Explode-Decay Dromions in the Non-Isospectral Davey-Stewartson I (DSI) Equation

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Abstract

In this letter, we report the existence of a novel type of explode-decay dromions, which are exponentially localized coherent structures whose amplitude varies with time, through Hirota method for a nonisospectral Davey-Stewartson equation I discussed recently by Jiang. Using suitable transformations, we also point out such solutions also exist for the isospectral Davey-Stewartson I equation itself for a careful choice of the potentials.

The identification of exponentially localized structures namely the so called "dromions" in the Davey-Stewartson I (DSI) equation [1] has given a new direction to the study of nonlinear partial differential equations (pdes) in (2+1) dimensions. The subsequent investigations of other physically and mathematically important (2+1) dimensional nonlinear pdes [2–10] have enriched the family of integrable nonlinear pdes, exhibiting considerable richness in the structure of the solutions. However, very little is known about non-isospectral (2+1) dimensional nonlinear pdes in this regard. Now one may ask, do exponentially localized structures exist in the later systems also, and if so, what are their characteristics? This letter is a modest attempt in this direction in which the existence of such localized solutions is pointed out for DSI equation with nonuniform or inhomogeneous terms.

In this letter, we consider a non-isospectral DSI equation introduced by Jiang [11] of the form

$$iq_t + \frac{1}{2}(q_{xx} + q_{yy}) + \left[u_1(\xi,t) + u_2(\eta,t) - \frac{1}{2}\partial_\xi^{-1}(qr)_\eta - \frac{1}{2}\partial_\eta^{-1}(qr)_\xi\right]q$$

$$-i(\omega_1 y + a_1)q_y - i\omega_1(xq)_x + 2(\omega_0 x + a_0)q = 0, \tag{1a}$$

$$ir_t - \frac{1}{2}(r_{xx} + r_{yy}) - \left[u_1(\xi,t) + u_2(\eta,t) - \frac{1}{2}\partial_\xi^{-1}(qr)_\eta - \partial_\eta^{-1}(qr)_\xi\right]r$$

$$-i(\omega_1 y + a_1)r_y - i\omega_1(xr)_x + 2(\omega_0 x + a_0)r = 0, \qquad \xi = x+y, \quad \eta = x-y, \tag{1b}$$

where ω_0, ω_1, a_0 and a_1 are all real constants and $u_1(\xi, t)$ and $u_2(\eta, t)$ are the so called boundaries (arbitrary functions). The above set of equations has been investigated via inverse scattering transform method [11] and an integro-differential equation for the time evolution of the scattering data by virtue of the time dependence of the scattering parameters has been brought out. Under the reduction $r = -q^*$, equation (1) reduces to the following form

$$iq_t + q_{\xi\xi} + q_{\eta\eta} + \left[u_1(\xi, t) + u_2(\eta, t) + \frac{1}{2}\partial_\xi^{-1}(|\,q\,|^2)_\eta + \frac{1}{2}\partial_\eta^{-1}(|\,q\,|^2)_\xi\right]q$$

$$-i\omega_1[\xi q_\xi + \eta q_\eta] - ia_1[q_\xi - q_\eta] + [2a_0 + \omega_0(\xi + \eta) - i\omega_1]q = 0. \tag{2}$$

In equation (2), the parameter a_0 can be scaled away by introducing the transformation $q = \hat{q}\exp(-2ia_0t)$. Then, the above equation can be equivalently written as

$$\frac{1}{2}\left(|\,q\,|^2\right)_\eta = U_\xi, \tag{3a}$$

$$\frac{1}{2}\left(|\,q\,|^2\right)_\xi = V_\eta, \tag{3b}$$

$$iq_t + q_{\xi\xi} + q_{\eta\eta} + (U+V)q - i\omega_1(\xi q_\xi + \eta q_\eta) - ia_1(q_\xi - q_\eta) + (\omega_0(\xi + \eta) - i\omega_1)q = 0. \tag{4}$$

As the complete integrability paralleling to that of dynamical systems under isospectral flows is not obvious for the nonisospectral problems in (2+1) dimensions, we address ourselves only to the nature of the solutions of the above equation rather than its integrability property. For this purpose, we bilinearize equation (3) and obtain the solutions using the Hirota method.

To bilinearise the above equation, we effect the following dependent variable transformation

$$q = \frac{G}{F}, \qquad U = 2\partial_{\eta\eta}\log F, \qquad V = 2\partial_{\xi\xi}\log F, \tag{5}$$

so that equations (3) and (4) get converted into the following Hirota form,

$$\left[iD_t + D_\xi^2 + D_\eta^2 - i\omega_1(\xi D_\xi + \eta D_\eta)\right.$$

$$\left. -ia_1(D_\xi - D_\eta) + (\omega_0[\xi + \eta] - i\omega_1)\right]G \cdot F = 0, \tag{6a}$$

$$2D_\xi D_\eta F \cdot F = |\,G\,|^2. \tag{6b}$$

We now introduce the following power series expansion

$$G = \epsilon g^{(1)} + \epsilon^3 g^{(3)} + \cdots, \tag{7a}$$

$$F = 1 + \epsilon^2 f^{(2)} + \epsilon^4 f^{(4)} + \cdots, \tag{7b}$$

into the bilinear form (6), where ϵ is a small parameter. Collecting the various powers of ϵ, we get the following set of equations,

$$O(\epsilon): \quad i(g^{(1)})_t + (g^{(1)})_{\xi\xi} + (g^{(1)})_{\eta\eta} - i\omega_1[\xi(g^{(1)})_\xi + \eta(g^{(1)})_\eta]$$

$$-ia_1[(g^{(1)})_\xi - (g^{(1)})_\eta] + (\omega_0(\xi + \eta) - i\omega_1)g^{(1)} = 0, \tag{8a}$$

$$O(\epsilon^2): \quad 4(f^{(2)})_{\xi\eta} = g^{(1)}g^{(1)*}, \tag{8b}$$

$$O(\epsilon^3): \quad \left[iD_t + D_\xi^2 + D_\eta^2 - i\omega_1(\xi D_\xi + \eta D_\eta)\right.$$
$$\left. -ia_1(D_\xi - D_\eta) + (\omega_0[\xi + \eta] - i\omega_1)\right] + \left(g^{(3)} + g^{(1)}.f^{(2)}\right) = 0, \tag{8c}$$

$$O(\epsilon^4): \quad 2D_\xi D_\eta(2f^{(4)} + f^{(2)}.f^{(2)}) = g^{(3)}g^{(1)*} + g^{(1)}g^{(3)*}, \tag{8d}$$

and so on. Solving (8a), we obtain the simplest "plane wave" solution

$$g^{(1)} = \sum_{j=1}^{N} e^{\chi_j}, \qquad \chi_j = k_j(t)\xi + l_j(t)\eta + \int \Omega_j(t)dt, \tag{9}$$

where the spectral parameters $k_j(t)$ and $l_j(t)$ evolve in an identical fashion obeying the time evolution equation

$$i(k_j)_t - i(\omega_1)k_j = i(l_j)_t - i(\omega_1)l_j = -\omega_0, \tag{10}$$

and the "dispersion" relation is given by

$$i\Omega_j(t) + (k_j(t))^2 + (l_j(t))^2 - ia_1(k_j(t) - l_j(t)) - i\omega_1 = 0. \tag{11}$$

To construct one soliton solution, we take $N = 1$ so that we have

$$g^{(1)} = e^{\chi_1}, \tag{12}$$

and hence the solution of (8b) becomes

$$f^{(2)} = e^{\chi_1 + \chi_1^* + 2\psi}, \qquad e^{2\psi} = \frac{1}{16k_{1R}(t)l_{1R}(t)}. \tag{13}$$

Substituting $g^{(1)}$ and $f^{(2)}$ in equations (8c) and (8d), one can show that $g^{(j)} = 0$ for $j \geq 3$ and $f^{(j)} = 0$ for $j \geq 4$ provided the spectral parameters obey the following equation

$$(k_{1R}l_{1R})_t = 2\omega_1 k_{1R}l_{1R}. \tag{14}$$

Considering the above equation alongwith equation (10), separating out the real and imaginary parts of the later, the time evolution of the spectral parameters can be obtained as

$$k_{1R}(t) = k_{1R}(0)e^{\omega_1 t}, \qquad l_{1R}(t) = l_{1R}(0)e^{\omega_1 t} \tag{15a}$$

and

$$k_{1I}(t) = k_{1I}(0)e^{\omega_1 t} - \frac{\omega_0}{\omega_1}, \qquad l_{1I}(t) = l_{1I}(0)e^{\omega_1 t} - \frac{\omega_0}{\omega_1}, \tag{15b}$$

where $k_{1R}(0)$, $k_{1I}(0)$, $l_{1R}(0)$ and $l_{1I}(0)$ are all constants. Using (5), (7), (12) and (13), the physical field variable q of equation (4) is driven by the envelope soliton (line soliton)

$$q = 2\sqrt{k_{1R}(t)l_{1R}(t)} \operatorname{sech}(\chi_{1R} + \psi)e^{i\chi_{1I}}. \tag{16}$$

It is interesting to note that the amplitude of the above soliton solution varies with time by virtue of equation (15). Similarly, the potentials U and V are driven by the line soliton solutions whose amplitude again varies with time as

$$U = (k_{1R}(t))^2 \operatorname{sech}^2(\chi_{1R} + \psi), \tag{17a}$$

$$V = (l_{1R}(t))^2 \operatorname{sech}^2(\chi_{1R} + \psi). \tag{17b}$$

As it is clear from equation (3) that the boundaries are specified by the arbitrary functions $u_2(\eta, t)$ and $u_1(\xi, t)$ which drive the potentials U and V even in the absence of the field q, as in the case of isospectral (uniform) DSI equation, one can expect the presence of exponentially localized solutions in the system provided one properly takes care of the time evolution of the spectral parameters. To generate a (1,1) dromion solution, we take the ansatz

$$F = \delta + \alpha e^{\chi_1 + \chi_1^*} + \beta e^{\chi_2 + \chi_2^*} + \gamma e^{\chi_1 + \chi_1^* + \chi_2 + \chi_2^*}, \qquad \chi_1 = k_1 \xi + \int \Omega_1(t) dt,$$

$$\chi_2 = l_1 \eta + \int \Omega_2(t) dt, \qquad \Omega_1(t) = ik_1^2 + a_1 k_1, \qquad \Omega_2(t) = il_1^2 - a_1 l_1. \tag{18}$$

where α, β, γ and δ are parameters. Substituting (18) into (6b), we obtain

$$G = \rho e^{\chi_1 + \chi_2}, \qquad \rho \quad \text{is real}, \tag{19a}$$

$$\rho^2 = 8k_{1R}(t)l_{1R}(t)(\delta\gamma - \alpha\beta). \tag{19b}$$

Hence, the dromion solution now becomes

$$q = \frac{\rho e^{\chi_1 + \chi_2}}{\delta + \alpha e^{\chi_1 + \chi_1^*} + \beta e^{\chi_2 + \chi_2^*} + \gamma e^{\chi_1 + \chi_1^* + \chi_2 + \chi_2^*}}. \tag{20}$$

It can be easily observed from the above solution that the amplitude of the dromion solution has to evolve in time obeying the time evolution equation

$$\rho_t - \omega_1 \rho = 0, \tag{21}$$

so that solution (20) satisfies equation (6). The above equation has the solution

$$\rho = \rho_0 e^{\omega_1 t}, \tag{22}$$

where ρ_0 is a constant. Thus, the amplitude of the dromion solution varies with time in the nonisospectral DSI equation unlike the isospectral DSI equation where it remains a constant. It should be mentioned that both the spectral parameters k_j and l_j as well as the amplitude of the dromion solution are governed by the same time evolution equation (differing only in integration constants). Thus, depending on the nature of the parameters involved, the amplitude of the dromion solution (20) either grows (explodes) or decays with time. A typical example is shown in Figs. 1–3. We call these types of solutions as "explode-decay dromions" which is reminiscent of the explode-decay solitons of the inhomogeneous (1+1) dimensional nonlinear pdes [12]. To our knowledge this seems to be the first instance of such a localized solution being realized in a (2+1) dimensional nonlinear pde.

Finally, we wish to point out that the above type of localized solutions with time varying amplitude do exist for the isospectral Davey-Stewartson I equation also. Using the following transformations

$$\hat{q} = q \exp\left(-i\left[\frac{1}{4}\omega_1\left(\xi^2 + \eta^2\right) + \frac{1}{4}a_1(\xi - \eta) + \frac{1}{2}a_1^2 t\right]\right), \tag{23a}$$

$$\hat{U} = U + \frac{1}{4}\omega_1^2\eta^2 - \left(\frac{1}{2}a_1\omega_1 - \omega_0\right)\eta, \tag{23b}$$

$$\hat{V} = V + \frac{1}{4}\omega_1^2\xi^2 - \left(\frac{1}{2}a_1\omega_1 - \omega_0\right)\xi, \tag{23c}$$

the nonisospectral DSI equation (3)–(4) reduces to the isospectral DSI equation

$$i\hat{q}_t + \hat{q}_{\xi\xi} + \hat{q}_{\eta\eta} + \left(\hat{U} + \hat{V}\right)\hat{q} = 0, \tag{24a}$$

$$\hat{U}_\xi = \frac{1}{2}\mid \hat{q} \mid_\eta^2, \tag{24b}$$

$$\hat{V}_\xi = \frac{1}{2}\mid \hat{q} \mid_\xi^2. \tag{24c}$$

It is well known that equation (24) admits exponentially localized solutions with constant amplitude by driving $\hat{U}$ and $\hat{V}$ by sech^2 potentials [1, 8]. However, by virtue of the above transformation (23), it is now evident that such localized solutions with time varying amplitude do exist for the isospectral DSI equation also for a careful choice of the potentials indicated by the transformations (23b) and (23c). However, this choice of the potential is not very obvious but for the above nonisospectral case by virtue of the transformations (23) and hence such localized solutions whose amplitude either grows or decays with time have eluded earlier observation.

In this letter, we have generated a new class of localized coherent structures to DSI equation with inhomogeneous terms known as "explode-decay dromions" whose amplitude varies with time unlike the basic dromions. It remains to be seen how the multiexplode-decay dromions would interact in the context of the variation of the spectral parameter and this remains as an open question. We have also indicated the possibility of the existence of such solutions for the isospectral DSI equation itself.

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Figure Captions

Fig. 1: The time evolution of explode-decay dromion at $t = -0.5$

Fig. 2: $\mid q \mid$ at $t = 0$

Fig. 3: $\mid q \mid$ at $t = 0.2$