Journal of Nonlinear Mathematical Physics 1998, V.5, N 2, 115–119. Letter

On a Class of Linearizable Monge-Ampère Equations

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Received November 21, 1997; Accepted November 25, 1997

Abstract

Monge-Ampère equations of the form, $u_{xx}u_{yy} - u_{xy}^2 = F(u, u_x, u_y)$ arise in many areas of fluid and solid mechanics. Here it is shown that in the special case $F = u_y^4 f(u, u_x/u_y)$, where f denotes an arbitrary function, the Monge-Ampère equation can be linearized by using a sequence of Ampère, point, Legendre and rotation transformations. This linearization is a generalization of three examples from finite elasticity, involving plane strain and plane stress deformations of the incompressible perfectly elastic Varga material and also relates to a previous linearization of this equation due to Khabirov [7].

1 Introduction

Monge-Ampère equations of the form

$$u_{xx}u_{yy} - u_{xy}^2 = F(x, y, u, u_x, u_y), \tag{1.1}$$

are well known to arise is many areas of science and engineering, especially areas relating to fluid mechanics (see for example von Mises [1] and Martin [2]). In three recent papers Hill and Arrigo [3, 4] and Arrigo and Hill [5], the present authors have shown that Monge-Ampère equations also arise in the context of finite elastic deformations. In particular, it is shown that certain plane strain, plane stress and axially symmetric deformations of the incompressible perfectly elastic Varga material (see Varga [6]) all give rise to Monge-Ampère equations of the form (1.1). Three of these may by linearized by a sequence of elementary transformations and the purpose of this brief communication is to show that the same sequence of transformations is also effective when $F = u_y^4 f(u, u_x/u_y)$, where f denotes an arbitrary function.

The recent work of the authors in finite elasticity may summarized as follows. For plane strain deformations of the form

$$x = x(X, Y), \qquad y = y(X, Y), \tag{1.2}$$

where (X, Y) and (x, y) denote respectively material and spatial plane rectangular Cartesian coordinates Hill and Arrigo [3, 4] show that for the Varga elastic material, exact finite elastic deformations can be determined from the following

$$x = U_X, \qquad y = U_Y, \tag{1.3}$$

$$x = V_\alpha, \qquad y = V_\beta, \tag{1.4}$$

$$x = \frac{W_X}{W_X^2 + W_Y^2}, \qquad y = -\frac{W_Y}{W_X^2 + W_Y^2}, \tag{1.5}$$

where $U(X, Y)$, $V(\alpha, \beta)$ and $W(X, Y)$ all satisfy Monge-Ampère equations, namely

$$U_{XX}U_{YY} - U_{XY}^2 = 1, \tag{1.6}$$

$$V_{\alpha\alpha}V_{\beta\beta} - V_{\alpha\beta}^2 = (\alpha^2 + \beta^2)^{-2}, \tag{1.7}$$

$$W_{XX}W_{YY} - W_{XY}^2 = (W_X^2 + W_Y^2)^2, \tag{1.8}$$

where (α, β) are intermediate coordinates which are defined by

$$\alpha = \frac{X}{X^2 + Y^2}, \qquad \beta = -\frac{Y}{X^2 + Y^2}. \tag{1.9}$$

In addition, for axially symmetric deformations of the form

$$r = r(R, Z), \qquad \theta = \Theta, \qquad z = z(R, Z), \tag{1.10}$$

where (R, Θ, Z) and (r, θ, z) denote respectively material and spatial cylindrical polar coordinates, Hill and Arrigo [3] and Arrigo and Hill [5] show that for the Varga elastic material, exact finite elastic deformations may be determined from

$$r = U_R, \qquad z = U_Z, \tag{1.11}$$

$$r = V_\alpha, \qquad z = V_\beta, \tag{1.12}$$

where $U(R, Z)$ and $V(\alpha, \beta)$ satisfy respectively the Monge-Ampère equations

$$U_{RR}U_{ZZ} - U_{RZ}^2 = \frac{R}{U_R}, \tag{1.13}$$

$$V_{\alpha\alpha}V_{\beta\beta} - V_{\alpha\beta}^2 = \frac{\alpha}{(\alpha^2 + \beta^2)^2 V_\alpha}, \tag{1.14}$$

where (α, β) are now the intermediate coordinates defined by

$$\alpha = \frac{R}{R^2 + Z^2}, \qquad \beta = -\frac{Z}{R^2 + Z^2}. \tag{1.15}$$

Finally, for plane stress deformations arising from the membrane theory of thin plane elastic sheets, which assumes a three dimensional deformation of the form

$$x = x(X, Y), \qquad y = y(X, Y), \qquad z = \lambda(X, Y)Z, \tag{1.16}$$

Arrigo and Hill [5] show that finite elastic solutions may be determined from

$$x = V_\alpha, \qquad y = V_\beta, \tag{1.17}$$

where α and β are precisely as defined in (1.9) and $V(\alpha, \beta)$ in this case satisfies the Monge-Ampère equation

$$V_{\alpha\alpha}V_{\beta\beta} - V_{\alpha\beta}^2 = (\alpha^2 + \beta^2)^{-2}(\alpha V_\alpha + \beta V_\beta - V)^{-1}. \tag{1.18}$$

Under the Legendre transformation

$$u = \alpha V_\alpha + \beta V_\beta - V, \qquad x = V_\alpha, \qquad y = V_\beta,$$

the above Monge-Ampère equations (1.7) and (1.18) can all be embodied in the general equation

$$u_{xx}u_{yy} - u_{xy}^2 = A(u_x^2 + u_y^2)^2, \tag{1.19}$$

where $A = 1$ for equation (1.7) and $A = u$ for equation (1.18) and of course $A = 1$ for equation (1.8). In Hill and Arrigo [4] and Arrigo and Hill [5] these special cases of equation (1.19), namely (1.7), (1.8) and (1.18) are shown to be linerazable under a combination of a Ampère, point, Legendre and rotational transformations. Remarkably, these Monge-Ampère equations belong to a much larger class of Monge-Ampère equations which can be linearized by precisely the same combination of transformations. The purpose of this paper is to show that the Monge-Ampère equation

$$u_{xx}u_{yy} - u_{xy}^2 = u_y^4 f(u, u_x/u_y), \tag{1.20}$$

where f denotes an arbitrary function, may be transformed to the linear equation

$$U_{XX} + f(X, Y)U_{YY} = 0, \tag{1.21}$$

under the contact transformation (2.2) and this result is established in the following section.

2 The basic linearization

Following Hill and Arrigo [4] and Arrigo and Hill [5], we consider the following sequence of transformations applied to the Monge-Ampère equation (1.1), namely

$$\begin{array}{ll} \text{(i)} & x = \alpha, \quad y = V_\beta, \quad u = V - \beta V_\beta, \\ \text{(ii)} & \alpha = \xi, \quad \beta = 1/\eta, \quad V = W/\eta, \\ \text{(iii)} & \xi = Z_\tau, \quad \eta = Z_\sigma, \quad W = \tau Z_\tau + \sigma Z_\sigma - Z, \\ \text{(iv)} & \tau = -Y, \quad \sigma = X, \quad Z = -U. \end{array} \tag{2.1}$$

The first transformation represents an Ampère transformation, the second is a simple point transformation, the third is a Legendre transformation, while the fourth represents a rotation and scaling transformation. Combining all four transformations we have

$$x = U_Y, \qquad y = U - YU_Y, \qquad u = X. \tag{2.2}$$

Now from (2.2), we find the first and second order partial derivatives according to the following

$$
\begin{aligned}
u_x &= Y/U_X, \qquad u_y = 1/U_X,\\
u_{xx} &= \left(Y^2U_{XY}^2 - Y^2U_{XX}U_{YY} - 2YU_XU_{XY} + U_X^2\right)U_X^{-3}U_{YY}^{-1},\\
u_{xy} &= \left(YU_{XY}^2 - YU_{XX}U_{YY} - U_XU_{XY}\right)U_X^{-3}U_{YY}^{-1},\\
u_{yy} &= \left(U_{XY}^2 - U_{XX}U_{YY}\right)U_X^{-3}U_{YY}^{-1},
\end{aligned}
\tag{2.3}
$$

which, on substitution into equation (1.1), and assuming that the function F is independent of x and y, yields the equation

$$U_{XX} + F(X, Y/U_X, 1/U_X)U_X^4U_{YY} = 0. \tag{2.4}$$

If now we require that equation (2.4) be linear, namely

$$U_{XX} + f(X, Y)U_{YY} = 0, \tag{2.5}$$

for some function $f(X, Y)$, then we require that the following equation

$$F(X, Y/U_X, 1/U_X)U_X^4 = f(X, Y), \tag{2.6}$$

holds identically. Now on transforming back to the original variables (x, y, u), using (2.1) and (2.3) we may deduce

$$F(u, u_x, u_y) = u_y^4 f(u, u_x/u_y), \tag{2.7}$$

which is the desired result.

We note that under only the Ampère transformation (i) of (2.1), that the Monge-Ampère equation (1.1) with $F = 1$ is linearizable to Laplaces' equation. We also comment that Khabirov [7] showed, using Lie contact symmetry analysis, that Monge-Ampère equations of the form

$$u_{xx}u_{yy} - u_{xy}^2 = F(x, y), \tag{2.8}$$

can be linearized provided that $F(x, y)$ is one of the following four cases, namely

$$F = 0, \quad F = 1, \quad F = f(x), \quad F = x^{-4}g(y/x),$$

where $f(x)$ and $g(y/x)$ denote arbitrary functions. From our point of view the final case is of particular interest since under the Legendre transformation,

$$x = U_X, \quad y = U_Y, \quad u = XU_X + YU_Y - U, \tag{2.9}$$

the Monge-Ampère equation

$$u_{xx}u_{yy} - u_{xy}^2 = x^{-4}g(y/x) = y^{-4}g^*(y/x), \tag{2.10}$$

transforms to the equation

$$U_{XX}U_{YY} - U_{XY}^2 = U_Y^4 G^*(U_Y/U_X), \tag{2.11}$$

where $G^* = 1/g^*$ and this equation is of the form (1.1) where F is as given in (2.7) except that in this case f is independent of u. Although this is a special case of the results presented here, Khabirov [7] does not provide the explicit contact transformation which linearizes equation (2.11).

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