

# Solutions of WDVV Equations in Seiberg-Witten Theory from Root Systems

*R. MARTINI and P.K.H. GRAGERT*

*Faculty of Mathematical Sciences, University of Twente,  
P.O. Box 217, 7500 AE Enschede, The Netherlands*

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## Abstract

We present a complete proof that solutions of the WDVV equations in Seiberg-Witten theory may be constructed from root systems. A generalization to weight systems is proposed.

## 1 Introduction

Recently in  $N = 2$  four-dimensional supersymmetric Yang-Mills theory (Seiberg-Witten effective theory) the following remarkable system of generalized WDVV equations emerged [1, 2]:

$$F_i F_k^{-1} F_j = F_j F_k^{-1} F_i \quad i, j, k = 1, \dots, n, \quad (1)$$

where  $F_i$  is the matrix

$$(F_i)_{mr} = \frac{\partial^3 F}{\partial a_i \partial a_m \partial a_r}$$

of third order derivatives of a function  $F(a_1, \dots, a_n)$ .

This system of nonlinear equations is satisfied by the Seiberg-Witten prepotential defining the low-energy effective action. Moreover the leading perturbative approximation to this exact Seiberg-Witten prepotential should satisfy this set of equations by itself. For instance for the gauge group  $SU(n)$  the expression

$$F_{\text{pert}} = \frac{1}{4} \sum_{i \leq i < j \leq n-1} (a_i - a_j)^2 \log(a_i - a_j)^2 + \frac{1}{2} \sum_{i=1}^{n-1} a_i^2 \log a_i^2$$

defines a solution of the generalized WDVV-system (1).

Of course other gauge groups may be considered and more general solutions may be proposed for classical Lie groups [3, 4]. So although extremely difficult to solve in general, this overdetermined system of nonlinear equations admit exact solutions. In this note we shall present a complete proof that a substantial class of solutions for the system (1) may be constructed from root systems of semisimple Lie algebras.

## 2 Solutions from root systems

Actually we have the following result.

**Theorem 2.1** *Let  $R$  be the root system of a semisimple Lie algebra  $\mathfrak{g}$ . Then the function*

$$F(a) = \frac{1}{4} \sum_{\alpha \in R} (\alpha, a)^2 \log(\alpha, a)^2 \quad (2)$$

*defined on the Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$  satisfies the generalized WDVV equations (1).*

*Here the bracket represents the Killing form of  $\mathfrak{g}$ .*

In order to prove this theorem we show that we can rewrite the system (1) into an equivalent form which is more suitable for our purposes.

**Proposition 2.1** *Let  $G = \sum_{i=1}^n c_i F_i$  be an invertible linear combination of the matrices  $F_i$  with coefficients  $c_i$  which may depend on  $a$ .*

*Then  $F$  is a solution of the WDVV-system (1) if and only if*

$$F_i G^{-1} F_j = F_j G^{-1} F_i \quad i, j = 1, \dots, n. \quad (3)$$

**Proof.** Suppose  $F$  satisfies the WDVV-system (1). Then by inverting these equations we obtain

$$F_j^{-1} F_k F_i^{-1} = F_i^{-1} F_k F_j^{-1}.$$

By taking linear combinations we get

$$F_j^{-1} G F_i^{-1} = F_i^{-1} G F_j^{-1}.$$

Inverting once more yields the equations (3). To prove the converse we set  $C_i = G^{-1} F_i$ . Then (3) implies that  $C_i$  and  $C_j$  commute. So

$$G^{-1} F_i F_k^{-1} F_j = C_i C_k^{-1} C_j = C_j C_k^{-1} C_i = G^{-1} F_j F_k^{-1} F_i.$$

Thus  $F$  is a solution of the WDVV-system (1).

We continue by proving our main result, theorem 2.1. Without restriction we may suppose that the root system is irreducible. Let  $\alpha_1, \dots, \alpha_n$  be a basis of the Cartan subalgebra  $\mathfrak{h}$  of the Lie algebra  $\mathfrak{g}$  consisting of simple roots. Moreover let  $a = \sum_{i=1}^n a_i \alpha_i$ .

For the linear combination  $G = \sum_{i=1}^n a_i F_i$ , where  $F$  is given in (2), we have

$$G_{km} = \sum_i a_i F_{ikm} = \sum_i a_i \sum_{\alpha \in R} \frac{(\alpha, \alpha_i)(\alpha, \alpha_k)(\alpha, \alpha_m)}{(\alpha, a)} = \sum_{\alpha \in R} (\alpha, \alpha_k)(\alpha, \alpha_m) = (\alpha_k, \alpha_m)$$

using the very definition of the Killing form. So in this case  $G$  equals the matrix of the Killing form on a basis of simple roots.

For this choice of  $G$  we have

$$(F_i G^{-1} F_j)_{rs} = 4 \sum_{\alpha, \beta \in R^+} \frac{(\alpha, \alpha_i)(\alpha, \alpha_r)(\alpha, \beta)(\beta, \alpha_j)(\beta, \alpha_s)}{(\alpha, a)(\beta, a)},$$

where  $R^+$  denotes the positive part of the root system. Consequently

$$(F_i G^{-1} F_j - F_j G^{-1} F_i)_{rs} = 4 \sum_{\alpha, \beta \in R^+} \frac{(\alpha, \beta)(\alpha, \alpha_r)(\beta, \alpha_s)[(\alpha, \alpha_i)(\beta, \alpha_j) - (\alpha, \alpha_j)(\beta, \alpha_i)]}{(\alpha, a)(\beta, a)}$$

We have to prove that this last expression vanishes, but by a close inspection we see that it is antisymmetric in  $r, s$ . Therefore we may also prove that

$$(F_i G^{-1} F_j - F_j G^{-1} F_i)_{rs} - (F_i G^{-1} F_j - F_j G^{-1} F_i)_{sr}$$

vanishes. This last expression equals

$$4 \sum_{\alpha, \beta \in R^+} \frac{(\alpha, \beta)[(\alpha, \alpha_i)(\beta, \alpha_j) - (\alpha, \alpha_j)(\beta, \alpha_i)][(\alpha, \alpha_r)(\beta, \alpha_s) - (\alpha, \alpha_s)(\beta, \alpha_r)]}{(\alpha, a)(\beta, a)}$$

which we abbreviate to

$$= \sum_{\{\alpha, \beta\}} t_{\{\alpha, \beta\}}, \quad (4)$$

where  $\{\alpha, \beta\}$  denotes an unordered pair of different roots  $\alpha, \beta$  in  $R^+$ . To finish our proof we consider two separate cases. First we consider the case that the Lie algebra  $\mathfrak{g}$  is simply-laced. In this case the roots have equal length which we suppose to be normalized so that the squared lengths equal 2.

Now consider a (unordered) pair of roots  $\{\alpha, \beta\}$  in  $R^+$  such that  $(\alpha, \beta) < 0$ . Then since the roots have equal length it follows (see table 1, p.45 in [5]) that  $(\alpha, \beta) = -1$  and therefore (lemma 2, p.45 in [5]) that  $\alpha + \beta$  is again a root in  $R^+$ . Moreover:  $(\alpha, \alpha + \beta) = 1$ ,  $(\beta, \alpha + \beta) = 1$ . Conversely if  $\{\alpha', \beta'\}$  is a pair of roots in  $R^+$  such that  $(\alpha', \beta') > 0$  then necessarily  $(\alpha', \beta') = 1$  and a small calculation shows that there is a unique pair of roots  $\{\alpha, \beta\}$  in  $R^+$  such that  $\{\alpha', \beta'\} = \{\alpha, \alpha + \beta\}$  or  $\{\alpha', \beta'\} = \{\beta, \alpha + \beta\}$ .

Consequently in this simply-laced case the sum  $\sum_{\{\alpha, \beta\}} t_{\{\alpha, \beta\}}$  may be split up into a sum of expressions of the form

$$t_{\{\alpha, \beta\}} + t_{\{\alpha, \alpha + \beta\}} + t_{\{\beta, \alpha + \beta\}}, \quad (5)$$

where  $\{\alpha, \beta\}$  represents an unordered pair of roots in  $R^+$  with  $(\alpha, \beta) = -1$ . Using the relation

$$\frac{1}{(\alpha, a)(\beta, a)} = \frac{1}{(\alpha, a)(\alpha + \beta, a)} + \frac{1}{(\beta, a)(\alpha + \beta, a)}$$

it is now easy to see that the expression (5) and with it the sum (4) vanishes. This completes the proof of the theorem in the simply-laced case.

In the non simply-laced case we have to consider also pair of roots of unequal length.

First observe that when the root system is of type  $G_2$  the dimension of the Cartan subalgebra equals 2 and therefore the theorem becomes trivial. So we may ignore this special case. In the other non simply-laced cases the ratio of the squared length of a long and a short root equals 2. We may assume the length of the short root to be equal to 1.

Now consider a pair  $\{\alpha, \beta\}$  of roots in  $R^+$  with  $\alpha$  a short and  $\beta$  a long root such that  $(\alpha, \beta) < 0$ . Then it follows (table 1, p. 45 in [5]) that  $(\alpha, \beta) = -1$ . We construct the

$\alpha$ -string through  $\beta$ . It consists of  $\beta, \beta + \alpha, \beta + 2\alpha$ . For the inner product of the roots  $\alpha, \beta, \beta + \alpha, \beta + 2\alpha$  we have

$$(\beta, \beta + \alpha) = (\beta + \alpha, \beta + 2\alpha) = (\alpha, \beta + 2\alpha) = 1$$

and

$$(\beta, \beta + 2\alpha) = (\alpha, \beta + \alpha) = 0.$$

We obtain three pairs  $\{\alpha + \beta, \beta\}, \{\alpha + \beta, 2\alpha + \beta\}, \{\alpha, 2\alpha + \beta\}$  of a short and a long root with inner product equal to 1. Conversely, a simple calculation shows that each pair  $\{\alpha', \beta'\}$  with  $\alpha'$  short,  $\beta'$  long and  $(\alpha', \beta') = 1$  is obtained uniquely in this way. Using the relation

$$\frac{1}{(\alpha, a)(\beta, a)} = \frac{1}{(\alpha + \beta, a)(\beta, a)} + \frac{1}{(\alpha + \beta, a)(2\alpha + \beta, a)} + \frac{1}{(\alpha, a)(2\alpha + \beta, a)}$$

it is now easily seen that

$$t_{\{\alpha, \beta\}} + t_{\{\alpha + \beta, \beta\}} + t_{\{\alpha + \beta, 2\alpha + \beta\}} + t_{\{\alpha, 2\alpha + \beta\}} = 0.$$

For pairs of equal length we may argue as in the simply-laced case. Consequently the sum  $\sum_{\{\alpha, \beta\}} t_{\{\alpha, \beta\}}$  split up into vanishing parts consisting of three or four terms. This completes the proof of the theorem in the non simply-laced case.

As is well-known a root system is the weight system of the adjoint representation of some semisimple Lie algebra  $\mathfrak{g}$ . So one may try to construct more general solutions in terms of weight systems.

Actually generalizing the expression (2) in a natural way we obtain for any representation  $\varphi$  of the Lie algebra  $\mathfrak{g}$  the function

$$F_\varphi(a) = \sum_{\lambda \in W} \lambda(a)^2 \log \lambda(a)^2 \tag{6}$$

defined on the Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$ . Here summation is over the set of weights of the representation  $\varphi$ .

From experience with concrete representations we know that unfortunately in general formula (6) does not satisfy the WDVV equations despite the fact that for some representations the WDVV equations indeed hold. For a short review from the point of view of physics see e.g. [4]. To our best knowledge at present no precise results are known in the literature. We hope to report on these questions in the near future.

## References

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