the variables v_x, v_{xx}, v_{xy}, v_{yy}, and v_{xyy}. Write

$$X_f = \sum_{i=0}^{4} f^{(i)}(t) X_i,$$

where X_i, $i = 0, \ldots, 4$, do not involve the function $f(t)$ or its derivatives. By the prolongation formula (2.3),

$$\mathrm{pr}X_f(v_{x^l y^m}) = \sum_{i=0}^{4} f^{(i)}\ \mathrm{pr}X_i(v_{x^l y^m}), \qquad l, m \geq 0.$$

Now

$$\mathrm{pr}X_3 = -\frac{4}{27}s^2\left(xy^2\frac{\partial}{\partial v} + y^2\frac{\partial}{\partial v_x} + 2xy\frac{\partial}{\partial v_y} + 2y\frac{\partial}{\partial v_{xy}} + 2x\frac{\partial}{\partial v_{yy}} + 2\frac{\partial}{\partial v_{xyy}}\right). \quad (3.19)$$

By (3.13)–(3.15), the Lie derivatives of the forms σ^t, σ^x, σ^y, $\Theta_{t^k x^l y^m}$ $(3k+l+2m \leq 5)$, and the variables v_{tx^l} $(l = 1, 2)$ only involve the term $f'(t)$ or the term $yf''(t)$ and no other terms in $f(t)$ or its derivatives. Since $f(t)$ is an arbitrary function, the coefficient of $f'''(t)$ in the expression for $\mathcal{L}_{\mathrm{pr}X_f}\omega$ must vanish. It therefore follows that the components $\hat{\omega}^{I_1\cdots I_s}_{i_1\cdots i_r}$ are invariant under $\mathrm{pr}X_3$. Recall that the $\hat{\omega}^{I_1\cdots I_s}_{i_1\cdots i_r}$ do not involve the variables x, y. Hence, on account of the formula (3.19) for $\mathrm{pr}X_3$, we see that the $\hat{\omega}^{I_1\cdots I_s}_{i_1\cdots i_r}$ can not depend on the variables v_x, v_{xy}, v_{yy}, and v_{xyy}. We can similarly use (3.12) to show that the $\hat{\omega}^{I_1\cdots I_s}_{i_1\cdots i_r}$ can not depend on v_{xx}. This concludes the proof of the Lemma. ∎

We let

$$\begin{aligned}
&\alpha^{1,1}_1 = \sigma^t \wedge \Theta_{xx}, \quad \alpha^{1,1}_2 = \sigma^t \wedge \Theta_y - \frac{2}{3}s^2\sigma^y \wedge \Theta_x,\\
&\alpha^{1,2}_1 = \sigma^t \wedge \Theta \wedge \Theta_x, \quad \alpha^{2,0}_1 = v_{tx}\sigma^t \wedge \sigma^y, \quad \alpha^{2,0}_2 = v_{xxxx}\sigma^t \wedge \sigma^y,\\
&\alpha^{2,0}_3 = v_{xxx}\sigma^t \wedge \sigma^x + v_{xxy}\sigma^t \wedge \sigma^y, \quad \alpha^{2,1}_1 = v_{xxx}\sigma^t \wedge \sigma^y \wedge \Theta,\\
&\alpha^{3,1}_1 = v_{tx}\sigma^t \wedge \sigma^x \wedge \sigma^y \wedge \Theta, \quad \alpha^{3,1}_2 = v_{xxxx}\sigma^t \wedge \sigma^x \wedge \sigma^y \wedge \Theta,\\
&\alpha^{2,2}_1 = \left(\sigma^t \wedge \sigma^y \wedge \Theta_t - \frac{3}{2}s^2\sigma^t \wedge \sigma^x \wedge \Theta_y - \sigma^x \wedge \sigma^y \wedge \Theta_x\right) \wedge \Theta.
\end{aligned} \quad (3.20)$$

One can verify without difficulty that, on account of the Lie derivative formulas (3.13)–(3.15) and the remark following the bracket relations (3.1), the forms $\alpha^{i,j}_k$ above are all Γ_{PKP} invariant.

Proposition 3.3. *The spaces $\Omega^{r,s}_{PKP}(J^\infty(E))$ of Γ_{PKP} invariant type (r, s) forms are spanned by the following forms.*

$s \geq 4$	0	0	0	0
$s = 3$	0	0	0	$\alpha_1^{1,1} \wedge \alpha_1^{2,2}$ $\alpha_2^{1,1} \wedge \alpha_1^{2,2}$
$s = 2$	0	$\alpha_1^{1,2}$	$\alpha_1^{2,2},\ d_H\alpha_1^{1,2},\ a_1^{1,1} \wedge \alpha_2^{1,1}$ $d_V\alpha_1^{2,1},\ \alpha_2^{1,1} \wedge \alpha_2^{1,1}$	$d_H\alpha_1^{2,2},\ d_H(a_1^{1,1} \wedge \alpha_2^{1,1})$ $d_Hd_V\alpha_1^{2,1},\ d_H(\alpha_2^{1,1} \wedge \alpha_2^{1,1})$
$s = 1$	0	$\alpha_1^{1,1}$ $\alpha_2^{1,1}$	$\alpha_1^{2,1},\ d_H\alpha_1^{1,1},\ d_V\alpha_1^{2,0}$ $d_H\alpha_2^{1,1},\ d_V\alpha_2^{2,0}$	$\alpha_1^{3,1},\ \alpha_2^{3,1},\ d_Hd_V\alpha_1^{2,0}$ $d_H\alpha_1^{2,1},\ d_Hd_V\alpha_2^{2,0}$
$s = 0$	1	0	$\alpha_1^{2,0},\ \alpha_2^{2,0},\ \alpha_3^{2,0}$	$d_H\alpha_1^{2,0},\ d_H\alpha_2^{2,0}$
	$r = 0$	$r = 1$	$r = 2$	$r = 3$

Proof. Due to Lemma 3.2 and the Lie derivative formulas (3.13)–(3.15), the proof reduces to routine computations. We illustrate the details by finding a basis for $\Omega^{2,1}_{PKP}(J^\infty(E))$ assuming that bases for $\Omega^{1,1}_{PKP}(J^\infty(E))$ and $\Omega^{2,0}_{PKP}(J^\infty(E))$ have already been found.

Let $\omega \in \Omega^{2,1}_{PKP}(J^\infty(E))$, and write

$$\omega = \omega_1\sigma^t \wedge \sigma^x + \omega_2\sigma^t \wedge \sigma^y + \omega_3\sigma^x \wedge \sigma^y, \tag{3.21}$$

where $\omega_i \in \Omega^{0,1}(J^\infty(E))$, $i = 1, 2, 3$. By the Γ_{PKP} invariance of ω,

$$W(\omega_1) = -4, \quad W(\omega_2) = -5, \quad W(\omega_3) = -3.$$

It now follows from Lemma 3.2 that ω_1 must be a linear combination of the forms

$$\Theta_t, \quad \Theta_{xy}, \quad \Theta_{xxx},$$

that ω_2 must be a linear combination of the forms

$$v_{xxx}\Theta, \quad \Theta_{tx}, \quad \Theta_{yy}, \quad \Theta_{xxy}, \quad \Theta_{xxxx},$$

and that ω_3 must be a linear combination of the forms

$$\Theta_y, \quad \Theta_{xx}.$$

Next note that

$$\begin{aligned} d_H\alpha_1^{1,1} &= -\sigma^t \wedge (\sigma^x \wedge \Theta_{xxx} + \sigma^y \wedge \Theta_{xxy}), \\ d_H\alpha_2^{1,1} &= -\sigma^t \wedge \sigma^x \wedge \Theta_{xy} - \frac{2}{3}s^2\sigma^t \wedge \sigma^y \wedge \left(\Theta_{tx} + \frac{3}{4}s^2\Theta_{yy}\right) - \frac{2}{3}s^2\sigma^x \wedge \sigma^y\Theta_{xx}, \\ d_V\alpha_1^{2,0} &= \sigma^t \wedge \sigma^y \wedge \Theta_{tx}, \quad d_V\alpha_2^{2,0} = \sigma^t \wedge \sigma^y \wedge \Theta_{xxxx}. \end{aligned} \tag{3.22}$$

are all Γ_{PKP} invariant type $(2, 1)$ forms. Thus, by subtracting a suitable linear combination of the form $\alpha_1^{2,1}$ and the forms in (3.22) from ω, we can assume that there are constants b^i_j so that

$$\omega_1 = b^1_1\Theta_t + b^2_1\Theta_{xy}, \quad \omega_2 = b^1_2\Theta_{yy} + b^2_2\Theta_{xxy}, \quad \omega_3 = b^1_3\Theta_y.$$

It remains to show that the invariance of ω in (3.21) with ω_1, ω_2, ω_3 as above forces $b_i^j = 0$ for all i, j. But this is an immediate consequence of the Lie derivative formulas (3.13)–(3.15). ∎

Proof of Theorem 1.1. One can verify without difficulty that, on account of (2.1), each of the forms $\alpha_1^{3,1}$, $\alpha_2^{3,1}$, $\alpha_1^{1,1} \wedge \alpha_1^{2,2}$ and $\alpha_2^{1,1} \wedge \alpha_2^{2,2}$ is fixed by the interior Euler operator I. Thus,

$$\begin{aligned} &\mathcal{F}^2_{PKP}(J^\infty(E)) = \mathbf{R} < \alpha_1^{3,1}, \alpha_2^{3,1} >, \\ &\mathcal{F}^4_{PKP}(J^\infty(E)) = \mathbf{R} < \alpha_1^{1,1} \wedge \alpha_1^{2,2}, \alpha_2^{1,1} \wedge \alpha_1^{2,2} >, \\ &\mathcal{F}^s_{PKP}(J^\infty(E)) = < 0 >, \quad s \neq 2, 4. \end{aligned} \tag{3.23}$$

Now the exactness of the augmented interior rows $(\Omega^{*,s}_{PKP}(J^\infty(E)), d_H)$, $s \geq 1$, of the Γ_{PKP} invariant variational bicomplex immediately follows from Proposition 3.3. This proves (i).

To prove (ii), we only need to verify that

$$\begin{aligned} &d_V \alpha_i^{1,1} = 0, \quad i = 1, 2, \qquad d_V \alpha_1^{1,2} = 0, \qquad d_V \alpha_3^{2,0} = -d_H \alpha_1^{1,1}, \\ &d_V \alpha_1^{3,1} = \frac{1}{2} d_H \alpha_1^{2,2}, \qquad d_V \alpha_2^{3,1} = d_H d_V \alpha_1^{2,1} - \frac{3}{2} s^2 d_H(\alpha_1^{1,1} \wedge \alpha_2^{1,1}). \end{aligned}$$

Then we can conclude with the help of Proposition 3.3 that the vertical cohomology spaces $H^{r,s}(\Omega_{PKP}(J^\infty(E)), d_V)$ are generated by the following forms.

$s \geq 4$	0	0	0	0
$s = 3$	0	0	0	$\alpha_1^{1,1} \wedge \alpha_1^{2,2}$ $\alpha_2^{1,1} \wedge \alpha_1^{2,2}$
$s = 2$	0	$\alpha_1^{1,2}$	$\alpha_1^{2,2}$, $\alpha_1^{1,1} \wedge \alpha_2^{1,1}$ $d_H \alpha_1^{1,2}$, $\alpha_2^{1,1} \wedge \alpha_2^{1,1}$	$d_H(\alpha_2^{1,1} \wedge \alpha_2^{1,1})$
$s = 1$	0	$\alpha_1^{1,1}, \alpha_2^{1,1}$	$d_H \alpha_2^{1,1}$	0
$s = 0$	1	0	0	0
	$r = 0$	$r = 1$	$r = 2$	$r = 3$

Now (ii) follows.

Finally, by (3.23), the differential δ reduces to the zero map when restricted to the spaces $\mathcal{F}^s_{PKP}(J^\infty(E))$, $s \geq 1$. Thus, by Proposition 3.3 and by (3.23),

$$\begin{aligned} &H^2(\mathcal{E}_{PKP}(J^\infty(E))) = \mathbf{R} < [\alpha_3^{2,0}] >, \\ &H^4(\mathcal{E}_{PKP}(J^\infty(E))) = \mathbf{R} < [\alpha_1^{3,1}], [\alpha_2^{3,1}] >, \\ &H^6(\mathcal{E}_{PKP}(J^\infty(E))) = \mathbf{R} < [\alpha_1^{1,1} \wedge \alpha_1^{2,2}], [\alpha_2^{1,1} \wedge \alpha_1^{2,2}] >, \\ &H^s(\mathcal{E}_{PKP}(J^\infty(E))) = < 0 >, \quad s \neq 2, 4, 6. \end{aligned} \tag{3.24}$$

Now (iii) follows. ∎

Proof of Corollary 1.2. We only need to remark that by (3.24) the cohomology space $H^4(\mathcal{E}_{PKP}(J^\infty(E)))$ is generated by the source forms

$$\Delta_{PKP} = \alpha_1^{3,1} + \alpha_2^{3,2} \quad \text{and} \quad \Delta_{u_{xxxx}} = \alpha_2^{3,2}. \qquad \blacksquare$$

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