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A Nonlinear Transformation of the Dispersive Long Wave Equations in (2+1) Dimensions and its Applications

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Abstract

A nonlinear transformation of the dispersive long wave equations in (2+1) dimensions is derived by using the homogeneous balance method. With the aid of the transformation given here, exact solutions of the equations are obtained.

1 Introduction: New results

Consider the dispersive long wave equations in $(2+1)$ dimensions,

$$u_{yt} + h_{xx} + \frac{1}{2}\left(u^2\right)_{xy} = 0, \tag{1.1}$$

$$h_t + (uh + u + u_{xy})_x = 0 \tag{1.2}$$

which were first obtained by Boiti *et al* [1] as a compatibility condition for a "weak" Lax pair. A Kac-Moody-Virasoro type Lie algebra for eqs. (1.1) and (1.2) were given by Paquin and Winternitz [2]. Moreover, Sen-yue Lou [3] showed that eqs. (1.1) and (1.2) do not pass the Painlevé test, both in the ARS algorithm and in the WTC approach. Equations (1.1) and (1.2) can be reduced to the $(1+1)$ dimensional model [4]

$$u_t + h_z + \frac{1}{2}\left(u^2\right)_z = 0, \tag{1.3}$$

$$h_t + (uh + u + u_{zz})_z = 0, \tag{1.4}$$

for $u = u(x+y,t) = u(z,t)$, $h = h(x+y,t) = h(z,t)$; the solitary wave solutions of which were obtained by the first author [5].

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In the present paper we show that if $\varphi = \varphi(x, y, t)$ is a nontrivial solution of the linear equation

$$\varphi_t \pm \varphi_{xx} = 0, \tag{1.5}$$

then the nonlinear transformation

$$u(x, y, t) = \pm \frac{2\varphi_x}{\varphi}, \qquad h(x, y, t) = -\frac{2\varphi_x \varphi_y}{\varphi^2} + \frac{2\varphi_{xy}}{\varphi} - 1, \tag{1.6}$$

satisfies eqs. (1.1) and (1.2).

At the time of writing, we have not seen the results like these in the literature. If this assertion is true, then the exact solutions of eqs. (1.1) and (1.2) can be easily obtained by the use of this transformation. In fact, we know that linear equation (1.5) can be handled by a variety of methods, and has many nontrivial solutions which may include arbitrary functions of y, so do eqs. (1.1) and (1.2) according to transformation (1.6). For example, the function

$$\varphi = 1 + \exp\left[a(y)x \mp a^2(y)t + b(y)\right], \tag{1.7}$$

satisfies eq. (1.5), where $a = a(y)$ and $b = b(y)$ are both arbitrary differentiable functions. Thus substituting (1.7) into (1.6), we obtain, after doing some computations, the exact solutions of eqs. (1.1) and (1.2) as follows:

$$u(x, y, t) = \pm a(y) \left\{ 1 + \tanh \frac{1}{2} \left[a(y)x \mp a^2(y)t + b(y)\right] \right\}, \tag{1.8}$$

$$\begin{aligned} h(x, y, t) = {} & \frac{1}{2} a(y) \left[a'(y)x \mp 2a(y)a'(y)t + b'(y)\right] \operatorname{sech}^2 \frac{1}{2} \left[a(y)x \mp a^2(y)t + b(y)\right] \\ & + a'(y) \tanh \frac{1}{2} \left[a(y)x \mp a^2(y)t + b(y)\right] + a'(y) - 1, \end{aligned} \tag{1.9}$$

where it is assumed that $\lim\limits_{y \to \pm\infty} a(y) = \text{const.}$; $\lim\limits_{y \to \pm\infty} b'(y) = \text{const.}$ so that u and h are bounded as y goes to infinity.

In particular, if $a = \text{const}$, $b = cy + d$, c and d are both arbitrary constants, then (1.8) and (1.9) become

$$u(x, y, t) = \pm a \left[1 + \tanh \frac{1}{2} \left(ax \mp a^2 t + cy + d\right) \right], \tag{1.8$'$}$$

$$h(x, y, t) = \frac{ac}{2} \operatorname{sech}^2 \frac{1}{2} \left[ax \mp a^2 t + cy + d\right] - 1, \tag{1.9$'$}$$

which coincide with the results in Ref. [6] (however, only one solution was given there). If taking $a = c$ and $x + y = z$, then (1.8$'$) and (1.9$'$) become the results in Ref.[5] (also, only one solution was given there).

2 Derivation of the nonlinear transformation

Now we use the homogeneous balance method to derive the nonlinear transformation (1.6). A function $\varphi = \varphi(x, y, t)$ is called a quasisolution of eqs. (1.1) and (1.2), if there exists a pair of functions $f = f(\varphi)$ and $g = g(\varphi)$ of one variable only, so that the following expressions

$$u(x,y,t) = \frac{\partial^{l+m+n} f(\varphi)}{\partial x^l \partial y^m \partial t^n} + \text{a suitable linear combination of some partial derivatives (their orders are all lower than } l+m+n\text{) with respect to } x, y \text{ and } t \text{ of } f(\varphi), \tag{2.1}$$

$$h(x,y,t) = \frac{\partial^{p+q+r} g(\varphi)}{\partial x^p \partial y^q \partial t^r} + \cdots, \tag{2.2}$$

which can be rewritten as

$$u(x,y,t) = f^{(l+m+n)}(\varphi)\varphi_x^l \varphi_y^m \varphi_t^n + \text{a polynomial of various partial derivatives of } \varphi(x,y,t)\text{(in spite of } f(\varphi) \text{ and its derivatives, the degree of which is lower than } l+m+n), \tag{2.1$'$}$$

$$h(x,y,t) = g^{(p+q+r)}(\varphi)\varphi_x^p \varphi_y^q \varphi_t^r + \cdots \tag{2.2$'$}$$

is actually a solution of eqs. (1.1) and (1.2). Here $f = f(\varphi)$, $g = g(\varphi)$, $\varphi = \varphi(x, y, t)$, and nonnegative integers l, m, n and p, q, r are all to be determined later. The unwritten parts in (2.2) and (2.2$'$) are similar to those in (2.1) and (2.1$'$), respectively. For the sake of simplicity, we omit these similar parts without explanation.

The homogeneous balance method mainly consists of three steps:

First step: Determine nonnegative integers l, m, n and p, q, r which determine the order of the highest (partial) derivative of $f(\varphi)$ and $g(\varphi)$ in (2.1) and (2.2), respectively. Then

$$(u^2)_{xy} = \left\{\left[f^{(l+m+n)}(\varphi)\right]^2\right\}'' \varphi_x^{2l+1}\varphi_y^{2m+1}\varphi_t^{2n} + \cdots, \tag{2.3}$$

$$h_{xx} = g^{(p+q+r+2)}(\varphi)\varphi_x^{p+2}\varphi_y^q\varphi_t^r + \cdots, \tag{2.4}$$

$$(uh)_x = \left[f^{(l+m+n)}(\varphi)g^{(p+q+r)}(\varphi)\right]' \varphi_x^{l+p+1}\varphi_y^{m+q}\varphi_t^{n+r} + \cdots, \tag{2.5}$$

$$u_{xxy} = f^{(l+m+n+3)}(\varphi)\varphi_x^{l+2}\varphi_y^{m+1}\varphi_t^n + \cdots. \tag{2.6}$$

In order that $(u^2)_{xy}$ and h_{xx} in eq. (1.1) can be partially balanced in view of (2.3) and (2.4), it should be required that the highest degrees in partial derivative for $\varphi(x, y, t)$ appearing in $(u^2)_{xy}$ and h_{xx} are equal; and similarly, in order that $(uh)_x$ and u_{xxy} in eq. (1.2) can be partially balanced in view of (2.5) and (2.6). It should also be required that the highest degrees in partial derivatives for $\varphi(x,y,t)$ appearing in $(uh)_x$ and u_{xxy} are equal. Thus we have

$$\left.\begin{array}{l} 2l+1=p+2,\ 2m+1=q,\ 2n=r \\ l+p+1=l+2,\ m+q=m+1,\ n+r=n \end{array}\right\} \\ \Longrightarrow l=1,\ m=n=0;\ p=q=1,\ r=0. \tag{2.7}$$

If the results in (2.7) are used, then the expressions (2.1) and (2.2) (or (2.1′) and (2.2′)) are of the form

$$u=f_x(\varphi)=f'\varphi_x, \qquad h=g_{xy}(\varphi)+A=g''\varphi_x\varphi_y+g'\varphi_{xy}+A, \tag{2.8}$$

where A is a constant to be determined.

Second step Determine functions $f=f(\varphi)$ and $g=g(\varphi)$ in expressions (2.1) and (2.2) ((2.1′) and (2.2′)). From (2.8) it is easy to deduce that

$$u_{yt}=f'''\varphi_x\varphi_y\varphi_t+f''(\varphi_{xt}\varphi_y+\varphi_x\varphi_{yt}+\varphi_{xy}\varphi_t)+f'\varphi_{xyt}, \tag{2.9}$$

$$\begin{aligned} h_{xx}=&\,g^{(4)}\varphi_x^3\varphi_y+g'''\left(3\varphi_x\varphi_{xx}\varphi_y+3\varphi_x^2\varphi_{xy}\right) \\ &+g''(\varphi_{xxy}\varphi_x+3\varphi_{xx}\varphi_{xy}+3\varphi_x\varphi_{xxy})+g'\varphi_{xxxy}, \end{aligned} \tag{2.10}$$

$$\begin{aligned} \tfrac{1}{2}(u^2)_{xy}=&\,(f''^2+f'f''')\varphi_x^3\varphi_y \\ &+f'f''(3\varphi_x^2\varphi_{xy}+2\varphi_x\varphi_{xx}\varphi_y)+f'^2(\varphi_{xx}\varphi_{xy}+\varphi_x\varphi_{xxy}), \end{aligned} \tag{2.11}$$

$$h_t=g'''\varphi_x\varphi_y\varphi_t+g''(\varphi_{xt}\varphi_y+\varphi_{yt}\varphi_x+\varphi_{xy}\varphi_t)+g'\varphi_{xyt}, \tag{2.12}$$

$$\begin{aligned} (uh)_x=&\,(f''g''+f'g''')\varphi_x^3\varphi_y+f'g''(2\varphi_x\varphi_{xx}\varphi_y+\varphi_x^2\varphi_{xy})+f''g'\varphi_x^2\varphi_{xy} \\ &+f'g''\varphi_x^2\varphi_{xy}+f'g'(\varphi_{xx}\varphi_{xy}+\varphi_x\varphi_{xxy})+Af''\varphi_x^2+Af'\varphi_{xx}, \end{aligned} \tag{2.13}$$

$$u_x=f''\varphi_x^2+f'\varphi_{xx}, \tag{2.14}$$

$$\begin{aligned} (u_{xy})_x=&\,f^{(4)}\varphi_x^3\varphi_y+f'''(3\varphi_x^2\varphi_{xy}+3\varphi_x\varphi_{xx}\varphi_y) \\ &+f''(3\varphi_{xx}\varphi_{xy}+3\varphi_x\varphi_{xxy}+\varphi_{xxx}\varphi_y)+f'\varphi_{xxxy}. \end{aligned} \tag{2.15}$$

Substituting (2.9)–(2.11) and (2.12)–(2.15) into the left hand sides of eqs. (1.1) and (1.2), respectively, and collecting all homogeneous terms of 4, 3, 2 and 1 degree in partial derivatives of $\varphi(x,y,t)$ together, yields

$$\begin{aligned} u_{yt}+h_{xx}+\tfrac{1}{2}(u^2)_{xy}=&\,(g^{(4)}+f''^2+f'f''')\varphi_x^3\varphi_y \\ &+[f'''\varphi_x\varphi_y\varphi_t+g'''(3\varphi_x\varphi_{xx}\varphi_y+3\varphi_x^2\varphi_{xy})+f'f''(3\varphi_x^2\varphi_{xy}+2\varphi_x\varphi_{xxy})] \\ &+[f''(\varphi_{xt}\varphi_y+\varphi_{yt}\varphi_x+\varphi_{xy}\varphi_t)+g''(\varphi_{xxx}\varphi_y+3\varphi_x\varphi_{xxy}+3\varphi_{xx}\varphi_{xy}) \\ &+f'^2(\varphi_{xx}\varphi_{xy}+\varphi_x\varphi_{xxy})]+(f'\varphi_{xyt}+g'\varphi_{xxxy}) \end{aligned} \tag{2.16}$$

and

$$
\begin{aligned}
h_t + (uh + u + u_{xy})_x &= (f^{(4)} + f''g'' + f'g''')\varphi_x^3\varphi_y \\
&+[g'''\varphi_x\varphi_y\varphi_t + f'g''(2\varphi_x\varphi_{xx}\varphi_y + \varphi_x^2\varphi_{xy}) + (f''g' + f'g'')\varphi_x^2\varphi_{xy} \\
&+f'''(3\varphi_x^2\varphi_{xy} + 3\varphi_x\varphi_{xx}\varphi_y)] + [g''(\varphi_{xt}\varphi_y + \varphi_{yt}\varphi_x + \varphi_{xy}\varphi_t) \\
&+g'f'(\varphi_{xx}\varphi_{xy} + \varphi_x\varphi_{xxy}) + f''(3\varphi_{xx}\varphi_{xy} + 3\varphi_x\varphi_{xxy} + \varphi_{xxx}\varphi_y) \\
&+(A+1)f''\varphi_x^2] + [g'\varphi_{xyt} + (A+1)f'\varphi_{xx} + f'\varphi_{xxxy}].
\end{aligned}
\tag{2.17}
$$

Setting the coefficients of $\varphi_x^3\varphi_y$ in (2.16) and (2.17) to zero, yields a system of ordinary differential equations for $f(\varphi)$ and $g(\varphi)$, namely

$$
\left.
\begin{aligned}
&g^{(4)} + f''^2 + f'f''' = 0, \\
&f^{(4)} + f''g'' + f'g''' = 0,
\end{aligned}
\right\}
\tag{2.18}
$$

which admits two solutions

$$
f(\varphi) = \pm 2\ln\varphi, \qquad g = 2\ln\varphi, \tag{2.19}
$$

and thereby

$$
g'g'' = -g''', \qquad g'^2 = -2g''. \tag{2.20}
$$

Third step: Determine the equation satisfied by the quasisolution $\varphi = \varphi(x,y,t)$ of eqs. (1.1) and (1.2). Using (2.18)–(2.20), the expressions (2.16) and (2.17) can be simplified as

$$
\begin{aligned}
&u_{yt} + \eta_{xx} + \frac{1}{2}(u^2)_{xy} \\
&\qquad = \pm\left[\varphi_x\varphi_y g''' + g''\left(\varphi_x\frac{\partial}{\partial y} + \varphi_y\frac{\partial}{\partial x} + \varphi_{xy}\right) + f'\frac{\partial^2}{\partial x\partial y}\right](\varphi_t \pm \varphi_{xx}),
\end{aligned}
\tag{2.21}
$$

$$
\begin{aligned}
&\eta_t + (u\eta + u + u_{xy})_x \\
&\qquad = \left[\varphi_x\varphi_y g''' + g''\left(\varphi_x\frac{\partial}{\partial y} + \varphi_y\frac{\partial}{\partial x} + \varphi_{xy}\right) + g'\frac{\partial^2}{\partial x\partial y}\right](\varphi_t \pm \varphi_{xx}),
\end{aligned}
\tag{2.22}
$$

provided that we take

$$
A = -1. \tag{2.23}
$$

In view of (2.21) and (2.22), it is easily deduced that if $\varphi = \varphi(x,y,t)$ satisfies the linear equation

$$
\varphi_t \pm \varphi_{xx} = 0,
$$

i.e., eq. (1.5), then the right hand sides of (2.21) and (2.22) vanish respectively. This means that (2.8) actually solves eqs. (1.1) and (1.2), provided that

$$
f = \pm 2\ln\varphi, \qquad g = 2\ln\varphi, \qquad (\text{ i.e., } (2.19))
$$

and

$$
A = -1. \qquad (\text{ i.e., } (2.23))
$$

Substituting (2.19) and (2.23) into (2.8), yields

$$
u = \pm\frac{2\varphi_x}{\varphi}, \qquad \eta = -\frac{2\varphi_x\varphi_y}{\varphi^2} + \frac{2\varphi_{xy}}{\varphi} - 1,
$$

which is the nonlinear transformation (1.6).

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