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Is the Classical Bukhvostov-Lipatov Model Integrable? A Painlevé Analysis

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Abstract

In this work we apply the Weiss, Tabor and Carnevale integrability criterion (Painlevé analysis) to the classical version of the two dimensional Bukhvostov-Lipatov model. We are led to the conclusion that the model is not integrable classically, except at a trivial point where the theory can be described in terms of two uncoupled sine-Gordon models.

1 Introduction

In a remarkable paper [1], Bukhvostov and Lipatov were able to map the partition function for interacting instantons and anti-instantons of the O(3) non-linear σ model onto a two component scalar field theory defined by the Lagrangian

$$\mathcal{L} = \sum_{i=1}^{2} \frac{1}{2}\partial_\nu \phi_i \partial^\nu \phi_i - \mu^2 \cos(\lambda_1\phi_1)\cos(\lambda_2\phi_2). \tag{1}$$

They further showed that the quantum version of the model above is exactly solvable provided the couplings λ_1, λ_2 are constrained by the following relation:

$$\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} = \frac{1}{\pi}. \tag{2}$$

The integrability in this case was proved via the bosonization technique [2] and the Bethe Ansatz [3].

We will call the model described by the Lagrangian (1) the Bukhvostov-Lipatov (BL) model[1]. The model (1) has been studied again recently in [4], where other integrable cases have been found, which are fundamental to an understanding of impurity problems in quantum wires. For a related model, see also [5].

An important open question, only partially addressed in [4], concerns the integrability of the classical equations of motion which can be derived from Eq.(1). This is crucial to assess whether one can apply semiclassical considerations to the corresponding quantum mechanical model. In the present paper we analyze this problem via the Weiss, Tabor and Carnevale (WTC) integrability criterion [6].

Some comments are due at this point. The Lagrangian (1) can be regarded as being of the general form (with complex couplings λ_a)

$$\mathcal{L} = \sum_a \frac{1}{2}\partial_\nu \vec{\Phi}^{(a)} \cdot \partial^\nu \vec{\Phi}^{(a)} - \frac{\mu^2}{4}\prod_a \left\{ \sum_{\vec{\alpha}^{(a)}} e^{\lambda_a \vec{\alpha}^{(a)} . \vec{\Phi}^{(a)}} \right\}, \tag{3}$$

where $\vec{\alpha}$ denotes the simple positive roots of the Lie algebra G and the index a labels the different algebras (possibly copies of the same one) that appear in the potential. Obviously the well known Toda Lagrangian is a special case of (3) for $a = 1$ and G being a finite or affine Lie algebra [7].

A lot of work has been done in the past twenty years in the field of integrable models. In particular, the Toda field theories have been thoroughly studied, both in their classical and quantum versions (for a review of the results see [8] and references therein). In particular, Yoshida has shown [9] that the integrable Toda field theories are characterized by the Painlevé property. More recently, the authors of [10] used the WTC criterion to examine integrability of the hyperbolic Toda field theories for which other techniques have not been applied so far. In [10], the autors concluded that the hyperbolic Toda field theories, although conformally invariant, are *not* integrable since they fail the Painlevé test. The BL model provides yet another example of theories belonging to the general class defined by (3).

The outline of the paper is the following: first, in Section 2, we briefly review the WTC algorithm and we apply it to the sine-Gordon (SG) model in a format best suited for the problem to follow. In Section 3, we present its application to the case of the BL model. Finally, Section 4 contains a discussion of the result.

2 The Painlevé property for a PDE

An ordinary differential equation (ODE) is said to possess the Painlevé property if all of its movable singularities are poles [11]. The connection between the Painlevé property and the integrability of an ODE had been noted since the work of S. Kowalevskaya [12] concerning the integrability of a rotating rigid body.

The relation between integrability and the absence of movable critical points was made more explicit through the work of Ablowitz, Ramani and Segur [13], who established the following conjecture: every ODE obtained by an exact reduction from a partial differential

[1]Strictly speaking, the term is usually used for the fermionic counterpart of Eq.(1) subject to the constraint (2). However, this slight abuse of terminology should not create any confusion.

equation (PDE) solvable via the inverse scattering transform possesses the Painlevé property. This led to the formulation of a three-step algorithm which allows one to test for the absence of multivalued movable singularities in the solutions of a given ODE.

The definition of the Painlevé property for PDEs and the corresponding generalization of the aforementioned algorithm was proposed by Weiss, Tabor and Carnevale [6]. This we will briefly review in the following subsection. For a comprehensive review see [14].

2.1 General description

It is well known [15] that the singularities of a function $f(z_1, z_2, \ldots, z_n)$ of $n > 1$ complex variables cannot be isolated; rather they occur along analytic manifolds of (complex) dimension $n-1$ determined by equations of the form

$$\chi\,(z_1, z_2, \ldots, z_n) = 0, \tag{4}$$

χ being an analytic function of its variables in a neighborhood of the singularity manifold defined by Eq.(4).

One says [6] that a given PDE possesses the Painlevé property if its solutions are single valued around the movable singularity manifold (4).

To test for the presence of the property one assumes that a solution $u(z_1, z_2, \ldots, z_n)$ of the PDE can be expanded around the singularity manifold (4) as follows

$$u = \chi^{-\alpha} \sum_{k=0}^{+\infty} u_k \chi^k, \tag{5}$$

where the coefficients $u_k(z_1, z_2, \ldots, z_n)$ are analytic in a neighbourhood of $\chi = 0$. One then substitutes the above expansion (5) in the PDE to determine the value(s) of α and the recurrence relations[2] among the u_k's.

If all the allowed values of α turn out to be integers and the set of recurrence relations consistently allows for the arbitrariness of initial conditions, then the given PDE is said to possess the Painlevé property and is conjectured to be integrable.

As an illustration of the method discussed above, we now analyze the well known SG equation.

2.2 An example: the sine-Gordon equation

The two-dimensional SG equation [2] arises as the dynamical equation from the Lagrangian

$$\mathcal{L}_{\mathrm{SG}} = \frac{1}{2}\partial_\nu\phi\partial^\nu\phi - \mu^2\cos(\lambda\phi).$$

In our notation $\nu = 1, 2$ and summation over repeated indices is understood unless otherwise indicated. The metric is $g_{\mu\nu} = \mathrm{diag}(1, -1)$. Also $\partial_\nu \equiv \partial/\partial x^\nu$. Introducing light-cone coordinates $x_\pm \equiv x_1 \pm x_2$, rescaling the field ϕ and fixing the mass scale so that $\mu^2 = 1$, we can write

$$\partial_+\partial_-\phi = \sin\phi.$$

[2]The recurrence relations are PDEs in the coefficients u_k.

In order to apply the algorithm we transform the equation above, following [10], in the equivalent system[3]:

$$\dot{A} = BC, \qquad \dot{B} = -AC, \qquad C' = A, \tag{6}$$

where the following new dependent variables have been defined:

$$A \equiv \sin\phi, \qquad B \equiv \cos\phi, \qquad C \equiv \dot{\phi},$$

and the following notation has been used:

$$\dot{X} \equiv \partial_- X, \qquad X' \equiv \partial_+ X. \tag{7}$$

Substituting in the system of PDEs (6) a series expansion for the functions A, B and C according to (5) one finds

$$A = \chi^{-2}\sum_{n=0}^{+\infty} A_n\chi^n, \quad B = \chi^{-2}\sum_{n=0}^{+\infty} B_n\chi^n, \quad C = \chi^{-1}\sum_{n=0}^{+\infty} C_n\chi^n,$$

with

$$A_0 = 2i\dot{\chi}\chi', \qquad B_0 = -2\dot{\chi}\chi', \qquad C_0 = 2i\dot{\chi}.$$

or

$$A_0 = -2i\dot{\chi}\chi', \qquad B_0 = -2\dot{\chi}\chi', C_0 = -2i\dot{\chi}.$$

We see that the solutions are single-valued around the singularity manifold $\chi = 0$. One finds two possible singular behaviors for A, corresponding to $e^{i\phi}$ or $e^{-i\phi}$ being singular. The recurrence relations for the coefficients are determined by the following equations

$$\begin{aligned}
&(n-2)\dot{\chi}A_n - C_0B_n - B_0C_n = \sum_{m=1}^{n-1} B_mC_{n-m} - \dot{A}_{n-1},\\
&(n-2)\dot{\chi}B_n + C_0A_n + A_0C_n = -\sum_{m=1}^{n-1} A_mC_{n-m} - \dot{B}_{n-1},\\
&(n-1)\chi'C_n - A_n = -C'_{n-1}.
\end{aligned} \tag{8}$$

One finds that the determinant of the coefficients vanishes for $n = 2$ and $n = 4$. For these values one can check that the relations (8) vanish identically. This, together with the undeterminacy in χ, accounts for a complete set of boundary conditions.

We are now ready to discuss the central topic in the paper.

[3]The notation we are using to discuss the simple SG case makes the problem more cumbersome than usual [6], but it establishes the conventions we will adopt in the model we want to study.

3 The Bukhvostov-Lipatov model

The dynamics of the model is characterized by the following Lagrangian [1]:

$$\mathcal{L}_{\rm BL} = \sum_{i=1}^{2} \frac{1}{2}\partial_\nu\phi_i\partial^\nu\phi_i - \mu^2\cos(\lambda_1\phi_1)\cos(\lambda_2\phi_2). \tag{9}$$

By appropriately fixing the mass scale, we can always set $\mu^2 = 4$, so that the equations of motion will read[4]

$$\begin{aligned} \partial_+\partial_-\phi_1 &= \lambda_1\sin(\lambda_1\phi_1)\cos(\lambda_2\phi_2), \\ \partial_+\partial_-\phi_2 &= \lambda_2\cos(\lambda_1\phi_1)\sin(\lambda_2\phi_2). \end{aligned} \tag{10}$$

Without loss of generality we can also assume that $\lambda_1, \lambda_2 > 0$.

As before, we introduce the new variables

$$A_i \equiv \sin(\lambda_i\phi_i), \qquad B_i \equiv \cos(\lambda_i\phi_i), \qquad C_i \equiv \partial_-\phi_i.$$

where $i = 1, 2$ and the repeated index is not summed over. With the notation defined in Eq.(7) the equations of motion will read

$$\dot{A}_i = \lambda_i B_i C_i, \qquad \dot{B}_i = -\lambda_i A_i C_i, \qquad C_1' = \lambda_1 A_1 B_2, \qquad C_2' = \lambda_2 A_2 B_1. \tag{11}$$

We find the following allowed leading behaviors:

$$A_i \sim A_0^{(i)}\chi^{-p_i}, \qquad B_i \sim B_0^{(i)}\chi^{-p_i}, \qquad C_i \sim C_0^{(i)}\chi^{-1},$$

where

$$p_1 \equiv \frac{2}{1+\left(\dfrac{\lambda_2}{\lambda_1}\right)^2}, \qquad p_2 \equiv \frac{2}{1+\left(\dfrac{\lambda_1}{\lambda_2}\right)^2}, \tag{12}$$

and the functions $B_0^{(1)}$, $B_0^{(2)}$ are constrained by the relation

$$B_0^{(1)}B_0^{(2)} = \frac{2\dot{\chi}\chi'}{\lambda_1^2+\lambda_2^2}. \tag{13}$$

The leading coefficients $A_0^{(i)}$, $B_0^{(i)}$, $C_0^{(i)}$ are additionally constrained. One can distinguish between four cases:

- **Case I:**

$$\begin{aligned} C_0^{(1)} &= i\frac{p_1}{\lambda_1}\dot{\chi}, \quad & C_0^{(2)} &= i\frac{p_2}{\lambda_2}\dot{\chi}, \\ A_0^{(1)} &= -iB_0^{(1)}, \quad & A_0^{(2)} &= -iB_0^{(2)}. \end{aligned} \tag{14}$$

- **Case II:**

$$\begin{aligned} C_0^{(1)} &= i\frac{p_1}{\lambda_1}\dot{\chi}, \quad & C_0^{(2)} &= -i\frac{p_2}{\lambda_2}\dot{\chi}, \\ A_0^{(1)} &= -iB_0^{(1)}, \quad & A_0^{(2)} &= iB_0^{(2)}. \end{aligned} \tag{15}$$

[4]Notice that via an appropriate rescaling of the fields one can introduce a single coupling λ_1/λ_2. For ease of notation we will keep writing the two couplings separately.

- **Case III:**

$$C_0^{(1)} = -i\frac{p_1}{\lambda_1}\dot{\chi}, \quad C_0^{(2)} = i\frac{p_2}{\lambda_2}\dot{\chi}, \qquad A_0^{(1)} = iB_0^{(1)}, \quad A_0^{(2)} = -iB_0^{(2)}. \tag{16}$$

- **Case IV:**

$$C_0^{(1)} = -i\frac{p_1}{\lambda_1}\dot{\chi}, \quad C_0^{(2)} = -i\frac{p_2}{\lambda_2}\dot{\chi}, \qquad A_0^{(1)} = iB_0^{(1)}, \quad A_0^{(2)} = iB_0^{(2)}. \tag{17}$$

We notice first af all from Eq.(12) that p_1 and p_2 are integer if and only if $\lambda_1 = \lambda_2$. In this case the equations (10) can be trivially decoupled: the fields $\phi_\pm \equiv \phi_1 \pm \phi_2$ both obey the SG equation.

In general it is conjectured that for PDEs the original full Painlevé property cannot be relaxed to its weaker form introduced in [16] for two-dimensional dynamical systems, where the leading behavior is allowed to be fractional. It is important to notice though that a new choice of the dependent variables might produce a pole type behavior: indeed, if $p_1 = m/n$ (with $m < 2n$), then $p_2 = (2n - m)/n$; it could be that the equations for the n-th power of the functions A_i, B_i, C_i satisfy the full Painlevé property.

We therefore proceed to apply the second step of the algorithm. We first notice that in all four cases above [equations (14)–(17)] one of the coefficients $B_k^{(0)}$ is left undetermined. This allows for a second initial condition arbitrariness, besides the position of the singularity manifold.

Substituting in the equations (11) an Ansatz of the type (5), we find the following recursion relations for the coefficients (for $n > 0$):

$$\begin{aligned}
&(n - p_i)\,\dot{\chi} A_n^{(i)} - \lambda_i C_0^{(i)} B_n^{(1)} - \lambda_i B_0^{(i)} C_n^{(i)} = \lambda_1 \sum_{m=1}^{n-1} B_{n-m}^{(i)} C_m^{(i)} - \dot{A}_{n-1}^{(i)} \\
&(n - p_i)\,\dot{\chi} B_n^{(i)} + \lambda_i C_0^{(i)} A_n^{(1)} + \lambda_i A_0^{(i)} C_n^{(i)} = -\lambda_1 \sum_{m=1}^{n-1} A_{n-m}^{(i)} C_m^{(i)} - \dot{B}_{n-1}^{(i)} \\
&(n - 1)\,\chi' C_n^{(1)} - \lambda_1 B_0^{(2)} A_n^{(1)} - \lambda_1 A_0^{(1)} B_n^{(2)} = \lambda_1 \sum_{m=1}^{n-1} A_{n-m}^{(1)} B_m^{(2)} - C_{n-1}^{(1)} \\
&(n - 1)\,\chi' C_n^{(2)} - \lambda_2 B_0^{(1)} A_n^{(2)} - \lambda_2 A_0^{(2)} B_n^{(1)} = \lambda_2 \sum_{m=1}^{n-1} A_{n-m}^{(2)} B_m^{(1)} - C_{n-1}^{(2)}
\end{aligned} \tag{18}$$

where the coefficients $A_0^{(i)}$, $B_0^{(i)}$, $C_0^{(i)}$ are constrained by (13)–(17).

The set of recurrence relations (18) will completely determine the coefficients $A_n^{(i)}$, $B_n^{(i)}$, $C_n^{(i)}$ unless for some value of n the determinant of the coefficients vanishes[5]. In this latter case, for the Ansatz (5) to be correct the equations (18) must consistently reduce to a set of trivial identities.

We therefore proceed to study the determinant D of the coefficients associated with equation (18). In all four cases (14)–(17) one finds

$$D = (\chi')^2\,(\dot{\chi})^4\,(n+1)n(n-1)(n-2)(n-2p_1)(n-2p_2).$$

[5] These values of n are referred to as resonances [14].

We therefore see that for general values of p_1 and p_2 one finds two resonances at $n = 1$ and $n = 2$. One can check that the rank of the corresponding 6×6 matrices is five. This means that one is allowed for only two extra arbitrary functions in the expansion (5). This implies a total of only four arbitrary coefficients, which is insufficient to allow for complete arbitrariness of initial conditions.

We still have to examine the possibility that the values of the coefficients λ_1 and λ_2 are such that the numbers $2p_1$, $2p_2$ lead to integer resonances. In this case some appropriate power of A_i and B_i would have a pole-like dominant behavior. This is the case provided

$$\left(\frac{\lambda_1}{\lambda_2}\right)^2 = \frac{1}{3}, \quad 1, \quad 3.$$

As mentioned before the case $(\lambda_1/\lambda_2)^2 = 1$ is trivially integrable (and consistently one finds that the rank of the coefficient matrices at the resonances is small enough to accomodate six initial conditions).

The other two cases lead to the appearence of an additional resonance at $n = 3$: if $(\lambda_1/\lambda_2)^2 = 1/3$ or 3, then the determinant D becomes

$$D = \left(\chi'\right)^2 \left(\dot{\chi}\right)^4 (n+1)n(n-1)^2(n-2)(n-3).$$

One finds room for a new arbitrary coefficient at the $n = 3$ level. The rank of the $n = 1$ matrix though remains equal to one, so that even for these values of the couplings the number of arbitrary coefficients is five, which is not enough to have integrability.

4 Discussion

The analysis above leads us to the conclusion that the BL model does not pass the Painlevé test and is therefore classically not integrable, except for the trivial case where $\lambda_1 = \pm\lambda_2$, where the model gives rise to two uncoupled SG equations. This is to be contrasted to the quantum theory of the same model which, as mentioned in the introduction, was shown to be integrable for particular choices of the couplings.

This should not come too surprising, though. Indeed, in the language of two-dimensional conformal field theory one can regard the Lagrangian (9) as a free field Lagrangian perturbed by a potential built out of a combination of vertex operators. It is indeed well known [17] that a two-dimensional conformal field theory may remain integrable after some relevant perturbation, which breaks the conformal invariance, is added.

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