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Neumann and Bargmann Systems Associated with an Extension of the Coupled KdV Hierarchy

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Abstract

An eigenvalue problem with a reference function and the corresponding hierarchy of nonlinear evolution equations are proposed. The bi-Hamiltonian structure of the hierarchy is established by using the trace identity. The isospectral problem is nonlinearized as to be finite-dimensional completely integrable systems in Liouville sense under Neumann and Bargmann constraints.

1 Introduction

A major difficulty in theory of integrable systems is that there is to date no completely systematic method for choosing properly an isospectral problem $\psi_x = M\psi$ so that the zero-curvature representation $M_t - \overline{N}_x + [M, \overline{N}] = 0$ is nontrivial. By inserting a reference function into AKNS and WKI isospectral problems, we have obtained successfully two new hierarchies [1, 2].

The coupled KdV hierarchy associated with the isospectral problem

$$\psi_x = M\psi, \qquad M = \begin{pmatrix} -\frac{1}{2}\lambda + \frac{1}{2}u & -v \\ 1 & \frac{1}{2}\lambda - \frac{1}{2}u \end{pmatrix} \tag{1.1}$$

is discussed by D. Levi, A. Sym and S. Wojciechowsk [3]. The isospectral problem (1.1) has been nonlinearized as finite-dimensional completely integrable systems in Liouville sense [4].

In this paper, we introduce the eigenvalue problem

$$\psi_x = M\psi, \qquad M = \begin{pmatrix} -\frac{1}{2}\lambda + \frac{1}{2}u & -v \\ f(v) & \frac{1}{2}\lambda - \frac{1}{2}u \end{pmatrix}, \tag{1.2}$$

where u and v are two scalar potentials, λ is a constant spectral parameter and $f(v)$ called reference function is an arbitrary smooth function. The bi-Hamiltonian structure of the corresponding hierarchy is established by using the trace identity [5, 6]. Since the reference function $f(v)$ in (1.2) can be chosen arbitrarily, many new hierarchies and their Hamiltonian forms are obtained.When $f=(-v)^{\beta}$ $(\beta \geq 0)$, the isospectral problem (1.2) is nonlinearized as finite-dimensional completely integrable sysstems in Liouville sense under Neumann and Bargmann constraints between the potentials and eigenfunctions.

2 Preliminaries

Consider the adjoint representation of (1.2)

$$N_x = MN - NM, \qquad N = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \sum_{j=0}^{\infty} \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix} \lambda^{-j} \tag{2.1}$$

which leads to

$$c_0 = b_0 = 0, \qquad a_0 = -\frac{1}{2}\alpha \quad \text{(constant)}, \tag{2.2}$$

$$c_1 = \alpha f(v), \qquad b_1 = -\alpha v, \qquad a_1 = 0, \tag{2.3}$$

$$c_2 = \alpha(f'(v)v_x + uf(v)), \qquad b_2 = \alpha(v_x - uv), \qquad a_2 = -\alpha v f(v), \tag{2.4}$$

$$a_j = -\partial^{-1}(vc_j + f(v)b_j), \tag{2.5}$$

$$\begin{pmatrix} c_1 \\ b_1 \end{pmatrix} = \alpha \begin{pmatrix} f(v) \\ -v \end{pmatrix}, \qquad \begin{pmatrix} c_{j+1} \\ b_{j+1} \end{pmatrix} = L \begin{pmatrix} c_j \\ b_j \end{pmatrix}, \qquad j = 1, 2, \ldots, \tag{2.6}$$

where $\partial = \dfrac{d}{dx}$, $\partial\partial^{-1} = \partial^{-1}\partial = 1$,

$$L = \begin{pmatrix} \partial + u + 2f\partial^{-1}v & 2f\partial^{-1}f \\ -2v\partial^{-1}v & -\partial + u - 2v\partial^{-1}f \end{pmatrix}.$$

It is easy from (1.2) and (2.1) to calculate that

$$\operatorname{tr}\left(N\frac{\partial M}{\partial \lambda}\right) = -a, \qquad \operatorname{tr}\left(N\frac{\partial M}{\partial u}\right) = a, \qquad \operatorname{tr}\left(N\frac{\partial M}{\partial v}\right) = -c + f'(v)b.$$

Noticing the trace identity [5, 6]

$$\left(\frac{\delta}{\delta u}, \frac{\delta}{\delta v}\right)(-a) = \frac{\partial}{\partial \lambda}(a, -c + f'(v)b),$$

hence we deduce that

$$\left(\frac{\delta}{\delta u}, \frac{\delta}{\delta v}\right) H_j = \left(G^{(1)}_{j-2}, G^{(2)}_{j-2}\right), \qquad H = \frac{a_{j+1}}{j}, \tag{2.7}$$

where

$$G^{(1)}_{j-2} = a_j, \qquad G^{(2)}_{j-2} = -c_j + f'(v)b_j. \tag{2.8}$$

3 The hierarchy and its Hamiltonian structure

Let ψ satisfy the isospectral problem (1.2) and the auxiliary problem

$$\psi_t = \overline{N}\psi, \qquad \overline{N} = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \tag{3.1}$$

where

$$A = A_m + \sum_{j=0}^{m-1} a_j \lambda^{m-j}, \qquad B = \sum_{j=1}^{m} b_j \lambda^{m-j}, \qquad C = \sum_{j=1}^{m} c_j \lambda^{m-j}.$$

The compatible condition $\psi_{xt} = \psi_{tx}$ between (1.1) and (3.1) gives the zero-curvature representation $M_t - \overline{N}_x + [M, \overline{N}] = 0$, from which we have

$$A_m = w(\partial + u)c_m + wf'(v)(\partial - u)b_m,$$

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \theta_0 L \begin{pmatrix} c_m \\ b_m \end{pmatrix} = \theta_0 \begin{pmatrix} c_{m+1} \\ b_{m+1} \end{pmatrix}, \tag{3.2}$$

where $w = \frac{1}{2}(vf'(v) + f)^{-1}$,

$$\theta_0 = \begin{pmatrix} 2\partial w & -2\partial w f'(v) \\ 2wv & 2wf \end{pmatrix}. \tag{3.3}$$

By (2.6) we know that Eqs.(3.2) are equivalent to the hierarchy of nonlinear evolution equations

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \theta_0 L^m \begin{pmatrix} \alpha f(v) \\ -\alpha v \end{pmatrix}, \qquad m = 1, 2, \ldots. \tag{3.4}$$

Let the potentials u and v in (1.2) belong to the Schwartz space $S(-\infty, +\infty)$ over $(-\infty, +\infty)$. Noticing (2.5) and (2.8) we get

$$\begin{pmatrix} c_j \\ b_j \end{pmatrix} = \theta_1 \begin{pmatrix} G^{(1)}_{j-2} \\ G^{(2)}_{j-2} \end{pmatrix}, \qquad \theta_1 = \begin{pmatrix} -2wf'(v)\partial & -2wf \\ -2w\partial & 2wv \end{pmatrix}. \tag{3.5}$$

Then the recursion relations (2.5), (2.6) and the hierarchy (3.2) can be written as

$$G_{-2} = -\frac{1}{2}\alpha(1, 0)^T, \quad G_{-1} = -\alpha(0, vf'(v) + f)^T, \quad G_0 = -\alpha(vf, uf + uvf'(v))^T,$$

$$KG_{j-1} = JG_j, \tag{3.6}$$

$$(u_t, v_t)^T = JG_{m-1} = KG_{m-2}, \tag{3.7}$$

where $J = \theta_0\theta_1$ and $K = \theta_0 L \theta_1$ are two skew-symmetric operators,

$$J = \begin{pmatrix} 0 & -2\partial w \\ -2w\partial & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix},$$

in which

$$\begin{cases} K_{11} = & -2\partial - 4\partial w(\partial f'(v) + f'(v)\partial)w\partial, \\ K_{12} = & -2\partial wu + 4\partial w(f'(v)\partial v - \partial f)w, \\ K_{21} = & -2wu\partial + 4w(f\partial - v\partial f'(v))w\partial, \\ K_{22} = & -4w(v\partial f + f\partial v)w. \end{cases}$$

From (2.7) we obtain the desired bi-Hamiltonian form of (3.7)

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = J \begin{pmatrix} \dfrac{\delta}{\delta u} \\ \dfrac{\delta}{\delta v} \end{pmatrix} H_{m+1} = K \begin{pmatrix} \dfrac{\delta}{\delta u} \\ \dfrac{\delta}{\delta v} \end{pmatrix} H_m. \tag{3.8}$$

4 Nonlinearization of the isospectral problem

Let λ_j and $\psi(x) = (q_j(x), p_j(x))^T$ be eigenvalue and the associated eigenfunction of (1.2). Through direct verification we know that the functional gradient $\nabla_{(u,v)}\lambda_j = \left(\dfrac{\delta\lambda_j}{\delta u}, \dfrac{\delta\lambda_j}{\delta v}\right)$ satisfies

$$\nabla_{(u,v)}\lambda_j = \left(q_j p_j, -p_j^2 - f'(v)q_j^2\right), \tag{4.1}$$

$$\theta_1\nabla\lambda_j = \begin{pmatrix} p_j^2 \\ -q_j^2 \end{pmatrix}, \qquad L\begin{pmatrix} p_j^2 \\ -q_j^2 \end{pmatrix} = \lambda_j \begin{pmatrix} p_j^2 \\ -q_j^2 \end{pmatrix} \tag{4.2}$$

in view of (1.2). Substituting the first expression of (4.2) into the second expression and acting with θ_0 upon once, we have

$$K\nabla\lambda_j = \lambda_j J\nabla\lambda_j. \tag{4.3}$$

So, the Lenard operator pair K, J and their gradient series G_j satisfy the basic conditions (3.6) and (4.3) given in Refs. [7, 8] for the nonlinearization of the eigenvalue problem (1.2).

Proposition 4.1. *When $f(v) = (-v)^\beta$ ($\beta \geq 0$), the isospectral problem (1.2) can be nonlinearized as to be a Neumann system.*

In fact, the Neumann constraint $G_{-1}|_{\alpha=1} = \sum_{j=1}^{N} \nabla\lambda_j$ gives

$$\langle q, p\rangle = 0, \langle p, p\rangle = (\beta+1)(-v)^\beta + \beta(-1)^{\beta-1}\langle q, q\rangle. \tag{4.4}$$

By differentiating (4.4) with respect to x and using (1.2), we have

$$\begin{cases} u = \dfrac{1}{\beta+1}\left(\dfrac{\langle\Lambda p, p\rangle}{\langle p, p\rangle} + \beta\dfrac{\langle\Lambda q, q\rangle}{\langle q, q\rangle}\right), \\ v = \langle q, q\rangle. \end{cases} \tag{4.5}$$

Substituting (4.5) into the equations for the eigenfunctions

$$\begin{pmatrix} q_{jx} \\ p_{jx} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\lambda_j + \frac{1}{2}u & -v \\ (-v)^\beta & \frac{1}{2}\lambda_j - \frac{1}{2}u \end{pmatrix} \begin{pmatrix} q_j \\ p_j \end{pmatrix}, \qquad j = 1, \ldots, N, \tag{4.6}$$

we obtain the Neumann system

$$\begin{cases} q_x = -\frac{1}{2}\Lambda q - \langle q, q\rangle p + \frac{1}{2(\beta+1)} \left(\frac{\langle \Lambda p, p\rangle}{\langle p, p\rangle} + \beta \frac{\langle \Lambda q, q\rangle}{\langle q, q\rangle} \right) q, \\ p_x = \frac{1}{2}\Lambda p + \langle p, p\rangle q - \frac{1}{2(\beta+1)} \left(\frac{\langle \Lambda p, p\rangle}{\langle p, p\rangle} + \beta \frac{\langle \Lambda q, q\rangle}{\langle q, q\rangle} \right) p, \\ \langle p, p\rangle = (-1)^\beta \langle q, q\rangle^\beta, \qquad \langle q, p\rangle = 0. \end{cases} \tag{4.7}$$

where $p = (p_1, \ldots, p_N)^T$, $q = (q_1, \ldots, q_N)^T$, $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$, and $\langle , \rangle$ stands for the canonical inner product in $\mathbf{R}^N$.

Proposition 4.2. *When* $f(v) = (-v)^\beta$ $(\beta \geq 0)$*, the isospectral problem* (1.2) *can be nonlinearized as to be a Bargmann system.*

In fact, the Bargmann constraint $G_0|_{\alpha=1} = \sum_{j=1}^{N} \nabla\lambda_j$ gives

$$\begin{cases} u = \frac{1}{\beta+1}\langle p, p\rangle \langle q, p\rangle^{-\frac{\beta}{\beta+1}} - \frac{\beta}{\beta+1}\langle q, q\rangle\langle q, p\rangle^{-\frac{1}{\beta+1}}, \\ v = -\langle q, p\rangle^{\frac{1}{\beta+1}}. \end{cases} \tag{4.8}$$

Substituting (4.8) into (4.6), we obtain the finite-dimensional Hamiltonian system

$$\begin{cases} q_x = -\frac{1}{2}\Lambda q + \langle q, p\rangle^{\frac{1}{\beta+1}} p + \frac{1}{2(\beta+1)}\langle p, p\rangle\langle q, p\rangle^{-\frac{\beta}{\beta+1}} q \\ \qquad - \frac{\beta}{2(\beta+1)}\langle q, q\rangle\langle q, p\rangle^{-\frac{1}{\beta+1}} q = \frac{\partial H}{\partial p}, \\ p_x = \frac{1}{2}\Lambda p - \frac{1}{2(\beta+1)}\langle p, p\rangle\langle q, p\rangle^{-\frac{\beta}{\beta+1}} p + \langle q, p\rangle^{\frac{\beta}{\beta+1}} \\ \qquad + \frac{\beta}{2(\beta+1)}\langle q, q\rangle\langle q, p\rangle^{-\frac{1}{\beta+1}} p = -\frac{\partial H}{\partial q}. \end{cases} \tag{4.9}$$

The Hamiltonian is

$$H = -\frac{1}{2}\langle \Lambda q, p\rangle + \frac{1}{2}\langle p, p\rangle\langle q, p\rangle^{\frac{1}{\beta+1}} - \frac{1}{2}\langle q, q\rangle\langle q, p\rangle^{\frac{\beta}{\beta+1}}.$$

5 Integrability of the Neumann system

The Poisson brackets of two functions in symplectic space $(\mathbf{R}^{2N}, dp \wedge dq)$ are defined as

$$(F, G) = \sum_{j=1}^{N} \left(\frac{\partial F}{\partial q_j}\frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j}\frac{\partial G}{\partial q_j} \right) = \langle F_q, G_p\rangle - \langle F_p, G_q\rangle.$$

The functions defined by $(m = 0, 1, 2, \ldots)$

$$F_m = -\frac{1}{2}\langle \Lambda^{m+1} q, p\rangle - \frac{1}{2}\sum_{i+j=m} \begin{vmatrix} \langle \Lambda^i q, q\rangle & \langle \Lambda^i q, p\rangle \\ \langle \Lambda^j p, q\rangle & \langle \Lambda^j p, p\rangle \end{vmatrix}$$

are in involution in pairs (see, [9]).

Consider the Moser constraint on the tangent bundle

$$TS^{N-1} = \left\{ (p, q) \in \mathbf{R}^{2N} | F = \langle q, p\rangle = 0, \ G = \frac{1}{2(\beta+1)}(\langle p, p\rangle - (-1)^\beta \langle q, q\rangle^\beta) = 0 \right\}.$$

Through direct calculations we have

$$(F, F_m) = 0, \qquad (F, G) = \langle p, p\rangle,$$

$$(F_m, G) = -\frac{1}{2(\beta+1)}\left(\langle \Lambda^{m+1} p, p\rangle + (-1)^\beta \beta \langle q, q\rangle^{\beta-1}\langle \Lambda^{m+1} q, q\rangle\right).$$

Thus the Lagrangian multipliers are

$$\mu_m = \frac{(F_m, G)}{(F, G)} = -\frac{1}{(\beta+1)}\left(\frac{\langle \Lambda^{m+1} p, p\rangle}{\langle p, p\rangle} + (-1)^\beta \beta \frac{\langle q, q\rangle^{\beta-1}}{\langle p, p\rangle}\langle \Lambda^{m+1} q, q\rangle\right).$$

Since $F = 0$ on the tangent bundle TS^{N-1}, the restriction of the canonical equation of $H^* = F_0 - \mu_0 F$ on TS^{N-1} is

$$\begin{cases} q_x = F_{0,p} - \mu_0 F_p|_{TS^{N-1}}, \\ p_x = -F_{0,q} + \mu_0 F_q|_{TS^{N-1}} \end{cases}$$

which is exactly the Neumann system (4.7).

Theorem 5.1. *The Neumann system* (4.7) $(TS^{N-1}, dp \wedge dq|_{TS^{N-1}}, H^* = F_0 - \mu_0 F)$ *is completely integrable in Liouville sense.*

Proof. Let $F_m^* = F_m - \mu_m F$, $m = 1, \ldots, N-1$, then it is easy to verify $(F_k^*, F_l^*) = 0$ on TS^{N-1}. Hence $\{F_m^*\}$ is an involutive system.

6 Integrability of the Bargmann system

Let

$$\Gamma_k = \sum_{\substack{j=1 \\ j \neq k}}^{N} \frac{B_{kj}^2}{\lambda_k - \lambda_j}, \tag{6.1}$$

where $B_{kj} = p_k q_j - p_j q_k$, we have (see Refs. [9, 10])

Lemma 6.1.

$$\left(\langle q, p\rangle, p_l^2\right) = 2p_l^2, \qquad \left(\langle q, p\rangle, q_l^2\right) = -2q_l^2, \tag{6.2}$$

$$\begin{aligned} &(p_k^2, \Gamma_l) = \frac{-4B_{lk}}{\lambda_l - \lambda_k} p_k p_l, \qquad (q_k^2, \Gamma_l) = \frac{-4B_{lk}}{\lambda_l - \lambda_k} q_k q_l, \\ &(q_k p_k, \Gamma_l) = \frac{-2B_{lk}}{\lambda_l - \lambda_k} (p_k q_l + q_k p_l). \end{aligned} \tag{6.3}$$

Lemma 6.2.

$$(\Gamma_k, \Gamma_l) = (\langle q, p\rangle, \Gamma_l) = (\langle q, p\rangle, q_l p_l) = 0, \tag{6.4}$$

$$\left(p_k^2, p_l^2\right) = \left(q_k^2, q_l^2\right) = (q_k p_k, q_l p_l) = 0, \tag{6.5}$$

$$\left(q_k p_k, p_l^2\right) = 2p_k p_l \delta_{kl}, \qquad \left(q_k^2, p_l^2\right) = 4q_k p_l \delta_{kl}, \qquad \left(q_k^2, p_l q_l\right) = 2q_k q_l \delta_{kl}. \tag{6.6}$$

Proposition 6.1. *Let*

$$E_k = \frac{1}{2}\langle q, p\rangle^{\frac{1}{\beta+1}} p_k^2 - \frac{1}{2}\langle q, p\rangle^{\frac{\beta}{\beta+1}} q_k^2 - \frac{1}{2}\lambda_k q_k p_k - \frac{1}{2}\Gamma_k,$$

the $E_1, \ldots, E_N$ *constitute an* N*-involutive system.*

Proof. Obviously $(E_k, E_l) = 0$ for $k = l$. Suppose $k \neq l$, in virtue of (6.4)–(6.6) and the property of Poisson bracket in $(\mathbf{R}^{2N}, dp \wedge dq)$, we have

$$\begin{aligned} 4(E_k, E_l) = &\frac{1}{\beta+1} p_k^2 \langle q, p\rangle^{\frac{1-\beta}{\beta+1}} \left(\langle q, p\rangle, p_l^2\right) + \frac{1}{\beta+1} p_l^2 \langle q, p\rangle^{\frac{1-\beta}{\beta+1}} \left(p_k^2, \langle q, p\rangle\right) \\ &- \frac{1}{\beta+1} p_k^2 \left(\langle q, p\rangle, q_l^2\right) - \frac{\beta}{\beta+1} q_l^2 \left(p_k^2, \langle q, p\rangle\right) - \langle q, p\rangle^{\frac{1}{\beta+1}} \left(p_k^2, \Gamma_l\right) \\ &- \langle q, p\rangle^{\frac{1}{\beta+1}} \left(\Gamma_k, p_l^2\right) - \frac{\beta}{\beta+1} q_k^2 \left(\langle q, p\rangle, p_l^2\right) - \frac{1}{\beta+1} p_l^2 \left(q_k^2, \langle q, p\rangle\right) \\ &+ \frac{\beta}{\beta+1} q_k^2 \langle q, p\rangle^{\frac{\beta-1}{\beta+1}} \left(\langle q, p\rangle, q_l^2\right) + \frac{\beta}{\beta+1} q_l^2 \langle q, p\rangle^{\frac{\beta-1}{\beta+1}} \left(q_k^2, \langle q, p\rangle\right) \\ &+ \langle q, p\rangle^{\frac{\beta}{\beta+1}} \left(q_k^2, \Gamma_l\right) + \langle q, p\rangle^{\frac{\beta}{\beta+1}} \left(\Gamma_k, q_l^2\right) + \lambda_k (q_k p_k, \Gamma_l) + \lambda_l (\Gamma_k, q_l p_l). \end{aligned}$$

Substituting (6.2) and (6.3) into the above equation yields $(E_k, E_l) = 0$.

Consider a bilinear function $Q_z(\xi, \eta)$ on $\mathbf{R}^N$:

$$Q_z(\xi, \eta) = \langle (z - \Lambda)^{-1}\xi, \eta\rangle = \sum_{k=1}^{N} \frac{\xi_k \eta_k}{z - \lambda_k} = \sum_{m=0}^{\infty} z^{-m-1} \langle \Lambda^m \xi, \eta\rangle.$$

The generating function of Γ_k is (see, [9, 10])

$$\begin{vmatrix} Q_z(q, q) & Q_z(q, p) \\ Q_z(p, q) & Q_z(p, p) \end{vmatrix} = \sum_{k=1}^{N} \frac{\Gamma_k}{z - \lambda_k}.$$

Hence the generating function of E_k is

$$\begin{aligned} &\frac{1}{2}\langle q, p\rangle^{\frac{1}{\beta+1}} Q_z(p, p) - \frac{1}{2}\langle q, p\rangle^{\frac{\beta}{\beta+1}} Q_z(q, q) - \frac{1}{2} Q_z(\Lambda q, p) \\ &\qquad - \frac{1}{2} \begin{vmatrix} Q_z(q, q) & Q_z(q, p) \\ Q_z(p, q) & Q_z(p, p) \end{vmatrix} = \sum_{k=1}^{N} \frac{E_k}{z - \lambda_k}. \end{aligned} \tag{6.7}$$

Substituting the Laurent expansion of Q_z and

$$(z-\lambda_k)^{-1} = \sum_{m=0}^{\infty} z^{-m-1}\lambda_k^m$$

in to both sides of (6.7) respectively, we have

Proposition 6.2. *Let*

$$F_m = \sum_{k=1}^{N} \lambda_k^m E_k, \qquad m = 0, 1, 2, \ldots$$

then

$$F_0 = \frac{1}{2}\langle q,p\rangle^{\frac{1}{\beta+1}}\langle p,p\rangle - \frac{1}{2}\langle q,p\rangle^{\frac{\beta}{\beta+1}}\langle q,q\rangle - \frac{1}{2}\langle \Lambda q,p\rangle,$$

$$F_m = \frac{1}{2}\langle q,p\rangle^{\frac{1}{\beta+1}}\langle \Lambda^m p,p\rangle - \frac{1}{2}\langle q,p\rangle^{\frac{\beta}{\beta+1}}\langle \Lambda^m q,q\rangle - \frac{1}{2}\langle \Lambda^{m+1} q,p\rangle - \frac{1}{2}\sum_{j=1}^{m}\left|\begin{array}{cc} \langle \Lambda^{j-1}q,q\rangle & \langle \Lambda^{j-1}q,p\rangle \\ \langle \Lambda^{m-j}p,q\rangle & \langle \Lambda^{m-j}p,p\rangle \end{array}\right|.$$

Moreover, $(F_k, F_l) = 0$.

Hence we arrive at the following theorem.

Theorem 6.1. *The Bargmann system defined by* (4.9) *is completely integrable in Liouville sense in the symplectic manifold* $(\mathbf{R}^{2N}, dp\wedge dq)$.

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