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# Bilinearization of Coupled Nonlinear Schrödinger Type Equations: Integrabilty and Solitons

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**Abstract**

Considering the coupled envelope equations in nonlinear couplers, the question of integrability is attempted. It is explicitly shown that Hirota's bilinear method is one of the simple and alternative techniques to Painlevé analysis to obtain the integrability conditions of the coupled nonlinear Schrödinger (CNLS) type equations. We also show that the coupled Hirota equation introduced by Tasgal and Potasek is the next hierarchy of the inverse scattering solvable CNLS equation. The results are in agreement with the known results.

## 1 Introduction

The propagation of temporal soliton envelopes in nonlinear optical media has been predicted and demonstrated experimentally [1]–[5]. This prediction arises from the opportunity of reducing the Maxwell equations which govern the propagation to a single, completely integrable, partial differential equation(PDE) where one is left with the time and the propagation distance as independent variables [1]. The best and well studied example is the nonlinear Schrödinger (NLS) equation

$$iq_z + q_{tt} + 2 \mid q \mid^2 q = 0 \tag{1}$$

which describes the wave propagation of picosecond-pulse envelopes $q(z,t)$ in a lossless single-mode fibre. Eq.(1) is one of the completely integrable $N$-soliton possessing systems. Eventhough eq.(1) may adequately describe the propagation in a single-mode waveguides, routing-, switching operations, and other optical effects which involve soliton pulses require the interaction between two or more modes [4]–[8]. In general, the coupled mode approach [7] still permits description of the pulse propagation in a multi-mode waveguide by means of vector versions of eq.(1). Although these systems of equations are no longer integrable, one may obtain quantitative information about the pulse propagation by resorting to numerical or perturbative methods. Situations of physical interest that can be described by CNLS equations include two parallel waveguides coupled through evansecnt field overlap, the coupling of two polarizations modes in uniform guides, and so on. Note that the study of the propagation of optical solitons in multi-mode nonlinear couplers, besides being important from the theoretical point of view, is also important in view of their possible applications. Recent advances have permitted proof that solitons are ideal candidates for performing all-optical switching operations in nonlinear couplers. In fact, their stability leads to the possibility of controlling the coupling of the whole pulse, by means of changing the input power of a single pulse, or the phase of the superimposed pulse.

For the past few years, the wave propagation in nonlinear couplers have been investigated from theoretical, numerical and experimental point of view [4]–[19]. From the detailed theoretical investigations, several integrable soliton possessing coupled NLS equations have been introduced [5, 9, 10, 12, 16, 19]. The experimental results reveals that these integrable soliton models are not accurate enough to explain the wave propagation in nonlinear couplers. On the otherhand, theoretically several new results and concepts have been introduced. Thus it is clear that to propagate exact soliton through nonlinear couplers, one has to look for different conditions among the parameters involved in the system. In this letter, we present new type of bilinear conditions, which may be very useful for the experimental realization of solitons in nonlinear couplers.

It is well known that Painlevé singularity structure analysis is one of the systematic and powerful method in nonlinear science to identify the integrability conditions of nonlinear partial differential equations (NPDEs) [18]. In recent years, this method has been applied to a very large number of NPDEs and also systematically estabilished the complete integrability properties like Lax pair, Bäcklund transformation, bilinear transformation, soliton solutions, and so on [13, 20, 21, 22]. For CNLS type equations, this method is found to be cumbersome and the construction of the Lax pair from this analysis is still an open question. Also, after getting the necessary condition(s) for integrability, one has to apply bilinear or other methods to prove the sufficient condition for the complete integrability. The main aim of this letter is to show explicitly that Hirota's bilinear approach [23] can be used as an alternative method to Painlevé analysis to identify the complete and partial integrability conditions of CNLS equations with many parameters. Using this method, we show that one can obtain the more general bilinear (integrability) conditions of coupled nonlinear equations in a very simple way.

## 2 Two Coupled Nonlinear Schrödinger Equation

There are several methods to derive a set of CNLS equations, depending on the physical situation that is being modeled. A fairly general and frequently studied CNLS equation is of the form [4, 5, 6, 8, 10, 12, 13]

$$iu_z = c_1 u_{tt} + 2\left(\alpha \mid u \mid^2 + \beta \mid v \mid^2\right) u, \tag{2}$$

$$iv_z = c_2 v_{tt} + 2\left(\beta \mid u \mid^2 + \gamma \mid v \mid^2\right) v. \tag{3}$$

Eqs.(2-3) are shown to be completely integarble for the following two cases:

$$(i): \ c_1 = c_2, \quad \alpha = \beta = \gamma; \qquad (ii): \ c_1 = -c_2, \quad \alpha = -\beta = \gamma. \tag{4}$$

To obtain the (bilinear)integrability conditions and soliton solutions, we introduce the following transformation [23]

$$u = \frac{G}{F}, \qquad v = \frac{H}{F}, \tag{5}$$

where $G$ and $H$ are complex functions and $F$ is a real function and the bilinear operator is defined by

$$D_t^m D_z^n = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial z'}\right)^n G(z,t)F(z',t') \mid z' = z, \ t' = t. \tag{6}$$

Substituting eq.(5) in eqs(2-3), we obtain

$$\left(iD_z - c_1 D_t^2\right) G \cdot F = 0, \qquad \left(iD_z - c_2 D_t^2\right) H \cdot F = 0, \tag{7}$$

$$D_t^2 F \cdot F = \frac{2}{c_1}\left(\alpha GG^* + \beta HH^*\right), \qquad D_t^2 F \cdot F = \frac{2}{c_2}\left(\gamma HH^* + \beta GG^*\right). \tag{8}$$

So the left hand sides of eq.(8) become equal. Hence the right hand sides of eq.(8) should also be equal which is true only when

$$c_2\alpha = c_1\beta, \qquad c_2\beta = c_1\gamma. \tag{9}$$

The above conditions can be obtained by equating the coefficients of $GG^*$ and $HH^*$ respectively in eq.(8). One can easily check that eq.(9) admits the already known integrability conditions (4). Solving eq.(9), we obtain

$$c_2 = \pm\sqrt{\frac{\gamma}{\alpha}}c_1, \qquad \alpha\gamma = \beta^2. \tag{10}$$

It should be noted that conditions (10) are more general than (4). As the bilinear conditions (9) or (10) are in terms of all parameters, we believe that these conditions may be very useful for the experimental generation of solitons in nonlinear couplers.

Having obtained the bilinear conditions and bilinear forms, our next aim is to obtain the soliton solutions. Now to generate the soliton solutions, we expand the dependent variables in terms of $\varepsilon$ and the solutions can be constructed as usual. For instance, if $G = \varepsilon G_1, H = \varepsilon H_1$ and $F = 1 + \varepsilon^2 F_2$ , the solutions are found to be

$$G_1 = e^{\eta_1}, \qquad H = e^{\eta_2 + \xi}, \qquad F_2 = \frac{\beta e^{\eta_2 + \eta_2^* + \xi + \xi^*} + \alpha e^{(\eta_1 + \eta_1^*)}}{c_1(k_1 + k_1^*)^2}, \tag{11}$$

where $\eta_1 = k_1(t - ic_1k_1z) + \eta_1^0$, $\eta_2 = k_1(t - ic_2k_1z) + \eta_1^0$ and $\xi$ is a complex constant. From the detailed investigations, we find that eqs.(2-3) admit the higher order soliton solutions for the conditions given in eq.(4) and the resultant solutions are in agreement with the resluts reported earlier [9, 14, 15]. So like fixing the parameters in the arbitrary analysis of Painlevé method, using bilinear method one can also get the complete integrability conditions from the generation of higher order soliton solutions.

## 3 Three Coupled NLS Equations

Considering the systems of three coupled NLS equations of the form [14]

$$iu_z = c_1 u_{tt} + 2\left(\alpha \mid u \mid^2 + \beta \mid v \mid^2 + \delta \mid w \mid^2\right) u, \tag{12}$$

$$iv_z = c_2 v_{tt} + 2\left(\beta \mid u \mid^2 + \gamma \mid v \mid^2 + \Gamma \mid w \mid^2\right) v, \tag{13}$$

$$iw_z = c_3 w_{tt} + 2\left(\delta \mid u \mid^2 + \Gamma \mid v \mid^2 + \Delta \mid w \mid^2\right) w. \tag{14}$$

Using the similar procedure the bilinear conditions are found to be of the form:

$$c_2c_3\alpha = c_1c_3\beta = c_1c_2\delta, \quad c_2c_3\beta = c_1c_3\gamma = c_1c_2\Gamma, \quad c_2c_3\delta = c_1c_3\gamma = c_1c_2\Delta. \tag{15}$$

Here also we find that eqs.(15) satisfy the integrability conditions reported through Painlevé analysis [14] and the soliton solutions can also be constructed.

## 4 Coupled Hirota equation

In this section, we will discuss the integrability of the coupled Hirota equation [16]. In order to increase the bit rates it is necessary to decrease the pulse width of the order of femtosecond. As pulse lengths become comparable to the wavelength, however, eqs.(2-3) are inadequate, as additional effects must now be considered. By considering the above facts, a generalized coupled Hirota equation reads as [16]

$$\begin{aligned} &iu_z + c_1 u_{tt} + 2\left(\alpha \mid u \mid^2 + \beta \mid v \mid^2\right) u \\ &\quad -i\epsilon\left[u_{ttt} + \left(2\mu_1 \mid u \mid^2 + \nu_1 \mid v \mid^2\right) u_t + \nu_1 u v^* v_t\right] = 0, \end{aligned} \tag{16}$$

$$\begin{aligned} &iv_z + c_2 v_{tt} + 2\left(\beta \mid u \mid^2 + \gamma \mid v \mid^2\right) v \\ &\quad -i\epsilon\left[v_{ttt} + \left(\nu_2 \mid u \mid^2 + 2\mu_2 \mid v \mid^2\right) v_t + \nu_2 u^* v u_t\right] = 0. \end{aligned} \tag{17}$$

When $c_1 = c_2 = \dfrac{\lambda}{2}$, $\alpha = \beta = \gamma = \lambda$, $\mu_1 = \nu_1 = \mu_2 = \nu_2 = 3$, eqs.(16-17) take the form of the equations in [16]. In eqs.(16-17), we have not included the linear effects, which can always be removed through suitable transformation in $u$ and $v$ [15] and, for our convenience, we introduce the new parameters $\nu_2$ and $\mu_2$. The bilinear forms of (16-17) are in the form

$$\left(iD_z + c_1 D_t^2 - i\epsilon D_t^3\right) G \cdot F = 0, \qquad \left(iD_z + c_2 D_t^2 - i\epsilon D_t^3\right) H \cdot F = 0, \tag{18}$$

$$D_t^2 F \cdot F = \frac{2}{c_1}\left(\alpha GG^* + \beta HH^*\right), \qquad D_t^2 F \cdot F = \frac{2}{3}\left(\mu_1 GG^* + \nu_1 HH^*\right), \tag{19}$$

$$D_t^2 F \cdot F = \frac{2}{c_2}\left(\gamma HH^* + \beta GG^*\right), \qquad D_t^2 F \cdot F = \frac{2}{3}\left(\nu_2 GG^* + \mu_2 HH^*\right), \tag{20}$$

The more general bilinear conditions are:

$$\frac{\alpha}{c_1} = \frac{\mu_1}{3} = \frac{\beta}{c_2} = \frac{\nu_2}{3}, \qquad \frac{\beta}{c_1} = \frac{\nu_1}{3} = \frac{\gamma}{c_2} = \frac{\mu_2}{3}. \tag{21}$$

Here also we find that eq.(21) admits the higher order soliton solutions for the conditions $c_1 = c_2$ , $\alpha = \beta = \gamma$, and $\mu_1 = \nu_1 = \mu_2 = \nu_2 = 3$ [16, 17]. Exact $N$-envelope soliton solutions of eqs.(16-17) can also be expressed in a similar way. It is interesting to note that, like CNLS, eqs.(16-17) also found to be completely integrable for another conditions $c_1 = -c_2$ , $\alpha = -\beta = \gamma$, and $\mu_1 = -\nu_1 = -\mu_2 = \nu_2 = 3$. It should be noted that the latter condition is one of the new integrable system obtained through this method. Recently, using multi-Hamiltonian formalism, we also verified the above complete integrability conditions and found that the above integrable cases are the next hierarchy of the CNLS equations with conditions (4) [18]. Hence, the coupled Hirota equation introduced by Tasgal and Potasek [16] is the next hierarchy of the IST solvable CNLS equation-I.

## 5 Coupled Higher Order NLS Equation

We consider the coupled higher order NLS equation of the form [19]

$$\begin{aligned} &iu_z + c_1 u_{tt} + 2\left(\alpha \mid u \mid^2 + \beta \mid v \mid^2\right) u \\ &\qquad -i\varepsilon \left[u_{ttt} + 6\left(\mid u \mid^2 + \mid v \mid^2\right) u_t + 3\left(\mid u \mid^2 + \mid v \mid^2\right)_t u\right] = 0, \end{aligned} \tag{22}$$

$$\begin{aligned} &iv_z + c_2 v_{tt} + 2\left(\beta \mid u \mid^2 + \gamma \mid v \mid^2\right) v \\ &\qquad -i\varepsilon \left[v_{ttt} + 6\left(\mid u \mid^2 + \mid v \mid^2\right) v_t + 3\left(\mid u \mid^2 + \mid v \mid^2\right)_t v\right] = 0. \end{aligned} \tag{23}$$

Equations (22-23) can be derived from the higher order NLS [3, 5] by considering the electromagnetic wave $\vec{E}$ as a sum of left and right polarized waves [19]. When $c_1 = c_2 = \dfrac{1}{2}$ and $\alpha = \beta = \gamma = 1$, the bilinear forms are found to be

$$\left(iD_z + \frac{1}{2}D_t^2 - i\varepsilon D_t^3\right) G \cdot F = 0, \quad \left(iD_z + \frac{1}{2}D_t^2 - i\varepsilon D_t^3\right) H \cdot F = 0, \tag{24}$$

$$D_t^2 F \cdot F = 4(GG^* + HH^*), \tag{25}$$

$$D_t G \cdot G^* = 0, \quad D_t G \cdot H^* = 0, \quad D_t G^* \cdot H = 0, \quad D_t G \cdot H = 0, \quad D_t H \cdot H^* = 0. \tag{26}$$

The one soliton solution of eqs.(22-23) is constructed as

$$u = \frac{a_1 k_1}{\sqrt{2(\mid a_1 \mid^2 + \mid a_2 \mid^2)}} \operatorname{sech}\left(k_1 t + \varepsilon k_1^3 z + \eta_1^0\right) \exp\left(\frac{ik_1^2 z}{2}\right), \tag{27}$$

$$v = \frac{a_2 k_1}{\sqrt{2(\mid a_1 \mid^2 + \mid a_2 \mid^2)}} \operatorname{sech}\left(k_1 t + \varepsilon k_1^3 z + \eta_1^0\right) \exp\left(\frac{ik_1^2 z}{2}\right), \tag{28}$$

where $a_1$ and $a_2$ are integration constants.

Thus in this letter, we have shown that Hirota's bilinear method is one of the alternative formalism to Painlevé singularity structure analysis to identify the integrability of coupled nonlinear Schrödinger equations with many parameters. This analysis is found to be very simple when compared with the Painlevé analysis. Also from the optical point of view, the general bilinear conditions obtained through bilinear form may be useful for the experimental generation of solitons in nonlinear couplers. Another advantage of this method is that one can, in a simple manner, construct the soliton solutions for all integrable cases. We have also pointed out that the coupled Hirota equation studied by Tasgal and Potasek is the next hierarchy of the IST solvable CNLS equation. We found that the conditions obtained through this method are in agreement with the results reported earlier. It will be interesting to investigate the nature of the solutions (at least two soliton solutions) for the new integrable cases. Work is in progress in this direction.

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