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Dynamical Correlation Functions for an Impenetrable Bose Gas with Neumann or Dirichlet Boundary Conditions

Takeo KOJIMA

Department of Mathematics, College of Science and Technology, Nihon University, 1-8, Kanda-Surugadai, Chiyoda Tokyo 101, Japan

E-mail: kojima@math.cst.nihon-u.ac.jp

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Abstract

We study the time and temperature dependent correlation functions for an impenetrable Bose gas with Neumann or Dirichlet boundary conditions $\langle\psi(x_1,0)\psi^\dagger(x_2,t)\rangle_{\pm,T}$. We derive the Fredholm determinant formulae for the correlation functions, by means of the Bethe Ansatz. For the special case $x_1 = 0$, we express correlation functions with Neumann boundary conditions $\langle\psi(0,0)\psi^\dagger(x_2,t)\rangle_{+,T}$, in terms of solutions of nonlinear partial differential equations which were introduced in [1] as a generalization of the nonlinear Schrödinger equations. We generalize the Fredholm minor determinant formulae of ground state correlation functions $\langle\psi(x_1)\psi^\dagger(x_2)\rangle_{\pm,0}$ in [2], to the Fredholm determinant formulae for the time and temperature dependent correlation functions $\langle\psi(x_1,0)\psi^\dagger(x_2,t)\rangle_{\pm,T}$, $t \in \mathbf{R}$, $T \geq 0$.

1 Introduction

In the standard treatment of quantum integrable models, one starts with a finite box and impose periodic boundary conditions, in order to ensure integrability. Recently, there has been increasing interest in exploring other possible boundary conditions compatible with integrability. These other possible boundary conditions are called "open boundary conditions".

With open boundary conditions, the works on the two dimensional Ising model are among the earliest. By the help of graph theoretical approach, B.M. McCoy and T.T. Wu [3] studied the two dimensional Ising model with open boundary conditions. They calculated the local magnetizations. E.K. Sklyanin [4] began the Bethe Ansatz approach to open boundary problems. M. Jimbo et.al. [5] studied the antiferromagnetic XXZ chains with open boundary conditions and derived an integrable representation of correlation functions, using Sklyanin's algebraic Bethe Ansatz framework and representation theoretical approach invented by Kyoto school [6, 7]. T. Kojima [2] studied the ground state correlation functions for an impenetrable Bose gas with open boundary conditions:

$$\langle \psi(x_1)\psi^\dagger(x_2)\rangle.$$

Kojima derived the Fredholm minor determinant representations for the ground state correlation functions by the help of fermions, which have the integral kernel:

$$\frac{\sin(\lambda-\mu)}{\lambda-\mu} \pm \frac{\sin(\lambda+\mu)}{\lambda+\mu}.$$

The integral intervals depend on the space parameter x_1, x_2. In this paper we study an impenetrable Bose gas with open boundary conditions. We are interested in the finite-temperature dynamical correlation functions:

$$\langle \psi(x_1,0)\psi^\dagger(x_2,t)\rangle_{\pm,T}.$$

We derive the Fredholm determinant representations for the dynamical correlation functions by the coordinate Bethe Ansatz, which have the integral kernel:

$$\sqrt{\vartheta(\lambda)}\,(L(\lambda,\mu) \pm L(\lambda,-\mu))\,\sqrt{\vartheta(\mu)},$$

where we have used

$$\begin{aligned} L(\lambda,\mu) = {} & \frac{e^{-\frac{1}{2}it(\lambda^2+\mu^2)}}{\lambda-\mu}\Big\{e^{it\lambda^2}\sin(x_1(\lambda-\mu)) + e^{it\mu^2}\sin(x_2(\lambda-\mu)) \\ & + \frac{2}{\pi}\,\text{P.V.}\int_{-\infty}^{\infty}\left(\frac{1}{s-\mu}-\frac{1}{s-\lambda}\right)e^{its^2}\sin((s-\mu)x_1)\sin((s-\lambda)x_2)ds\Big\}. \end{aligned} \tag{1.1}$$

Here the notation P.V. represents Cauchy's principle value and the measure $\vartheta(\lambda)$ is given by

$$\vartheta(\lambda) = \frac{1}{1+\exp\left(\frac{\lambda^2-h}{T}\right)}. \tag{1.2}$$

For an impenetrable Bose gas without boundaries, V. Korepin and N. Slavnov [8] has derived the Fredholm determinant formulae for the dynamical correlation functions.

To describe the $2n$ point dynamical correlation functions for an impenetrable Bose gas without boundaries, N. Slavnov [1] introduced a system of nonlinear partial differential equations, which becomes the nonlinear Schrödinger equation in the simplest case. The generalization of the nonlinear Schrödinger equations has $2n$ time variables t_j, $(1 \le j \le 2n)$ and $2n$ space variables x_j, $(1 \le j \le 2n)$. In this paper, we consider the dynamical correlation functions with Neumann boundary conditions for the special case that one space parameter has the value $x_1 = 0$: $\langle \psi(0,0)\psi^\dagger(x,t)\rangle_{T,+}$. We express the dynamical correlation functions in terms of a solution of Slavnov's generalization of the nonlinear Schrödinger equations. The differential equations, which describe four-point correlation functions without boundaries:

$$\langle \psi(x_1,t_1)\psi^\dagger(x_2,t_2)\psi(x_3,t_3)\psi^\dagger(x_4,t_4)\rangle_T,$$

describe the dynamical correlation functions with Neumann boundary conditions:

$$\langle\psi(0,0)\psi^\dagger(x,t)\rangle_{T,+}.$$

Now a few words about the organization of the paper. In Section 2 we formulate the problem and summarize the main results. In Section 3 we obtain the determinant formulae for the field form factors. In Section 4 we obtain the Fredholm determinant representation for the dynamical correlation functions. In Section 5 we consider the completely integrable differential equation which describes the 2-point dynamical correlation functions with Neumann boundary conditions. In Section 6 we consider the special case that time $t = 0$ and derive the Fredholm minor determinant representations for the finite-temperature fields correlation functions. We show that our Fredholm formulae coincides with the one which has been obtained [2] at temperature $T = 0$.

2 Formulation and Results

The purpose of this section is to formulate the problem and summarize the main results. The Hamiltonian of our model is given by

$$H = \int_0^L dx\left(\partial_x\psi^\dagger\partial_x\psi + c\psi^\dagger\psi^\dagger\psi\psi - h\psi^\dagger\psi\right) + h_0\left(\psi^\dagger(0)\psi(0) - \psi^\dagger(L)\psi(L)\right).$$

Here the fields $\psi(x)$ and $\psi^\dagger(x)$ $(x\in\mathbf{R})$ are canonical Bose fields given by

$$[\psi(x),\psi^\dagger(y)] = \delta(x-y), \qquad [\psi(x),\psi(y)] = [\psi^\dagger(x),\psi^\dagger(y)] = 0, \qquad (x,y\in\mathbf{R}),$$

and $L > 0$ is the size of box. The parameters $h > 0$ and $h_0 \in \mathbf{R}$ represent the chemical potential and the boundary chemical potential respectively. We only consider the case of the coupling constant $c = \infty$, so-called "impenetrable case". The Hamiltonian H acts on the Fock space of the Bose fields defined by the following relations between the Fock vacuum $|0\rangle$ and the Bose fields:

$$\langle 0|\psi^\dagger(x) = 0,\ \psi(x)|0\rangle = 0, \qquad \langle 0|0\rangle = 1.$$

A N-particle state vector $|\Psi_N\rangle$ is given by

$$|\Psi_N\rangle = \int_0^L dz_1\ldots\int_0^L dz_N\psi_N(z_1,\ldots,z_N)\psi^\dagger(z_1)\ldots\psi^\dagger(z_N)|0\rangle,$$

where the integrand $\psi_N(z_1,\ldots,z_N)$ is a $\mathbf{C}$-valued function. The eigenvector problem: $H|\Psi_N\rangle = E_N|\Psi_N\rangle$, $(E_N\in\mathbf{R})$, is equivalent to the quantum mechanics problem defined by the following four conditions of the integrand function $\psi_N(z_1,\ldots,z_N)$.

1. The wave function $\psi_N = \psi_N(z_1,\ldots,z_N)$ satisfies the free-particle Schrödinger equation in the case of variables $0 < z_i \neq z_j < L$:

$$-\sum_{j=1}^N\left(\frac{\partial}{\partial z_j}\right)^2\psi_N(z_1,\ldots,z_N) = E_N\cdot\psi_N(z_1,\ldots,z_N),$$

$(0 < z_i \neq z_j < L,\ E_N\in\mathbf{R})$.

2. The wave function ψ_N is symmetric with respect to the variables:

$$\psi_N(z_1, \ldots, z_N) = \psi_N(z_{\sigma(1)}, \ldots, z_{\sigma(N)}), \qquad (\sigma \in S_N).$$

3. The wave function ψ_N satisfies the integrable open boundary conditions:

$$\left(\frac{\partial}{\partial z_j} - h_0\right) \psi_N \Big|_{z_j=0} = 0,$$
$$\left(\frac{\partial}{\partial z_j} + h_0\right) \psi_N \Big|_{z_j=L} = 0, \qquad (j = 1, \ldots, N).$$

4. The wave function ψ_N vanishes whenever the coordinates coincide:

$$\psi_N(z_1, \ldots, z_i, \ldots, z_j, \ldots, z_N)\big|_{z_i=z_j} = 0.$$

This condition corresponds to the condition: $c \to \infty$.

The wave functions ψ_N which satisfy the above four conditions were constructed [2]. They are parameterized by the spectral parameters

$$\psi_N(z_1, \ldots, z_N|\lambda_1, \ldots, \lambda_N)$$
$$= Cons. \prod_{1\le j<k\le N} \mathrm{sgn}(z_j - z_k) \det_{1\le j,k\le N} \left(\lambda_j \cos(\lambda_j z_k) + h_0 \sin(\lambda_j z_k)\right).$$

Here the function $\mathrm{sgn}(x) = \dfrac{x}{|x|}$ and the spectral parameters $0 \le \lambda_1 < \lambda_2 < \cdots < \lambda_N$ are determined by the so-called Bethe Ansatz equations:

$$\lambda_j = \frac{\pi}{L} I_j, \qquad (I_j \in \mathbf{N},\ j = 1, 2, \ldots, N). \tag{2.1}$$

Because the coupling constant $c \to \infty$, the Bethe Ansatz equations become simple. The constant factor "$Cons.$" is determined by

$$\langle \Psi_N(\lambda_1, \ldots, \lambda_N)|\Psi_N(\lambda_1, \ldots, \lambda_N)\rangle = (2L)^N.$$

The eigenvalue $E_N(\{\lambda\})$:

$$H|\Psi_N(\lambda_1, \ldots, \lambda_N)\rangle_\epsilon = E_N(\{\lambda\})|\Psi_N(\lambda_1, \ldots, \lambda_N)\rangle_\epsilon,$$

is given by

$$E_N(\{\lambda\}) = \sum_{j=1}^{N} (\lambda_j^2 - h).$$

We assume that the set $\{|\Psi_N(\lambda_1, \ldots, \lambda_N)\rangle\}_{\mathrm{all}\{\lambda\}_N\ N\in\mathbf{N}}$ is a basis of physical space of this model. Here the index all$\{\lambda\}_N$ represents all the solutions of the Bethe Ansatz equations (2.1). This type assumption is usually called "Bethe Ansatz". The following lemma is a foundation of our analysis.

Lemma 2.1 *If the boundary condition h_0 takes the special value $h_0 = 0, \infty$, the eigenvectors $|\Psi(\{\lambda\})\rangle$ satisfy orthogonality relations*

$$\langle \Psi_N(\lambda_1, \ldots, \lambda_N) | \Psi_N(\mu_1, \ldots, \mu_N) \rangle = (2L)^N \prod_{j=1}^{N} \delta_{\lambda_j, \mu_j}, \qquad (h_0 = 0, \infty). \tag{2.2}$$

Here $\delta_{\lambda,\mu}$ is Kronecker Delta.

To prove the above lemma, we have used the Bethe-Ansatz equations of the spectral parameters. In the sequel we use the orthogonality relations of the eigenstates, therefore we concentrate our attentions to the case of the special boundary conditions: $h_0 = 0, \infty$. The boundary conditions $h_0 = 0$ and $h_0 = \infty$ are called Neumann, Dirichlet, respectively. In the sequel we use the following abberivations

$$|\Psi_N(\lambda_1, \ldots, \lambda_N)\rangle_+ \quad \text{for Neumann}, \qquad |\Psi_N(\lambda_1, \ldots, \lambda_N)\rangle_- \quad \text{for Dirichlet}.$$

The constant "*Cons.*" is given by

$$Cons. = \begin{cases} \dfrac{2^N}{\sqrt{(1+\delta_{\lambda_1,0})N!}} \left(\displaystyle\prod_{j=1}^{N} \lambda_j \right)^{-1}, & \text{for Neumann}, \\ \dfrac{1}{\sqrt{N!}} \left(\dfrac{2i}{h_0} \right)^N, & \text{for Dirichlet}. \end{cases}$$

By using the orthogonal relations (2.2) and the so-called "Bethe Ansatz", we arrive at the completeness relation:

$$id = \sum_{N=0}^{\infty} \sum_{\text{all}\{\lambda\}_N} \frac{|\Psi_N(\lambda_1, \ldots, \lambda_N)\rangle_\epsilon \, {}_\epsilon\langle \Psi_N(\lambda_1, \ldots, \lambda_N)|}{{}_\epsilon\langle \Psi_N(\lambda_1, \ldots, \lambda_N) | \Psi_N(\lambda_1, \ldots, \lambda_N)\rangle_\epsilon}. \tag{2.3}$$

The Bose fields $\psi(x,t)$, $\psi^\dagger(x,t)$ are developed by the time t by

$$i\partial_t \psi = [\psi, H], \qquad i\partial_t \psi^\dagger = [\psi^\dagger, H].$$

More explicitly the time dependence of the Bose fields are written by

$$\psi(x,t) = e^{iHt}\psi(x)e^{-iHt}, \qquad \psi^\dagger(x,t) = e^{iHt}\psi^\dagger(x)e^{-iHt}.$$

In this paper we are interested in the dynamical correlation functions $\langle \psi(x_1,t_1)\psi^\dagger(x_2,t_2)\rangle_{\epsilon,T}$ defined by the following way. For the nonzero temperature $T > 0$, the dynamical correlation functions for N state are defined by the summation of the every states:

$$\begin{aligned} &\langle \psi(x_1,t_1)\psi^\dagger(x_2,t_2)\rangle_{\epsilon,N,T} \\ &\quad = \left\{ \sum_{\text{all}\{\lambda\}_N} \exp\left(-\frac{E_N(\{\lambda\})}{T} \right) \right\}^{-1} \left\{ \sum_{\text{all}\{\lambda\}_N} \exp\left(-\frac{E_N(\{\lambda\})}{T} \right) \right. \\ &\quad \left. \times \frac{{}_\epsilon\langle \Psi_N(\lambda_1, \ldots, \lambda_N) | \psi(x_1,t_1)\psi^\dagger(x_2,t_2) | \Psi_N(\lambda_1, \ldots, \lambda_N)\rangle_\epsilon}{{}_\epsilon\langle \Psi_N(\lambda_1, \ldots, \lambda_N) | \Psi_N(\lambda_1, \ldots, \lambda_N)\rangle_\epsilon} \right\}, \end{aligned}$$

where the index $\epsilon = \pm$ represents the boundary conditions. Now the index "+" represents the dynamical correlation functions with Neumann boundary conditions. The index "−" represents the dynamical correlation functions with Dirichlet boundary conditions. For the ground state case $T = 0$, the dynamical correlation functions for N state are defined by the vacuum expectation value of the ground state:

$$\langle\psi(x_1,t_1)\psi^\dagger(x_2,t_2)\rangle_{\epsilon,N,0}$$
$$= \frac{{}_\epsilon\langle\Psi_N(\lambda_1,\ldots,\lambda_N)|\psi(x_1,t_1)\psi^\dagger(x_2,t_2)|\Psi_N(\lambda_1,\ldots,\lambda_N)\rangle_\epsilon}{{}_\epsilon\langle\Psi_N(\lambda_1,\ldots,\lambda_N)|\Psi_N(\lambda_1,\ldots,\lambda_N)\rangle_\epsilon},$$

where the spectral parameters $(\lambda_1,\ldots,\lambda_N)$ are given by

$$\lambda_j = \begin{cases} \frac{\pi}{L}(j-1), & \text{for Neumann,} \\ \frac{\pi}{L}j, & \text{for Dirichlet.} \end{cases}$$

In this paper we are interested in the thermodynamic limit of the correlation functions, for the temperature $T \geq 0$ and the time $t_1, t_2 \in \mathbf{R}$. For the nonzero temperature $T > 0$, the dynamical correlation functions in the thermodynamic limit are defined by

$$\langle\psi(x_1,t_1)\psi^\dagger(x_2,t_2)\rangle_{\epsilon,T} = \lim_{\substack{N,L\to\infty \\ \frac{N}{L}=D(T)}} \langle\psi(x_1,t_1)\psi^\dagger(x_2,t_2)\rangle_{\epsilon,N,T}.$$

Here the density $D(T) = \frac{N}{L}$ [9] is given by

$$D(T) = \frac{1}{\pi}\int_0^\infty \vartheta(\lambda)d\lambda, \tag{2.4}$$

where the Fermi weight $\vartheta(\lambda)$ is defined in (1.2). For the ground state $T = 0$, the dynamical correlation function in the thermodynamic limit is defined by

$$\langle\psi(x_1,t_1)\psi^\dagger(x_2,t_2)\rangle_{\epsilon,0} = \lim_{\substack{N,L\to\infty \\ \frac{N}{L}=D(0)}} \langle\psi(x_1,t_1)\psi^\dagger(x_2,t_2)\rangle_{\epsilon,N,0}.$$

Here the density $D(0) = \frac{N}{L}$ can be chosen arbitrary. In this paper we give the Fredholm determinant representations for the dynamical correlation functions $\langle\psi(x_1,t_1)\psi^\dagger(x_2,t_2)\rangle_{\epsilon,T}$, $(T \geq 0, \epsilon = \pm)$. Because the following relation hols:

$$\langle\psi(x_1,0)\psi^\dagger(x_2,t_2-t_1)\rangle_{\epsilon,T} = \langle\psi(x_1,t_1)\psi^\dagger(x_2,t_2)\rangle_{\epsilon,T},$$

we only need one time parameter $t = t_2 - t_1$, to describe correlation functions. In the sequel, we use the abbreviation $t = t_2 - t_1$. Let us set

$$\tau(s|x,t) = its^2 - ixs, \tag{2.5}$$

$$G(x) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{\tau(s|x,t)}ds, \tag{2.6}$$

and

$$P(\lambda|x_1,x_2) = e^{-\frac{1}{2}it\lambda^2}\left\{e^{\tau(\lambda|x_1,t)} - \frac{2}{\pi}\,\text{P.V.}\int_{-\infty}^\infty \frac{1}{s-\lambda}e^{\tau(s|x_1,t)}\sin(x_2(s-\lambda))ds\right\}. \tag{2.7}$$

In Section 4 we derive the following formulae.

Theorem 2.2 *In the thermodynamic limit $N, L \to \infty$, such that $\frac{N}{L} = D$, the ground state dynamical correlation functions have the Fredholm determinant representations*

$$\langle \psi(x_1, 0)\psi^\dagger(x_2, t)\rangle_{\epsilon,0}$$
$$= e^{-iht}\left(G(x_1 - x_2) + \epsilon G(x_1 + x_2) + \frac{1}{2\pi}\frac{\partial}{\partial\alpha}\right)\det\left(1 - \frac{2}{\pi}\widehat{V}_\epsilon - \alpha\widehat{A}_\epsilon\right)\Big|_{\alpha=0},$$

where the function $G(x)$ is given in (2.6). Here the integral operators $\widehat{V}_\epsilon$ and $\widehat{A}_\epsilon$ are defined by

$$(\widehat{V}_\epsilon f)(\lambda) = \int_0^q V_\epsilon(\lambda, \mu) f(\mu) d\mu, \qquad (\widehat{A}_\epsilon f)(\lambda) = \int_0^q A_\epsilon(\lambda, \mu) f(\mu) d\mu,$$

where the Fermi sphere $q = \pi D$ and the integral kernel are given by Neumann or Dirichlet sum:

$$V_\epsilon(\lambda, \mu) = L(\lambda, \mu) + \epsilon L(\lambda, -\mu),$$

$$A_\epsilon(\lambda, \mu) = \epsilon\left(P(\lambda|x_1, x_2) + \epsilon P(-\lambda|x_1, x_2)\right)\left(P(\mu|x_2, x_1) + \epsilon P(-\mu|x_2, x_1)\right).$$

Here we have used the function $L(\lambda, \mu)$ defined in (1.1) and the function $P(\lambda|x_1, x_2)$ defined in (2.7). Here we can choose the density $D > 0$ arbitrary.

We have succeeded to write the integral kernel by elementary functions:

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{e^{\tau(s|y,t)}}{s - \lambda} ds,$$

and trigonometric functions. In Section 4, we consider the finite temperature case, too.

Theorem 2.3 *In the thermodynamic limit: $N, L \to \infty$, such that $\frac{N}{L} = D(T)$ (2.4), the finite temperature dynamical correlation functions have the Fredholm determinant representations*

$$\langle \psi(x_1, 0)\psi^\dagger(x_2, t)\rangle_{\epsilon,T} = e^{-iht}\left(G(x_1 - x_2) + \epsilon G(x_1 + x_2) + \frac{1}{2\pi}\frac{\partial}{\partial\alpha}\right)$$
$$\times \det\left(1 - \frac{2}{\pi}\widehat{V}_{\epsilon,T} - \alpha\widehat{A}_{\epsilon,T}\right)\Big|_{\alpha=0}.$$

Here the temperature $T > 0$ and the integral operators $\widehat{V}_{\epsilon,T}$ and $\widehat{A}_{\epsilon,T}$ are defined by

$$(\widehat{V}_{\epsilon,T} f)(\lambda) = \int_0^\infty V_\epsilon(\lambda, \mu)\vartheta(\mu) f(\mu) d\mu, \qquad (\widehat{A}_{\epsilon,T} f)(\lambda) = \int_0^\infty A_\epsilon(\lambda, \mu)\vartheta(\mu) f(\mu) d\mu,$$

where the Fermi weight $\vartheta(\lambda)$ is given in (1.2).

In Section 5, we derive the differential equations for correlation functions for the case $x_1 = 0$. To describe $2n$ point dynamical correlation functions for an impenetrabel Bose gas without boundaries, N. Slavnov [1] introduced a system of nonlinear partial differential equations, which becomes the nonlinear Schrödinger equation in the simplest case. In this paper, we express the dynamical correlation functions $\langle\psi(0,0)\psi^\dagger(x,t)\rangle_{T,+}$, $(T \geq 0)$ in terms of Slavnov's generalization of the nonlinear Schrödinger equations [1]. For $x_1 = 0$ and Dirichlet boundary case:

$$\langle\psi(0,0)\psi^\dagger(x_2,t)\rangle_{T,-} = 0,$$

because the wave functions become to zero. We consider the case $x_1 = 0$ and Neumann boundary conditions. First we consider $T = 0$ case. Let us set

$$\left(\widehat{W}f\right)(\lambda) = \int_0^q W(\lambda,\mu)f(\mu)d\mu, \qquad (q = \pi D), \tag{2.8}$$

$$W(\lambda,\mu) = \frac{\sin(x(\lambda-\mu))}{\lambda-\mu} + \frac{\sin(x(\lambda+\mu))}{\lambda+\mu}. \tag{2.9}$$

Theorem 2.4 *The correlation functions for an impenetrable Bose gas with Neumann boundaries at the ground state are given by the following formulaes:*

$$\langle\psi(0,0)\psi^\dagger(x,t)\rangle_{0,+} = 2e^{-iht}\det\left(1 - \frac{2}{\pi}\widehat{W}\right) b_{1,4}\left(\begin{array}{cccc} 0 & 0 & -x & x \\ 0 & 0 & t & t \end{array}\right).$$

Here the integral operator $\widehat{W}$ is defined in (2.8) and the function $b_{1,4}$ is a component of matrix b defined in (5.5).

The matrix b defined in (5.5) satisfies a set of partial differential equations introduced in [1]:

$$\frac{\partial}{\partial t_j}L_k - \frac{\partial}{\partial y_k}M_j + [L_k, M_j] = 0, \qquad (1 \leq j,k \leq 4). \tag{2.10}$$

Here we have used

$$L_j(\mu) = \mu P_j + [b, P_j], \qquad M_j(\mu) = -\mu L_j(\mu) + \frac{\partial b}{\partial y_j},$$

where we have used the matrix P_j whose components are defined by

$$(P_j)_{l,m} = i\delta_{l,j}\delta_{m,j}. \tag{2.11}$$

The diffential equations (2.10) describe the logarithmic derivatives of the four point correlation functions without boundaries, too:

$$\langle\psi(y_1,t_1)\psi^\dagger(y_2,t_2)\psi(y_3,t_3)\psi^\dagger(y_4,t_4)\rangle_0.$$

In Section 5, we consider the finite temperature case, too. Let us consider the finite temperature case $T > 0$. Let us set

$$\left(\widehat{W}_T f\right)(\lambda) = \int_0^\infty W_T(\lambda,\mu)f(\mu)d\mu, \tag{2.12}$$

$$W_T(\lambda,\mu) = W(\lambda,\mu)\vartheta(\mu), \tag{2.13}$$

where the kernel $W(\lambda,\mu)$ is defined in (2.9) and $\vartheta(\mu)$ is defined in (1.2).

Theorem 2.5 *The correlation functions for an impenetrable Bose gas with Neumann boundaries are given by the following formulaes:*

$$\langle\psi(0,0)\psi^\dagger(x,t)\rangle_{T,+} = 2e^{-iht}\det\left(1-\frac{2}{\pi}\widehat{W}_T\right) b^T_{1,4}\left(\begin{array}{cccc} 0 & 0 & -x & x \\ 0 & 0 & t & t \end{array}\right),$$

Here the integral operator $\widehat{W}_T$ is defined in (2.12) and the function $b^T_{1,4}$ is a component of the matrix b^T defined in (5.6).

The matrix b^T satisfy a set of diffential equations (2.10), too. (We substitute b to b^T.)

T. Kojima [2] derived the Fredholm minor determinants formulae for the ground state correlation functions: $\langle\psi(x_1)\psi^\dagger(x_2)\rangle_{0,\epsilon}$. In Section 6 of this paper, we consider the special case for the time $t=0$ of our Fredholm determinant formulae:

$$\langle\psi(x_1,0)\psi^\dagger(x_2,0)\rangle_{T,\epsilon}, \qquad (\epsilon=\pm)$$

and derive the Fredholm minor determinant formulae for temperature $T\geq 0$. This Fredholm minor determinant formulae for $T=0$ coincide with the one which has been obtained [2]. Let us set

$$\left(\hat{\theta}^{(y_1,y_2)}_{\epsilon,T} f\right)(\xi) = \int_0^\infty \left((\mathrm{E}(y_1-\xi')+\mathrm{E}(y_2-\xi'))\theta_{\epsilon,T}(\xi,\xi')\right) f(\xi')d\xi', \tag{2.14}$$

where

$$\theta_{\epsilon,T}(\xi,\eta) = \int_0^\infty \vartheta(\nu)\left\{\cos((\xi-\eta)\nu)+\epsilon\cos((\xi+\eta)\nu)\right\}d\nu,$$

and $\vartheta(\lambda)$ is defined in (1.2). Here $\mathrm{E}(\xi)$ represents the step function

$$\mathrm{E}(\xi) = \left\{\begin{array}{ll} 1, & \text{for} \quad \xi\geq 0, \\ 0, & \text{for} \quad \xi<0. \end{array}\right.$$

Theorem 2.6 *For the temperature $T\geq 0$, the field correlation functions have the first Fredholm minor determinants representations*

$$\langle\psi(x_1)\psi^\dagger(x_2)\rangle_{\epsilon,T} = \frac{1}{2}\det\left(1-\frac{2}{\pi}\widehat{\theta}^{(x_1,x_2)}_{\epsilon,T}\,\middle|\,\begin{array}{c} x_2 \\ x_1 \end{array}\right),$$

where the integral operator $\widehat{\theta}^{(x_1,x_2)}_{\epsilon,T}$ is defined in (2.14).

Theorem 2.7 *[2] The ground state correlation functions have the first Fredholm minor determinant formulae*

$$\langle\psi(x_1)\psi^\dagger(x_2)\rangle_{\epsilon,0} = \frac{1}{2}\det\left(1-\frac{2}{\pi}\widehat{K}^{(x_1,x_2)}_{\epsilon}\,\middle|\,\begin{array}{c} x_2 \\ x_1 \end{array}\right).$$

Here the integral operator is defined by

$$\left(\hat{K}^{(x_1,x_2)}_{\epsilon} f\right)(\xi) = \int_{x_1}^{x_2} K_\epsilon(\xi,\xi')f(\xi')d\xi',$$

where

$$K_\epsilon(\xi,\eta) = \frac{\sin D(\xi-\eta)}{\xi-\eta} + \epsilon\frac{\sin D(\xi+\eta)}{\xi+\eta}.$$

Here the density $D=\dfrac{N}{L}$ can be chosen arbitrary.

3 Form Factors

The purpose of this section is to derive the determinant formulae for the form factors. First we prepare a lemma.

Lemma 3.1 *For the sequences* $\{f_{j,k}\}_{j=1,\ldots,N+1,\ k=1,\ldots,N}$ *and* $\{g_j\}_{j=1,\ldots,N+1}$*, the following holds:*

$$\sum_{\sigma\in S_{N+1}} \mathrm{sgn}\,\sigma f_{\sigma(N+1)} \prod_{j=1}^{N} g_{\sigma(j),j} = \left(f_{N+1}+\frac{\partial}{\partial\alpha}\right) \det_{1\le j,k\le N}\left(g_{j,k}-\alpha f_j\cdot g_{N+1,k}\right)\Big|_{\alpha=0}. \quad (3.1)$$

Proof. Consider the coset decomposition:

$$S_{N+1}=S_N(N+1)\cup S_N(N)\cdot(N,N+1)\cup\cdots\cup S_N(1)\cdot(1,N+1),$$

where $S_N(j)$ is permutations of $(1,\ldots,j-1,N+1,j+1,\ldots,N)$. Rewrite the left side of the equation (3.1) with respect to the coset decomposition:

$$(L.H.S.)=\sum_{j=1}^{N} g_j \sum_{\tau\in S_N(j)} \mathrm{sgn}(\tau\cdot(j,N+1)) \prod_{\substack{k=1\\k\neq j}}^{N} f_{\tau(k),k}\cdot f_{\tau(N+1),j}$$

$$+g_{N+1}\sum_{\tau\in S_N(N+1)} \mathrm{sgn}(\tau\cdot(N+1,N+1))\prod_{k=1}^{N} f_{\tau(k),k}=(R.H.S.)$$

Q.E.D.

Now let us consider the field form factor:

$${}_\epsilon\langle\Psi_{N+1}(\lambda_1,\ldots,\lambda_{N+1})|\psi^\dagger(x)|\Psi_N(\mu_1,\ldots,\mu_N)\rangle_\epsilon$$

$$=\sqrt{N+1}\int_0^L dz_1\ldots\int_0^L dz_N \psi^*_{N+1}(z_1,\ldots,z_N,x|\lambda_1,\ldots,\lambda_{N+1})$$

$$\times\psi_N(z_1,\ldots,z_N|\mu_1,\ldots,\mu_N)$$

$$=\frac{1}{\sqrt{(1+\delta_{\lambda_1,0})(1+\delta_{\mu_1,0})}}\sum_{\sigma\in S_{N+1}} \mathrm{sgn}\ \sigma(e^{-i\lambda_{\sigma(N+1)}x}+\epsilon e^{i\lambda_{\sigma(N+1)}x})$$

$$\times\prod_{j=1}^{N}\left\{\int_0^L dz\ \mathrm{sgn}\ (z-x)(e^{-i\lambda_{\sigma(j)}z}+\epsilon e^{i\lambda_{\sigma(j)}z})(e^{i\mu_j z}+\epsilon e^{-i\mu_j z})\right\}.$$

To derive the third line, we have used a simple fact:

$$\sum_{\sigma,\tau\in S_N} \mathrm{sgn}\ \sigma\tau\prod_{j=1}^{N} f_{\sigma(j),\tau(j)}=N!\sum_{\sigma\in S_N}\mathrm{sgn}\ \sigma\prod_{j=1}^{N} f_{\sigma(j),j}.$$

Using lemma 3.1, we arrive at the determinant formulae for the form factors.

Lemma 3.2 *The field form factors have the determinant formula*

$$ {}_\epsilon\langle\Psi_{N+1}(\lambda_1,\cdots,\lambda_{N+1})|\psi^\dagger(x)|\Psi_N(\mu_1,\ldots,\mu_N)\rangle_\epsilon = \left(C_\epsilon(x|\lambda_{N+1}) + \frac{\partial}{\partial\alpha}\right) $$
$$ \times \det_{1\le j,k\le N}\left(I_\epsilon(x|\lambda_j,\mu_k) - \alpha C_\epsilon(x|\lambda_j) I_\epsilon(x|\lambda_{N+1},\mu_k)\right)\Big|_{\alpha=0}. $$

Here we have used

$$ C_\epsilon(x|\lambda) = \frac{1}{\sqrt{1+\delta_{\lambda,0}}}(e^{-i\lambda x} + \epsilon e^{i\lambda x}), \tag{3.2} $$

$$ \begin{aligned} I_\epsilon(x|\lambda,\mu) = {} & \frac{1}{\sqrt{(1+\delta_{\lambda,0})(1+\delta_{\mu,0})}}\left\{\frac{4}{\lambda-\mu}\sin(x(\lambda-\mu))\right. \\ & \left. + \epsilon\frac{4}{\lambda+\mu}\sin(x(\lambda+\mu)) - 2L(\delta_{\lambda,\mu} + \epsilon\delta_{\lambda,0}\delta_{\mu,0})\right\}. \end{aligned} \tag{3.3} $$

We can write the field form factors without using integrals.

From the relation $\psi^\dagger(x,t) = e^{iHt}\psi^\dagger(x)e^{-iHt}$, the dynamical form factors are given by

$$ \begin{aligned} & {}_\epsilon\langle\Psi_{N+1}(\lambda_1,\ldots,\lambda_{N+1})|\psi^\dagger(x,t)|\Psi_N(\mu_1,\ldots,\mu_N)\rangle_\epsilon \\ & = \exp\left\{it\left(-h + \sum_{j=1}^{N+1}\lambda_j^2 - \sum_{j=1}^{N}\mu_j^2\right)\right\} \\ & \times {}_\epsilon\langle\Psi_{N+1}(\lambda_1,\ldots,\lambda_{N+1})|\psi^\dagger(x)|\Psi_N(\mu_1,\ldots,\mu_N)\rangle_\epsilon. \end{aligned} \tag{3.4} $$

4 Correlation Functions

The purpose of this section is to derive the Fredholm determinant formulas of the dynamical correlation functions $\langle\psi(x_1,t_1)\psi^\dagger(x_2,t_2)\rangle_{\epsilon,T}$. First we consider the vacuum expectation values of fields operators. Using the completeness relation (2.3), the vacuum expectation values of two fields are given by

$$ \frac{{}_\epsilon\langle\Psi_N(\mu_1,\ldots,\mu_N)|\psi(x_1,t_1)\psi^\dagger(x_2,t_2)|\Psi_N(\mu_1,\ldots,\mu_N)\rangle_\epsilon}{{}_\epsilon\langle\Psi_N(\mu_1,\ldots,\mu_N)|\Psi_N(\mu_1,\ldots,\mu_N)\rangle_\epsilon} $$
$$ = \sum_{\mathrm{all}\{\lambda\}_{N+1}} \frac{{}_\epsilon\langle\Psi_N(\{\mu\})|\psi(x_1,t_1)|\Psi_{N+1}(\{\lambda\})\rangle_\epsilon\ {}_\epsilon\langle\Psi_{N+1}(\{\lambda\})|\psi^\dagger(x_2,t_2)|\Psi_N(\{\mu\})\rangle_\epsilon}{{}_\epsilon\langle\Psi_N(\{\mu\})|\Psi_N(\{\mu\})\rangle_\epsilon\ {}_\epsilon\langle\Psi_{N+1}(\{\lambda\})|\Psi_{N+1}(\{\lambda\})\rangle_\epsilon}. $$

Using the equation (3.4) and the following relations:

$$ {}_\epsilon\langle\Psi_N(\{\mu\})|\psi(x_1,t_1)|\Psi_{N+1}(\{\lambda\})\rangle_\epsilon = {}_\epsilon\langle\Psi_{N+1}(\{\lambda\})|\psi^\dagger(x_1,t_1)|\Psi_N(\{\mu\})\rangle_\epsilon^*, $$
$$ {}_\epsilon\langle\Psi_N(\{\lambda\})|\Psi_N(\{\lambda\})\rangle_\epsilon = (2L)^N, $$

we obtain

$$ \frac{1}{(N+1)!}\left(\frac{1}{2L}\right)^{2N+1} e^{-i(t_2-t_1)(h+\sum_{j=1}^N\mu_j^2)} \sum_{\lambda_1\in\frac{\pi}{L}\mathbf{N}}\cdots\sum_{\lambda_{N+1}\in\frac{\pi}{L}\mathbf{N}} e^{i(t_2-t_1)\sum_{j=1}^{N+1}\lambda_j^2} $$
$$ \times {}_\epsilon\langle\Psi_{N+1}(\{\lambda\})|\psi^\dagger(x_1)|\Psi_N(\{\mu\})\rangle_\epsilon^*\ {}_\epsilon\langle\Psi_{N+1}(\{\lambda\})|\psi^\dagger(x_2)|\Psi_N(\{\mu\})\rangle_\epsilon. $$

The translation invariance of time holds

$$
\begin{aligned}
&{}_{\epsilon}\langle \Psi_N(\{\mu\})|\psi(x_1,t_1)\psi^{\dagger}(x_2,t_2)|\Psi_N(\{\mu\})\rangle_{\epsilon} \\
&\quad = {}_{\epsilon}\langle \Psi_N(\{\mu\})|\psi(x_1,0)\psi^{\dagger}(x_2,t_2-t_1)|\Psi_N(\{\mu\})\rangle_{\epsilon}.
\end{aligned}
$$

In the sequel we set the abberiviation $t=t_2-t_1$. Remember a following simple fact. *For sequences* $\{f_{j_1,\ldots,j_n}\}_{j_1,\ldots,j_n\in I}, \{g_{j_1,\ldots,j_n}\}_{j_1,\ldots,j_n\in I}$, ($I$: some index set), *the following holds*

$$
\sum_{j_1,\ldots,j_n\in I} (\mathrm{Sym}\ f)_{j_1,\ldots,j_n} (\mathrm{Sym}\ g)_{j_1,\ldots,j_n} = \sum_{j_1,\ldots,j_n\in I} f_{j_1,\ldots,j_n}\ (\mathrm{Sym}\ g)_{j_1,\ldots,j_n}.
$$

Here we have used

$$
(\mathrm{Sym}\ f)_{j_1,\ldots,j_n} = \frac{1}{n!}\sum_{\sigma\in S_n} f_{j_{\sigma(1)},\ldots,j_{\sigma(n)}}.
$$

The form factors have the determinant formulae in lemma 3.2 and

$$
{}_{\epsilon}\langle \Psi_{N+1}(\{\lambda\})|\psi^{\dagger}(x)|\Psi_N(\{\mu\})\rangle_{\epsilon} = \sum_{\sigma\in S_{N+1}} C_{\epsilon}(x|\lambda_{\sigma(N+1)})\prod_{j=1}^{N} I_{\epsilon}(x|\lambda_{\sigma(j)},\mu_j),
$$

We obtain

$$
\begin{aligned}
& e^{-it\left(h+\sum_{j=1}^{N}\mu_j^2\right)}\left(\frac{1}{2L}\right)^{2N+1} \\
&\quad \times \sum_{\lambda_1\in\frac{\pi}{L}\mathbf{N}}\cdots\sum_{\lambda_{N+1}\in\frac{\pi}{L}\mathbf{N}} \left(e^{it\lambda_{N+1}^2}C_{\epsilon}^{*}(x_1|\lambda_{N+1})C_{\epsilon}(x_2|\lambda_{N+1})+\frac{\partial}{\partial\alpha}\right) \\
&\quad \times \det_{1\le j,k\le N}\Big(e^{it\lambda_j^2}I_{\epsilon}(x_1|\lambda_j,\mu_k)I_{\epsilon}(x_2|\lambda_j,\mu_j)-\alpha e^{it\lambda_j^2}C_{\epsilon}^{*}(x_1|\lambda_j)I_{\epsilon}(x_2|\lambda_j,\mu_j) \\
&\quad \times\ e^{it\lambda_{N+1}^2}C_{\epsilon}(x_2|\lambda_{N+1})I_{\epsilon}(x_1|\lambda_{N+1},\mu_k)\Big)\Big|_{\alpha=0}.
\end{aligned}
$$

The j th line of the above matrix only depends on λ_j not on $\lambda_k, (k\neq j)$, therefore we can insert the summations $\sum_{\lambda_1}\cdots\sum_{\lambda_{N+1}}$ into the matrix. Now we arrive at the following.

Proposition 4.1 *The vacuum expectation values of two fields have the determinant formulas*

$$
\begin{aligned}
&\frac{{}_{\epsilon}\langle \Psi_N(\mu_1,\ldots,\mu_N)|\psi(x_1,0)\psi^{\dagger}(x_2,t)|\Psi_N(\mu_1,\ldots,\mu_N)\rangle_{\epsilon}}{{}_{\epsilon}\langle \Psi_N(\mu_1,\ldots,\mu_N)|\Psi_N(\mu_1,\ldots,\mu_N)\rangle_{\epsilon}} \\
&\quad = e^{-ith}\left(\frac{1}{2L}\sum_{s\in\frac{\pi}{L}\mathbf{N}}\epsilon e^{its^2}C_{\epsilon}(x_1|s)C_{\epsilon}(x_2|s)+\frac{\partial}{\partial\alpha}\right) \\
&\quad \times \det_{1\le j,k\le N}\left(\left(\frac{1}{2L}\right)^2\sum_{s\in\frac{\pi}{L}\mathbf{N}} e^{its^2}J_{\epsilon}(x_1|s,\mu_k)J_{\epsilon}(x_2|s,\mu_j)\right.
\end{aligned}
$$

$$-\alpha\epsilon\frac{1}{2L}\left(\frac{1}{2L}\sum_{s\in\frac{\pi}{L}\mathbf{N}} e^{its^2}C_\epsilon(x_1|s)J_\epsilon(x_2|s,\mu_j)\right)$$
$$\times\left(\frac{1}{2L}\sum_{s\in\frac{\pi}{L}\mathbf{N}} e^{its^2}C_\epsilon(x_2|s)J_\epsilon(x_1|s,\mu_k)\right)\Bigg)\Bigg|_{\alpha=0}.$$

Here we have used

$$J_\epsilon(x|s,\mu)=e^{-\frac{1}{2}it\mu^2}I_\epsilon(x|s,\mu).$$

and functions $C_\epsilon(x|s)$ and $I_\epsilon(x|s,\mu)$ are defined in (3.2) and (3.3), respectively.

The size of the above matrix depends on the state number N, however, the element of the matrix does not depend on N. By calculations, we obtain

$$\frac{1}{2L}\sum_{s\in\frac{\pi}{L}\mathbf{N}}\epsilon e^{its^2}C_\epsilon(x_1|s)C_\epsilon(x_2|s)$$
$$=\frac{1}{2L}\sum_{s\in\frac{\pi}{L}\mathbf{Z}} e^{its^2-is(x_1-x_2)}+\epsilon\frac{1}{2L}\sum_{s\in\frac{\pi}{L}\mathbf{Z}} e^{its^2-is(x_1+x_2)}, \tag{4.1}$$

$$\frac{1}{2L}\sum_{s\in\frac{\pi}{L}\mathbf{N}} e^{its^2}C_\epsilon(x_1|s)J_\epsilon(x_2|s,\mu)=\frac{e^{-\frac{1}{2}it\mu^2}}{\sqrt{1+\delta_{\mu,0}}}\Bigg[\Bigg\{\frac{2}{\pi}\left(\frac{\pi}{L}\right)$$
$$\times\sum_{s\in\frac{\pi}{L}\mathbf{Z}}\frac{e^{its^2-isx_1}}{s-\mu}\sin(x_2(s-\mu))-e^{it\mu^2-i\mu x_1}\Bigg\}+\epsilon\{\mu\leftrightarrow(-\mu)\}\Bigg], \tag{4.2}$$

and

$$\left(\frac{1}{2L}\right)^2\sum_{s\in\frac{\pi}{L}\mathbf{N}} e^{its^2}J_\epsilon(x_1|s,\mu)J_\epsilon(x_2|s,\lambda)=\delta_{\lambda,\mu}-\frac{2}{\pi}\left(\frac{\pi}{L}\right)\frac{e^{-\frac{1}{2}it(\lambda^2+\mu^2)}}{\sqrt{(1+\delta_{\lambda,0})(1+\delta_{\mu,0})}}$$
$$\times\Bigg[\frac{1}{\lambda-\mu}\Bigg\{e^{it\lambda^2}\sin(x_1(\lambda-\mu))+e^{it\mu^2}\sin(x_2(\lambda-\mu))$$
$$-\frac{2}{\pi}\left(\frac{\pi}{L}\right)\sum_{s\in\frac{\pi}{L}\mathbf{Z}} e^{its^2}\left(\frac{1}{s-\lambda}-\frac{1}{s-\mu}\right)\sin((s-\mu)x_1)\sin((s-\lambda)x_2)\Bigg\} \tag{4.3}$$
$$+\epsilon\frac{1}{\lambda+\mu}\{\mu\leftrightarrow(-\mu)\}\Bigg].$$

It is straightforwards to take the thermodynamic limit of the right hand side of the equations (4.1), (4.2) and (4.3). We arrive at Theorem 2.2. Next we consider the finite temperature thermodynamics. By statistical mechanics arguments, at temperature $T>0$, the thermodynamic equilibrium distribution of the spectral parameters is given by the Fermi weight $\vartheta(\lambda)$ (1.2):

$$\lim\left(\frac{\pi}{L}\right)\frac{1}{\lambda_{j+1}-\lambda_j}=\vartheta(\lambda_j).$$

Therefore the density is given by

$$D(T)=\frac{N}{L}=\frac{1}{\pi}\int_0^\infty \vartheta(\lambda)d\lambda.$$

Now we arrive at Theorem 2.3.

5 Differential equations

In this section we will study the most interesting case $\langle\psi(0,0)\psi^\dagger(x,t)\rangle_{\epsilon,T}$, which only appears in open boundary model. For Dirichlet boundary case $\epsilon=-$, $\langle\psi(0,0)\psi^\dagger(x,t)\rangle_{-,T}=0$, because the wave functions become zero. We will consider Neumann boundary case $\langle\psi(0,0)\psi^\dagger(x,t)\rangle_{+,T}$ and derive the differentaial equations which describe the dynamical correlation functions.

5.1 Preparations

First we will consider the zero temperature and general x_1,x_2 case.

By using the relation:

$$\frac{\partial}{\partial\alpha}\det\left(1-\frac{2}{\pi}\hat{V}_\epsilon-\alpha\widehat{A}_\epsilon\right)\Big|_{\alpha=0}=-\det\left(1-\frac{2}{\pi}\hat{V}_\epsilon\right)\mathrm{Tr}\left(\left(1-\frac{2}{\pi}\hat{V}_\epsilon\right)^{-1}\widehat{A}_\epsilon\right)$$

we obtain the following formulae

$$\langle\psi(x_1,0)\psi^\dagger(x_2,t)\rangle_{\epsilon,0}=e^{-iht}\det\left(1-\frac{2}{\pi}\hat{V}_\epsilon\right)$$
$$\times\left(G(x_1-x_2)+\epsilon G(x_1+x_2)-\frac{1}{2\pi}\mathrm{Tr}\left(\left(1-\frac{2}{\pi}\hat{V}_\epsilon\right)^{-1}\widehat{A}_\epsilon\right)\right).$$

Define the integral operator $\widehat{R}_\epsilon$ by

$$\left(\widehat{R}_\epsilon f\right)(\lambda)=\int_0^q R_\epsilon(\lambda,\mu)f(\mu)d\mu.$$

The kernel function $R_\epsilon(\lambda,\mu)$ is characterized by the following integral equation:

$$\left(1-\frac{2}{\pi}\widehat{V}_\epsilon\right)\left(1+\frac{2}{\pi}\widehat{R}_\epsilon\right)=1.$$

Define the integral operator $\widehat{S}$ and $\widehat{L}$ by

$$\left(\widehat{S}f\right)(\lambda)=\int_{-q}^q S(\lambda,\mu)f(\mu)d\mu,\qquad \left(\widehat{L}f\right)(\lambda)=\int_{-q}^q L(\lambda,\mu)f(\mu)d\mu,$$

where kernel function $L(\lambda,\mu)$ is defined in (1.1). The kernel function $S(\lambda,\mu)$ is characterized by the following integral equation:

$$\left(1-\frac{2}{\pi}\widehat{L}\right)\left(1+\frac{2}{\pi}\widehat{S}\right)=1.$$

Lemma 5.1 *The kernel functions are related by the following linear relation*

$$R_\epsilon(\lambda,\mu) = S(\lambda,\mu) + \epsilon S(\lambda,-\mu).$$

Proof. The following characteristic relation holds:

$$S(\lambda,\mu) - \frac{2}{\pi}\int_0^q \left(L(\lambda,\nu)S(\nu,\mu) + L(\lambda,-\nu)S(-\nu,\mu)\right) d\nu = L(\lambda,\mu).$$

Using the relations $\epsilon^2 = 1$, $(\epsilon = \pm)$ and $L(\lambda,-\mu) = L(-\lambda,\mu)$, we obtain the following characteristic relation:

$$(S(\lambda,\mu) + \epsilon S(\lambda,-\mu)) - \frac{2}{\pi}\int_0^q (S(\lambda,\nu) + \epsilon S(\lambda,-\nu))\,(L(\nu,\mu) + \epsilon L(\nu,-\mu))\, d\nu$$
$$= L(\lambda,\mu) + \epsilon L(\lambda,-\mu).$$

Q.E.D.

By using lemma 5.1, we obtain

$$\mathrm{Tr}\left(\left(1 - \frac{2}{\pi}\widehat{V}_\epsilon\right)^{-1}\widehat{A}_\epsilon\right) = \mathrm{Tr}\left(\left(1 + \frac{2}{\pi}\widehat{R}_\epsilon\right)\widehat{A}_\epsilon\right)$$
$$= \mathrm{Tr}\left(\left(1 + \frac{2}{\pi}\widehat{S}\right)\widehat{U}\right) + \epsilon\ \mathrm{Tr}\left(\left(1 + \frac{2}{\pi}\widehat{S}\right)\widehat{U}\widehat{Asy}\right).$$

Here we have used

$$\left(\widehat{U}f\right)(\lambda) = \int_{-q}^{q} U(\lambda,\mu)f(\mu)d\mu, \qquad \text{where} \quad U(\lambda,\mu) = P(\lambda|x_1,x_2)P(\mu|x_2,x_1). \quad (5.1)$$

Here we have used

$$\left(\widehat{Asy}f\right)(\lambda) = f(-\lambda).$$

In the sequel of this section we will consider the special case that $x_1 = 0$, $x_2 = x$ and $\epsilon = +$. The following simplication occurs:

$$L(\lambda,\mu)|_{x_1=0,x_2=x} = e^{\frac{1}{2}it(-\lambda^2+\mu^2)}\frac{\sin(x(\lambda-\mu))}{\lambda-\mu}.$$

Therefore $\det\left(1 - \frac{2}{\pi}\widehat{V}_+\right)\Big|_{x_1=0,x_2=x}$ dose not depend on time variable t:

$$\det\left(1 - \frac{2}{\pi}\widehat{V}_+\right)\Big|_{x_1=0,x_2=x} = \det\left(1 - \frac{2}{\pi}\widehat{W}\right),$$

where the operator $\widehat{W}$ is defined in (2.8). By using the relation:

$$P(-\lambda|x_1,x_2) = P(\lambda|-x_1,x_2),$$

we obtain the simplication:

$$\mathrm{Tr}\left(\left(1-\frac{2}{\pi}\hat{V}_+\right)^{-1}\widehat{A}_+\right)=2\ \mathrm{Tr}\left(\left(1+\frac{2}{\pi}\widehat{S}\right)\widehat{U}\right)\Big|_{x_1=0,x_2=x}.$$

We arrive at formulae

$$\langle\psi(0,0)\psi^\dagger(x,t)\rangle_{+,0}$$
$$=2e^{-iht}\det\left(1-\frac{2}{\pi}\hat{W}\right)\left(G(x)-\frac{1}{2\pi}\mathrm{Tr}\left(\left(1+\frac{2}{\pi}\widehat{S}\right)\widehat{U}\right)\right)\Big|_{x_1=0,x_2=x}.$$

5.2 Differential Equations

In this section we will find the partial differential equations of variables t and x. By discussion in the previous subsection, it is enough to consider the factor:

$$G(x_1+x_2)-\frac{1}{2\pi}\mathrm{Tr}\left(\left(1+\frac{2}{\pi}\widehat{S}\right)\widehat{U}\right).$$

It is convenient to consider the problem in more general situation. We introduce the auxiliary functions $G_p(\lambda)$ and the auxiliary vectors $e_p^L(\lambda)$ and $e_p^R(\mu)$ defined by

$$G_p(\lambda)=\frac{1}{2\pi}\ \mathrm{P.V.}\int_{-\infty}^{\infty}\frac{1}{s-\lambda}e^{\tau(s|y_{2p}-y_{2p-1}\ ,t_{2p}-t_{2p-1})}ds,$$

$$e_p^L(\lambda)=\begin{pmatrix}-e^{it_{2p-1}\lambda^2-iy_{2p-1}\lambda} & e^{it_{2p-1}\lambda^2-iy_{2p-1}\lambda}G_p(\lambda)\end{pmatrix},$$

$$e_p^R(\mu)=\frac{2}{\pi}\begin{pmatrix}e^{-it_{2p}\mu^2+iy_{2p}\mu}G_p(\mu)\\ e^{-it_{2p}\mu^2+iy_{2p}\mu}\end{pmatrix}.$$

Let us set the integral operators $\widehat{K}_p$ by

$$\left(\widehat{K}_pf\right)(\lambda)=\int_{-\infty}^{\infty}K_p(\lambda,\mu)f(\mu)d\mu,$$

where we have used the kernel defined by

$$K_p(\lambda,\mu)=\frac{\pi}{2}\frac{1}{\lambda-\mu}e_p^L(\lambda)e_p^R(\mu).$$

Let us set

$$E^L(\lambda)=\left(E_1^L(\lambda)\ E_2^L(\lambda)\ E_3^L(\lambda)\ E_4^L(\lambda)\right)=\begin{pmatrix}e_1^L(\lambda) & \left(\left(1+\frac{2}{\pi}\widehat{K}_1\right)e_2^L\right)(\lambda)\end{pmatrix},$$

$$E^R(\mu)=\begin{pmatrix}E_1^R(\mu)\\ E_2^R(\mu)\\ E_3^R(\mu)\\ E_4^R(\mu)\end{pmatrix}=\begin{pmatrix}\left(e_1^R\left(1+\frac{2}{\pi}\widehat{K}_2\right)\right)(\mu)\\ e_2^R(\mu)\end{pmatrix}.$$

By direct calculations we obtain the closed differential equations of the above vectors:

$$\frac{\partial}{\partial y_j}E^R(\mu) = l_j(\mu)E^R(\mu), \qquad \frac{\partial}{\partial t_j}E^R(\mu) = m_j(\mu)E^R(\mu),$$

$$l_j(\mu) = \mu P_j + [Q, P_j], \qquad m_j(\mu) = -\mu l_j(\mu) + \frac{\partial Q}{\partial y_j}.$$

Here we have used the matrix P_j defined in (2.11) and

$$Q = \begin{pmatrix} -\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\tau(s|y_2-y_1,t_2-t_1)}ds\sigma_+ & -\int_{-\infty}^{\infty} e_1^R(s)e_2^L(s)ds \\ 0 & -\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\tau(s|y_4-y_3,t_4-t_3)}ds\sigma_+ \end{pmatrix}, \tag{5.2}$$

where $\tau(s|y,t)$ is defined in (2.5). Define the integral operator $\widehat{M}$ by

$$\left(\widehat{M}f\right)(\lambda) = \int_0^q M(\lambda,\mu)f(\mu)d\mu,$$

where we have used the kernel defined by

$$M(\lambda,\mu) = -\frac{\pi}{2}\frac{E^L(\lambda)E^R(\mu)}{\lambda-\mu}. \tag{5.3}$$

By direct calculations we obtain the following Propositions.

Proposition 5.2 *The kernel $L(\lambda,\mu)$ is the special case of the kernel $M(\lambda,\mu)$*

$$L(\lambda,\mu|t,x_1,x_2) = M\left(\lambda,\mu \,\middle|\, \begin{matrix} -x_1 & x_1 & -x_2 & x_2 \\ 0 & 0 & t & t \end{matrix}\right).$$

Define the vectors $F^L(\lambda)$ and $F^R(\mu)$ by the integral equations

$$F^L(\lambda) = \left(F_1^L(\lambda) \;\; F_2^L(\lambda) \;\; F_3^L(\lambda) \;\; F_4^L(\lambda)\right) = -\left(\left(1-\frac{2}{\pi}\widehat{M}\right)^{-1} E^L\right)(\lambda),$$

$$F^R(\mu) = \begin{pmatrix} F_1^R(\mu) \\ F_2^R(\mu) \\ F_3^R(\mu) \\ F_4^R(\mu) \end{pmatrix} = \left(E^R\left(1-\frac{2}{\pi}\widehat{M}\right)^{-1}\right)(\mu).$$

By usual calculation procedure described in [10], we obtain the closed differential equations

$$\frac{\partial}{\partial y_j}F^R(\mu) = L_j(\mu)F^R(\mu), \qquad \frac{\partial}{\partial t_j}F^R(\mu) = M_j(\mu)F^R(\mu), \tag{5.4}$$

$$L_j(\mu) = \mu P_j + [b, P_j], \qquad M_j(\mu) = -\mu L_j(\mu) + \frac{\partial b}{\partial y_j}.$$

Here we have used matrix b defined by

$$b = B + Q, \tag{5.5}$$

where Q is defined in (5.2) and the matrix B is defined by

$$B_{j,k} = B_{j,k}\left(\begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \\ t_1 & t_2 & t_3 & t_4 \end{array}\right) = \int_{-q}^{q} F_j^R(\lambda)E_k^L(\lambda)d\lambda.$$

The compatibility condition of the above differential equations (5.4) yields the differential equations (2.10).

Proposition 5.3 *A factor of correlation functions can be written by an element of the matrix b*

$$G(x_1+x_2) - \frac{1}{2\pi}\mathrm{Tr}\left(\left(1+\frac{2}{\pi}\widehat{S}\right)\widehat{Q}\right) = b_{1,4}\left(\begin{array}{cccc} -x_1 & x_1 & -x_2 & x_2 \\ 0 & 0 & t & t \end{array}\right).$$

Proof. The kernel $U(\lambda,\mu)$ (5.1) is related to the vectors $E^R(\lambda)$ and $E^L(\mu)$

$$Q(\lambda,\mu) = -2\pi\, E_1^R(\lambda)E_4^L(\mu)\big|_{y_1=-y_2=-x_1;y_3=-y_4=-x_2;t_1=t_2=0;t_3=t_4=t}.$$

By using Proposition 5.2, we arrive at the following

$$\mathrm{Tr}\left(\left(1+\frac{2}{\pi}\widehat{S}\right)\widehat{U}\right) = -2\pi B_{1,4}\left(\begin{array}{cccc} -x_1 & x_1 & -x_2 & x_2 \\ 0 & 0 & t & t \end{array}\right).$$

Q.E.D.

Now we arrive at Theorem 2.4. For finite temperature case $T>0$, we prepare some functions. Let us set

$$\left(M_T f\right)(\lambda) = \int_{-\infty}^{\infty} M(\lambda,\mu)\vartheta(\mu)d\mu,$$

where $\vartheta(\mu)$ is defined in (1.2) and $M(\lambda,\mu)$ is defined in (5.3). Define the vectors $F^L(\lambda)_T$ and $F^R(\mu)_T$ by the integral equations

$$F^L(\lambda)_T = -\left(\left(1-\frac{2}{\pi}\widehat{M_T}\right)^{-1}E^L\right)(\lambda), \qquad F^R(\mu)_T = \left(E^R\left(1-\frac{2}{\pi}\widehat{M_T}\right)^{-1}\right)(\mu).$$

Define the matrix b^T by

$$b^T = B^T + Q, \tag{5.6}$$

where Q is defined in (5.2) and the matrix B^T is defined by

$$B_{j,k}^T = B_{j,k}^T\left(\begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \\ t_1 & t_2 & t_3 & t_4 \end{array}\right) = \int_{-\infty}^{\infty} F_j^R(\lambda)_T E_k^L(\lambda)_T d\lambda.$$

By the similar discussion as temperature $T=0$ case, we arrive at Theorem 2.5.

6 The time-independent Case

The purpose of this section is to derive the Fredholm minor determinant representations for finite-temperature fields correlation functions:

$$\langle \psi(x_1)\psi^\dagger(x_2)\rangle_{\epsilon,T}.$$

Our Fredholm minor determinant representations coincide with the one which has been obtained in [2]. When we take the limit $t \to 0$, the following simplifications occur:

$$G(x) \to 0, \qquad L(\lambda,\mu) \to \frac{1}{\lambda-\mu}\left(\sin(x_1(\lambda-\mu)) + \sin(x_2(\lambda-\mu))\right),$$

$$P(\lambda|x_1,x_2) \to e^{-ix_1\lambda}.$$

Therefore we obtain

$$\langle \psi(x_1)\psi(x_2)\rangle_{\epsilon,T} = \frac{1}{2\pi}\left(\frac{\partial}{\partial\alpha}\right)\det\left(1-\frac{2}{\pi}\widehat{\tilde{V}}_{\epsilon,T} - \alpha\widehat{\tilde{W}}^{(x_1,x_2)}_{\epsilon,T}\right)\bigg|_{\alpha=0}$$

$$= -\frac{1}{2\pi}\det\left(1-\frac{2}{\pi}\widehat{\tilde{V}}_{\epsilon,T}\right)\mathrm{Tr}\left[\left(1-\frac{2}{\pi}\widehat{\tilde{V}}_{\epsilon,T}\right)^{-1}\widehat{\tilde{W}}^{(x_1,x_2)}_{\epsilon,T}\right].$$

Here the integral operators are given by

$$\left(\widehat{\tilde{V}}_{\epsilon,T}f\right)(\lambda) = \int_0^\infty \tilde{V}_{\epsilon,T}(\lambda,\mu)f(\mu)d\mu, \qquad \left(\widehat{\tilde{W}}^{(\xi,\eta)}_{\epsilon,T}f\right)(\lambda) = \int_0^\infty \tilde{W}^{(\xi,\eta)}_{\epsilon,T}(\lambda,\mu)f(\mu)d\mu,$$

where the integral kernels are given by

$$\tilde{V}_{\epsilon,T}(\lambda,\mu) = \sqrt{\vartheta(\lambda)}\left[\frac{1}{\lambda-\mu}\left\{\sin(x_1(\lambda-\mu)) + \sin(x_2(\lambda-\mu))\right\}\right.$$

$$\left. + \epsilon\frac{1}{\lambda+\mu}\left\{\sin(x_1(\lambda+\mu)) + \sin(x_2(\lambda+\mu))\right\}\right]\sqrt{\vartheta(\mu)},$$

$$\tilde{W}^{(\xi,\eta)}_{\epsilon,T}(\lambda,\mu) = \sqrt{\vartheta(\lambda)}\epsilon(e^{i\xi\lambda} + \epsilon e^{-i\xi\lambda})(e^{i\eta\mu} + \epsilon e^{-i\eta\mu})\sqrt{\vartheta(\mu)}.$$

Pay attention to the Fourier transforms:

$$f(\lambda) = \frac{1}{2\pi\sqrt{\vartheta(\lambda)}}\int_{-\infty}^{\infty} d\xi e^{i\lambda\xi}\varphi(\xi), \qquad \varphi(\xi) = \int_{-\infty}^{\infty} d\lambda\sqrt{\vartheta(\lambda)}e^{-i\lambda\xi}f(\lambda).$$

The following identity holds for functions $f_\epsilon(\epsilon\lambda) = \epsilon f(\epsilon\lambda)$:

$$\int_0^\infty d\mu f_\epsilon(\mu)\sqrt{\vartheta(\lambda)\vartheta(\mu)}\left\{\frac{1}{\lambda-\mu}\sin(x(\lambda-\mu)) + \epsilon\frac{1}{\lambda+\mu}\sin(x(\lambda+\mu))\right\}$$

$$= \frac{1}{2\pi\sqrt{\vartheta(\lambda)}}\int_{-\infty}^{\infty} d\xi e^{i\lambda\xi}\left(\int_0^x d\xi'\theta_{\epsilon,T}(\xi',\xi)\right)\int_{-\infty}^{\infty} d\mu f_\epsilon(\mu)\sqrt{\vartheta(\mu)}e^{-i\xi'\mu},$$

where

$$\theta_{\epsilon,T}(\xi,\eta) = \int_0^\infty \vartheta(\nu)\left\{\cos((\xi-\eta)\nu) + \epsilon\cos((\xi+\eta)\nu)\right\}d\nu.$$

Therefore we arrive at

$$\det\left(1-\frac{2}{\pi}\widehat{\tilde{V}}_{\epsilon,T}\right)=\det\left(1-\frac{2}{\pi}\left(\hat{\theta}^{(x_1,x_2)}_{\epsilon,T}\right)\right),$$

where the integral operator $\hat{\theta}^{(y_1,y_2)}_{\epsilon,T}$ is defined by

$$\left(\hat{\theta}^{(y_1,y_2)}_{\epsilon,T}f\right)(\xi)=\int_0^\infty\left((\mathrm{E}(y_1-\xi')+\mathrm{E}(y_2-\xi'))\theta_{\epsilon,T}(\xi,\xi')\right)f(\xi')d\xi'.$$

Here $\mathrm{E}(\xi)$ represents the step function

$$\mathrm{E}(\xi)=\begin{cases}1, & \text{for}\quad \xi\geq 0,\\ 0, & \text{for}\quad \xi<0.\end{cases}$$

Let us set

$$\Delta_\epsilon(\xi,\eta)=\frac{\det\left(1-\frac{2}{\pi}\left(\hat{\theta}^{(x_1,x_2)}_{\epsilon,T}\right)\middle|\begin{array}{c}\eta\\ \xi\end{array}\right)}{\det\left(1-\frac{2}{\pi}\left(\hat{\theta}^{(x_1,x_2)}_{\epsilon,T}\right)\right)}.$$

Here we have used the following notation of the r-th Fredholm minor determinants:

$$\det\left(1-\lambda\widehat{K}_I\middle|\begin{array}{ccc}\xi_1 & \cdots & \xi_r\\ \eta_1 & \cdots & \eta_r\end{array}\right)$$

$$=\sum_{n=0}^\infty\frac{(-\lambda)^{n+r}}{n!}\int_I d\lambda_1\cdots\int_I d\lambda_n K_{n+r}\left(\begin{array}{cccccc}\xi_1 & \cdots & \xi_r & \lambda_1 & \cdots & \lambda_n\\ \eta_1 & \cdots & \eta_r & \lambda_1 & \cdots & \lambda_n\end{array}\right),$$

where we have used

$$K_m\left(\begin{array}{ccc}\xi_1 & \cdots & \xi_m\\ \eta_1 & \cdots & \eta_m\end{array}\right)=\det_{1\leq j,k\leq m}\left(K(\xi_j,\eta_k)\right).$$

The integral operator $\widehat{K}_I$ is defined by using the integral kernel $K(\lambda,\mu)$ and the integral interval I:

$$(\widehat{K}_I f)(\lambda)=\int_I K(\lambda,\mu)f(\mu)d\mu.$$

From the above definition of the Fredholm minor determinants, the function $\Delta_\epsilon(\xi,\eta)$ satisfies the integral equation:

$$\Delta_\epsilon(\xi,\eta)-\frac{2}{\pi}\int_{-x_1}^{x_2}\theta_{\epsilon,T}(\xi,\xi')\Delta_\epsilon(\xi',\eta)d\xi'=-\frac{2}{\pi}\theta_{\epsilon,T}(\xi,\eta).$$

Let us take the Fourier transforms of this integral equation

$$\frac{1}{2\pi\sqrt{\vartheta(\lambda)}}\int_{-\infty}^\infty d\xi e^{i\lambda\xi}\Delta_\epsilon(\xi,\eta)-\frac{2}{\pi}\int_0^\infty d\mu\tilde{V}_{\epsilon,T}(\lambda,\mu)\frac{1}{2\pi\sqrt{\vartheta(\mu)}}\int_{-\infty}^\infty d\xi' e^{i\mu\xi'}\Delta_\epsilon(\xi',\eta)$$

$$=-\frac{1}{\pi}\sqrt{\vartheta(\lambda)}(e^{i\lambda\eta}+\epsilon e^{-i\lambda\eta}).$$

Therefore we obtain

$$\Delta_\epsilon(\xi,\eta)=-\frac{1}{\pi}\mathrm{Tr}\left[\left(1-\frac{2}{\pi}\widehat{\tilde{V}}_{\epsilon,T}\right)^{-1}\widehat{\tilde{W}}^{(\xi,\eta)}_{\epsilon,T}\right].$$

Now we arrive at Theorem 2.6. In much the same way as with finite temperature case $T>0$ we arrive at Theorem 2.7.

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References

[1] Slavnov N., Differential Equations for Multipoint Correlation Functions in a One-Dimensional Impenetrable Bose Gas, *Theor. Math. Phys.*, 1996, V.106, N 1, 131–142.

[2] Kojima T., Ground-State Correlation Functions for an Impenetrable Bose Gas with Neumann or Dirichlet Boundary Conditions, *J. Stat. Phys.*, 1997, V.88, N 3/4, 713–743.

[3] McCoy B. and Wu T., Theory of Toeplitz Determinants and the Spin Correlation Functions of the Ising Model 4, *Phys. Rev.*, 1967, V.162, 436–475.

[4] Sklyanin E., Boundary Conditions for Integrable Quantum Systems, *J. Phys. A*, 1988, V.21, 2375–2389.

[5] Jimbo M., Kedem R., Kojima T., Konno H. and Miwa T., XXZ Chain with a Boundary, *Nucl. Phys. B*, 1995, V.441[FS], 429–453.

[6] Davis B., Foda O., Jimbo M., Miwa T. and Nakayashiki A., Diagonalization of the XXZ Hamiltonian by Vertex Operators, *Commun. Math. Phys.*, 1993, V.151, 89–153.

[7] Jimbo M. and Miwa T., Algebraic Analysis of Solvable Lattice Models, CBMS Regional Conference Series in Mathematics, Vol.85, AMS, 1994.

[8] Korepin V. and Slavnov N., The Time Dependent Correlation Function of an Impenetrable Bose Gas as a Fredholm Minor. I, *Commun. Math. Phys.*, 1990, V.129, 103–113.

[9] Yang C.N. and Yang C.P., Thermodynamics of a One-Dimensional System of Bosons with Repulsive Delta-Function Interactions, *J. Math. Phys.*, 1969, V.10, 1115–1122.

[10] Korepin V., Bogoliubov N. and Izergin A., Quantum Inverse Scattering Method and Correlation Functions, Cambridge University Press, 1993.