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Semiclassical Solutions of the Nonlinear Schrödinger Equation

A.V. SHAPOVALOV † and *A.Yu. TRIFONOV* ‡

† *Department of Theor. Physics, Tomsk State University, Tomsk 634050, Russia*
E-mail: shpv@phys.tsu.ru

‡ *Department of Math. Physics, Tomsk Polytechnical University, Tomsk 634034, Russia*
E-mail: trifonov@phtd.tpu.edu.ru

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Abstract

A concept of semiclassically concentrated solutions is formulated for the multidimensional nonlinear Schrödinger equation (NLSE) with an external field. These solutions are considered as multidimensional solitary waves. The center of mass of such a solution is shown to move along with the bicharacteristics of the basic symbol of the corresponding linear Schrödinger equation. The leading term of the asymptotic WKB-solution is constructed for the multidimensional NLSE. Special cases are considered for the standard one-dimensional NLSE and for NLSE in cylindrical coordinates.

1 Introduction

Soliton phenomena is an attractive field of present day research in nonlinear physics and mathematics. Essential ingredients in the soliton theory are the nonlinear Schrödinger equation (NLSE) and its variants appearing in a wide spectrum of problems. Examples are coupled nonlinear optics [1, 2, 3, 4], superconductivity [5, 6], and excitation in lattice systems [7].

More exactly, solitons are identified with a certain class of reflectionless solutions of the equations integrable via the Inverse Scattering Transform method (see, for example, [8]). Such equations, including NLSE, are named soliton equations. At every instant a soliton is localized in a restricted spatial region with its centroid moving like a particle. The particle-like properties of solitons are also manifested in their elastic collisions.

Soliton equations make up a narrow class of nonlinear equations, whereas a wider set of nonlinear equations, being nonintegrable in the framework of the IST, possess soliton-like solutions. They are localized in some sense, propagate with small energy losses, and collide with a varied extent of inelasticity. These solutions are termed solitary waves (SWs), quasisolitons, soliton-like solutions, etc. to differentiate them from the solitons in the above exact meaning. The stability of the localized form of solitons and SWs and their elastic collisions have led to interesting physical applications.

It is of interest to study the influence of external fields on the soliton propagation. To do that it is necessary to modify the original solitary equation by introducing variable coefficients representing an external field potential that breaks the IST-integrability. This problem was considered by variational methods [9], the theory of soliton perturbations (see, for example, reviews [10, 11]), or by using an appropriate ansatz [4, 12].

An investigation of soliton-like states is a separate problem for multidimensional models. The IST constructions are unsuitable for them except for the D4 self-dual Yang-Mills equations and their reductions in $1 \leq D \leq 3$ (see, for example, [13] for details). The problem is more complicated in view of the singular behavior of NLSE in 2D [14]. Nevertheless, soliton-like solutions for NLSE on a plane can exist, as shown in [15] in terms of a suitable ansatz.

Since the methods for constructing exact solutions for multidimensional models are restricted if compared to the cases of 2D models, approximate methods should be used. An effective approach to the problem can be developed based on the WKB-method [16].

Consider the generalized nonlinear Schrödinger equation for a "matter" field $\Psi(\vec{x},t)$:

$$\begin{aligned} -i\hbar\frac{\partial}{\partial t}\Psi + \hat{\mathcal{H}}_{\rm nl}(\Psi) = \Big\{ -i\hbar\frac{\partial}{\partial t} + \frac{1}{2m}\Big(-i\hbar\nabla - \vec{\mathcal{A}}(\vec{x},t)\Big)^2 & \\ +V(\vec{x},t) - 2r|\Psi(\vec{x},t)|^2 \Big\}\Psi(\vec{x},t) = 0. & \end{aligned} \tag{1}$$

Here, $\vec{x} \in \mathbb{R}^n$, $t \in \mathbb{R}^1$; $V(\vec{x},t)$, $\vec{\mathcal{A}}(\vec{x},t)$ are given functions; m is a real constant, r is a real parameter of nonlinearity; $\hbar$ is Planck's constant playing the role of an asymptotic parameter; $|\Psi|^2 = \Psi^*\Psi$, Ψ^* is the complex conjugate of Ψ.

In such a form equation (1) was studied in Refs. [5, 15]. Since the field of application of (1) is wider, the physical meaning of the quantities entering in (1) may be quite different from the quantum-mechanical one. In particular, in the one-dimensional problem of the propagation of an optical pulse [2, 3] we have to assume for equation (1) $\vec{x} = x \in \mathbb{R}^1$ with x being a normed temporal variable and t being a normed space coordinate along which the pulse propagates. The function $\Psi(x,t)$ is an envelope of the pulse field. Equation (1) takes the form

$$\Big\{ -i\hbar\frac{\partial}{\partial t} + \frac{1}{2m}\Big(i\hbar\frac{\partial}{\partial x} + \mathcal{A}(x,t)\Big)^2 + V(x,t) - 2r|\Psi(x,t)|^2 \Big\}\Psi(x,t) = 0. \tag{2}$$

The functions $V(x,t)$ and $\mathcal{A}(x,t)$ simulate the heterogeneity of the medium. The symmetry of equation (2) was considered in Ref. [17], and asymptotical solutions where studied for (2) with $\mathcal{A} = 0$, $r = r(x,t)$ in Ref. [18].

In the present work we have formulated the concept of semiclassically localized solutions for (1), following the ideas of Ref. [19]. These solutions are the multidimensional analogues of the soliton, like the solutions for (1). The particle-like properties of a solitary wave are described in terms of the wave centroid. The latter is shown to move along with the bicharacteristics of the basic symbol of the correspondent *linear* Schrödinger equation. We construct asymptotical WKB-solutions for equation (1) and consider some examples.

2 Semiclassical concentraited solutions

Soliton solutions are known to show particle properties. In classical mechanics, a particle is completely described by its phase orbit. Therefore, it is natural to introduce a similar concept for the soliton-like solutions of the nonlinear Schrödinger equation (1). The way by which to introduce the phase orbit seems to be obvious enough. It is based on the fact that in quantum mechanics, the first moments of a state $\Psi(\vec{x},t)$ play the role of the phase orbit for the quantum system. Let now $\Psi(\vec{x},t)$ be a solution of NLSE (1). The generalized position operators are $\hat{\vec{x}}(=\vec{x})$ and their conjugate momentum variables are $\hat{\vec{p}}(=-i\hbar\nabla)$,

$$[\hat{x}_k,\hat{p}_s]=i\hbar\delta_{k,s},\qquad k,s=\overline{1,n}.$$

The mean value of an operator $\hat{A}$ by the solution Ψ is defined as

$$\langle A\rangle=\frac{\langle\Psi|\hat{A}|\Psi\rangle}{\|\Psi\|^2}.\tag{3}$$

Here, $\|\Psi\|^2=\langle\Psi|\Psi\rangle$;

$$\langle\Psi|\hat{A}(t)|\Psi\rangle=\int\Psi^*(\vec{x},t)\hat{A}(t)\Psi(\vec{x},t)d\vec{x},$$

is a function of time for every operator $\hat{A}(t)$ and it parametrically depends on $\hbar$,

$$\langle\vec{x}\,\rangle=\vec{x}(t,\hbar),\qquad\langle\vec{p}\,\rangle=\vec{p}(t,\hbar).\tag{4}$$

If there exist the limits:

$$\lim_{\hbar\to0}\vec{x}(t,\hbar)=\vec{x}(t),\qquad\lim_{\hbar\to0}\vec{p}(t,\hbar)=\vec{p}(t),\tag{5}$$

then $\vec{x}(t)$ and $\vec{p}(t)$ are natural to be named the phase orbit of the classical system corresponding to the given solution Ψ. It is obvious that both the mean values (4) and the limit values (5) depend on the solution Ψ in the general case. Hence, the choice of the solution Ψ meets the requirement for the expressions (5) to be a solution of the classical equations of motion. By analogy with quantum mechanics (see Ref. [19]), we can define the soliton-like solutions asymptotic in $\hbar\to0$ as follows.

Definition 1 *Let $z(t)=\{(\vec{x}(t),\vec{p}(t)),\ 0\leqslant t\leqslant T\}$ be an arbitrary phase orbit in $\mathbb{R}^{2n}$. We name the solution $\Psi(\vec{x},t,\hbar)$ of equation (1) as a semiclassically concentrated solution (SCS) of the class $\mathbb{CS}_S(z(t),N)$ ($\Psi\in\mathbb{CS}_S(z(t),N)$) if:*

(i) *there exist the generalized limits* [1]

$$\lim_{\hbar\to0}\frac{|\Psi(\vec{x},t,\hbar)|^2}{\|\Psi\|^2}=\delta(\vec{x}-\vec{x}(t)),\qquad\lim_{\hbar\to0}\frac{|\tilde{\Psi}(\vec{p},t,\hbar)|^2}{\|\Psi\|^2}=\delta(\vec{p}-\vec{p}(t));$$

(ii) *there exist the centered moments*

$$\Delta^{(k)}_{\alpha,\beta}(t,\hbar)=\langle\hat{\Delta}^{(k)}_{\alpha,\beta}\rangle,\qquad0\leqslant k\leqslant N.$$

[1] By generalized limit we mean the passage to the limit standardly defined in the distribution theory (see, for example, Ref. [20]).

Here, α and β are multiindices, $|\alpha| + |\beta| = k$, $0 \leqslant k \leqslant N$, and $\hat{\Delta}^{(k)}_{\alpha,\beta}$ is an operator with the symmetrized (Weyl) symbol $\Delta^{(k)}_{\alpha,\beta}(\vec{p}, \vec{x}) = (\vec{p} - \vec{p}(t))^{\alpha}(\vec{x} - \vec{x}(t))^{\beta}$. Recall that the multiindex α is a vector of the form $\alpha = (\alpha_1, \ldots, \alpha_n)$, where $\alpha_j \geqslant 0$ are integer numbers. In addition, $|\alpha| = \sum\limits_{j=1}^{n} \alpha_j$ and for a vector $\vec{\zeta} = (\zeta_1, \ldots, \zeta_n) \in \mathbb{R}^n$ we suppose $\vec{\zeta}^{\alpha} = \prod\limits_{j=1}^{n} \zeta_j^{\alpha_j}$.

Note that the vectors $\vec{x}(t)$ and $\vec{p}(t)$ in Definition 1 are by no means connected with each other. The vector $z(t) = (\vec{p}(t), \vec{x}(t))$ is named the classical phase orbit of the system.

Definition 1 specifies the concept of solitary waves for asymptotic solutions of NLSE (1).

Theorem 1 *If a solution Ψ of (1) is semiclassically concentrated ($\Psi \in \mathbb{CS}_S(z(t), N)$), then $z(t) = (\vec{p}(t), \vec{x}(t))$ is a solution of a classical Hamilton system with the Hamiltonian*

$$\mathcal{H}_{\rm cl}(\vec{p}, \vec{x}, t) = \frac{1}{2m}\left(\vec{p} - \vec{\mathcal{A}}(\vec{x}, t)\right)^2 + V(\vec{x}, t).$$

Proof. Let $\hat{A}$ be an operator, then the Ehrenfest theorem [21] is true for the mean value of $\hat{A}$:

$$\frac{d}{dt}\langle A\rangle = \langle \frac{\partial \hat{A}}{\partial t}\rangle + \frac{i}{\hbar}\langle[\hat{\mathcal{H}}_{nl}, \hat{A}]\rangle.$$

In particular, for the operators $\hat{\vec{p}}$ and $\hat{\vec{x}}$ we have:

$$\frac{d}{dt}\langle \vec{p}\rangle = \frac{i}{\hbar}\langle[\hat{\mathcal{H}}_{\rm nl}, \hat{\vec{p}}]\rangle, \qquad \frac{d}{dt}\langle \vec{x}\,\rangle = \frac{i}{\hbar}\langle[\hat{\mathcal{H}}_{\rm nl}, \hat{\vec{x}}\,]\rangle. \tag{6}$$

Using the obvious relations

$$\langle[|\Psi(\vec{x}, t)|^2, \hat{\vec{p}}]\rangle = \langle[|\Psi(\vec{x}, t)|^2, \hat{\vec{x}}\,]\rangle = 0,$$

we have from (6):

$$\frac{d}{dt}\langle \vec{p}\rangle = \frac{i}{\hbar}\langle[\hat{\mathcal{H}}_{\rm l}, \hat{\vec{p}}]\rangle, \qquad \frac{d}{dt}\langle \vec{x}\,\rangle = \frac{i}{\hbar}\langle[\hat{\mathcal{H}}_{\rm l}, \hat{\vec{x}}\,]\rangle, \tag{7}$$

where

$$\hat{\mathcal{H}}_{\rm l} = \frac{1}{2m}\left(i\hbar\nabla + \vec{\mathcal{A}}(\vec{x}, t)\right)^2 + V(\vec{x}, t).$$

With Definition 1 of the SCS further proof coincides with a similar one for the linear case [19].

Remark. Emphasize that, as follows from the theorem, the centroid of the SCS Ψ moves along the bicharacteristics of the linear Schrödinger equation.

3 Asymptotic solutions

In Section 2 we have discussed the definition and the basic features of SCS. Here, we study a theoretical possibility of construction of the WKB-asymptotic semiclassically concentrated solutions of equation (1) on a limited time domain $0 < t < T$ with $\hbar$-independent T.

Taking into account the form of the one-soliton solution of NLSE (see, for example, [8]), let us try solution of equation (1) in the form

$$\Psi = \rho(\theta, \vec{x}, t, \hbar) \exp\left[\frac{i}{\hbar} S(\vec{x}, t, \hbar)\right]. \tag{8}$$

Here, $\theta = \hbar^{-1}\sigma(\vec{x}, t, \hbar)$ is a "fast" variable; $\sigma(\vec{x}, t, \hbar)$, $\rho(\theta, \vec{x}, t, \hbar)$, and $S(\vec{x}, t, \hbar)$ are real functions regular in $\hbar$, that is:

$$S(\vec{x}, t, \hbar) = S(\vec{x}, t) + \hbar S_1(\vec{x}, t) + \cdots .$$

The solution (8) is assumed to be localized according to Definition (1).

The derivative operators $\partial/\partial t$ and ∇ act on the function (8) as follows:

$$-i\hbar\frac{\partial}{\partial t} = -i\hbar\frac{\partial}{\partial t}\Big|_{\theta=\mathrm{const}} - i\sigma_{,t}\frac{\partial}{\partial\theta}, \qquad -i\hbar\nabla = -i\hbar\nabla|_{\theta=\mathrm{const}} - i(\nabla\sigma)\frac{\partial}{\partial\theta},$$

where $\sigma_{,t} = \partial\sigma/\partial t$.

Henceforth we put $\partial/\partial t|_{\theta=\mathrm{const}} \equiv \partial_t$, $\nabla|_{\theta=\mathrm{const}} \equiv \nabla$, $\partial/\partial\theta \equiv \partial_\theta$.

Substituting (8) into (1) we find:

$$\begin{aligned}\exp\left(\frac{i}{\hbar}S\right)\Big\{ & -i\hbar\partial_t + S_{,t} - i\sigma_{,t}\partial_\theta + V - \frac{\hbar^2}{2m}\nabla^2 - \frac{\hbar}{2m}(\nabla^2\sigma)\partial_\theta - \frac{\hbar}{m}(\nabla\sigma\cdot\nabla)\partial_\theta \\ & -i\frac{\hbar}{2m}(\nabla^2 S) - \frac{1}{2m}(\nabla\sigma)^2\partial^2_{\theta\theta} - i\frac{\hbar}{m}(\nabla S\cdot\nabla) - i\frac{1}{m}(\nabla S\cdot\nabla\sigma)\partial_\theta + \frac{1}{2m}(\nabla S)^2 \\ & +i\frac{\hbar}{m}(\vec{A}\cdot\nabla) + i\frac{1}{m}(\vec{A}\cdot\nabla\sigma)\partial_\theta - \frac{1}{m}(\vec{A}\cdot\nabla S) + i\frac{\hbar}{2m}(\nabla\vec{A}) + \frac{1}{2m}\vec{A}^2 - 2r\rho^2\Big\}\rho = 0.\end{aligned}$$

Let us gather $\hbar$-free terms in this equation and put their sum to zero. In the obtained equation we separate real and imaginary parts and then separate the "fast" variable θ from others. As a result we come to the following system of equations which determines the leading term of the asymptotic solution:

$$\sigma_{,t} + \frac{1}{m}\langle\left(\nabla S - \vec{\mathcal{A}}\right), \nabla\sigma\rangle = 0, \tag{9}$$

$$S_{,t} + V + \frac{1}{2m}\left(\nabla S - \vec{\mathcal{A}}\right)^2 = r\tilde{b}(t, \vec{x}), \tag{10}$$

$$\frac{1}{2m}(\nabla\sigma)^2\rho_{,\theta\theta} + 2\rho^3 = r\tilde{b}\rho. \tag{11}$$

Here, $\langle\vec{a}, \vec{b}\rangle$ denotes the Euclidean scalar product of the vectors: $\sum\limits_{j=1}^{n} a_j b_j$; the function $\tilde{b}(t, \vec{x})$ appears as a "separation parameter" in separating the "fast" variable θ, and $\tilde{b}(t, \vec{x})$ is to be determined in what follows.

Let us look for $\rho(\theta, \vec{x}, t, \hbar)$ in the class of functions satisfying the conditions:

$$\lim_{\theta\to\infty}\rho(\theta, \vec{x}, t, \hbar) = \lim_{\theta\to\infty}\rho_{,\theta}(\theta, \vec{x}, t, \hbar) = 0. \tag{12}$$

Integrating equation (11) in view of (12) we obtain:

$$\rho_{,\theta} = \sqrt{\frac{2mr}{(\nabla\sigma)^2}}\sqrt{\tilde{b}-\rho^2}\rho. \tag{13}$$

Let us put $r = \varkappa^2 > 0$ that corresponds to the existence of soliton solutions in the case when equation (1) is reduced to the standard NLSE (one-dimensional with $V(\vec{x},t) =$, $\vec{\mathcal{A}}(\vec{x},t) = 0$) [8]. Then $\tilde{b} = b^2 > 0$ and further integration of (13) results in

$$\rho = \frac{b}{\cosh\left\{b\varkappa\sqrt{2m(\nabla\sigma)^{-2}}\left(\frac{1}{\hbar}\sigma(\vec{x},t) + \sigma_1(\vec{x},t)\right)\right\}}, \tag{14}$$

where the function $\sigma_1(\vec{x},t)$ appears as a “separation constant” which is to be determined later.

By definition, the “fast” variable θ must have the structure $\frac{1}{\hbar}\sigma(\vec{x},t)$. For equation (14) to correspond this condition it is necessary to set

$$b\varkappa\sqrt{\frac{2m}{(\nabla\sigma)^2}} = \mathrm{const} = \alpha.$$

Without loss of generality we can put $\alpha = 1$ and then we have

$$\rho = \sqrt{\frac{(\nabla\sigma)^2}{2m\varkappa^2}}\,\frac{1}{\cosh\left(\frac{1}{\hbar}\sigma(\vec{x},t) + \sigma_1(\vec{x},t)\right)}. \tag{15}$$

Equation (10) takes the form

$$S_{,t} + V + \frac{1}{2m}\left(\nabla S - \vec{\mathcal{A}}\right)^2 = \frac{1}{2m}(\nabla\sigma)^2,$$

that, together with (9), is equivalent to the single complex Hamilton-Jacobi equation

$$(S+i\sigma)_{,t} + V + \frac{1}{2m}\left[\nabla(S+i\sigma) - \vec{\mathcal{A}}\right]^2 = 0. \tag{16}$$

Thus, for the leading term of the asymptotic expansion (8),

$$\Psi(\vec{x},t,\hbar) = \Psi^0(\vec{x},t,\hbar) + O(\hbar),$$

we have:

$$\Psi^0 = \rho(\theta,\vec{x},t,\hbar)\exp\left[\frac{i}{\hbar}S(\vec{x},t) + S_1(\vec{x},t)\right], \tag{17}$$

where ρ has the form (15) and the functions $\sigma_1(\vec{x},t), S_1(\vec{x},t)$ are determined from successive approximations. The function (17) can be represented in the form:

$$\Psi(\vec{x},t,\hbar) = 2\sqrt{\frac{(\nabla\sigma)^2}{2m\varkappa^2}}\,\frac{\Psi_0(\vec{x},t,\hbar)}{1+|\Psi_0(\vec{x},t,\hbar)|^2}, \tag{18}$$

where

$$\Psi_0(\vec{x},t,\hbar) = \exp\left\{\frac{i}{\hbar}\left[S(\vec{x},t) + i\sigma(\vec{x},t) + \hbar(S_1(\vec{x},t) + i\sigma_1(\vec{x},t))\right]\right\}.$$

It can easily to show that Ψ_0 is an asymptotic solution of the *linear* Schrödinger equation:

$$\left\{-i\hbar\frac{\partial}{\partial t}+V(\vec{x},t)+\frac{1}{2m}\left(-i\hbar\nabla-\vec{\mathcal{A}}(\vec{x},t)\right)^2\right\}\Psi_0(\vec{x},t)=O(\hbar^\alpha), \tag{19}$$

where $\alpha = 1$. Let us write a function Ψ, which satisfies equation (1) to an accuracy of $O(\hbar^2)$, in the form

$$\Psi=\Psi^0(1+\hbar\Psi^1). \tag{20}$$

Here, Ψ^0 is determined by expression (17) and the function $\Psi^1(\theta,\vec{x},t)$ is to be determined. As can be seen from (15), it is convenient to take the variable θ in the form

$$\theta=\frac{1}{\hbar}\sigma(\vec{x},t)+\sigma_1(\vec{x},t). \tag{21}$$

Let us denote $\operatorname{Re}\Psi^1(\theta,\vec{x},t)=u(\theta,\vec{x},t)$, $\operatorname{Im}\Psi^1(\theta,\vec{x},t)=v(\theta,\vec{x},t)$. Substituting (20) into (1) and setting to zero summands at the equal powers of $\hbar$. Extract real and imaginary parts in the obtained equations and find a system of equations for function (20). This system includes the Hamilton-Jacobi equation (16) for the functions $S(\vec{x},t)$, $\sigma(\vec{x},t)$ and also the following equations determining the functions $S_1(\vec{x},t)$ and $\sigma_1(\vec{x},t)$:

$$S_{1,t}+\frac{1}{m}\langle\left(\nabla S-\vec{\mathcal{A}}\right),\nabla S_1\rangle-\frac{1}{m}\langle\nabla\sigma,\nabla\sigma_1\rangle+\frac{1}{2m}\Delta\sigma+\frac{1}{2m}\langle\nabla\sigma,\nabla\rangle\log(\nabla\sigma)^2=0, \tag{22}$$

$$\begin{aligned}&\sigma_{1,t}+\frac{1}{m}\langle\left(\nabla S-\vec{\mathcal{A}}\right),\nabla\sigma_1\rangle+\frac{1}{m}\langle\nabla\sigma,\nabla S_1\rangle\\&\quad-\frac{1}{2}\left[\frac{1}{m}\langle\nabla,\left(\nabla S-\vec{\mathcal{A}}\right)\rangle+\left[\log(\nabla\sigma)^2\right]_{,t}+\frac{1}{m}\langle\left(\nabla S-\vec{\mathcal{A}}\right),\nabla\log(\nabla\sigma)^2\rangle\right]=0.\end{aligned} \tag{23}$$

The functions $u(\theta,\vec{x},t)$, $v(\theta,\vec{x},t)$ are given by the expressions

$$\begin{aligned}\rho(\theta,\vec{x},t)u(\theta,\vec{x},t)&=\sqrt{\frac{2m}{\varkappa^2(\nabla\sigma)^2}}\frac{1}{\cosh\theta}\left\{C_1(\vec{x},t)\tanh\theta+\frac{1}{2m}\langle\nabla\sigma,\nabla\sigma_1\rangle\right.\\&\left.+\frac{1}{12m}\left[\Delta\sigma+\langle\nabla\sigma,\nabla\log(\nabla\sigma)^2\rangle\right]\left(\sinh\theta\cosh\theta-\epsilon\cosh^2\theta\right)\right\},\end{aligned} \tag{24}$$

$$\begin{aligned}\rho(\theta,\vec{x},t)v(\theta,\vec{x},t)&=\sqrt{\frac{2m}{\varkappa^2(\nabla\sigma)^2}}\left\{\frac{C_1(\vec{x},t)}{\cosh\theta}+\frac{1}{4}\left[\frac{1}{m}\langle\nabla,\nabla S-\vec{\mathcal{A}}\rangle\right.\right.\\&\left.\left.+\left(\partial_t+\frac{1}{m}\langle\left(\nabla S-\vec{\mathcal{A}}\right),\nabla\rangle\right)\log(\nabla\sigma)^2\right](\epsilon\sinh\theta-\cosh\theta)\right\}.\end{aligned} \tag{25}$$

Here, $\epsilon=\operatorname{sign}(\sigma)$ and the function $C_1(\vec{x},t)$ is determined by successive approximations.

Remark. From relations (22) and (23) it follows that (19) is accurate to $O(\hbar^2)$.

Relation (18) can be considered as a transformation connecting the asymptotic solutions of the nonlinear equation (1) with the solution of the linear equation (19) for $\alpha = 2$.

So, the leading term Ψ^0 of the asymptotic solution of equation (1) is completely fixed by expressions (15), (17), (20), and (21), where the functions $S(\vec{x},t)$, $\sigma(\vec{x},t)$, $S_1(\vec{x},t)$, and $\sigma_1(\vec{x},t)$ are determined by the system of equations (16), (22), and (23).

4 Special solutions

For a more detailed study of the above asymptotic solutions (17), let us consider some special cases of these solutions.

4.1 One-dimensional NLSE

Let us put $\vec{x} = x \in \mathbb{R}^1$, $\Delta = \partial^2/\partial x^2$ and $\vec{\mathcal{A}} = V = 0$ in (1); then equation (1) takes the form

$$\left[i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + 2\varkappa^2|\Psi(x,t,\hbar)|^2\right]\Psi(x,t,\hbar) = 0. \tag{26}$$

The leading term of the asymptotic solution (20) is as follows:

$$\Psi(x,t) = \rho(x,t,\hbar)\exp\left[\frac{i}{\hbar}(S(x,t) + \hbar S_1(x,t))\right], \tag{27}$$

where

$$\rho(x,t,\hbar) = \sqrt{\frac{\sigma_{,x}^2}{2m\varkappa^2}}\,\frac{1}{\cosh\left(\frac{1}{\hbar}\sigma(x,t) + \sigma_1(x,t)\right)}. \tag{28}$$

Equations (16), (22), and (23) become:

$$(S + i\sigma)_{,t} + \frac{1}{2m}(S + i\sigma)_{,x}^2 = 0, \tag{29}$$

$$S_{1,t} + \frac{1}{m}S_{,x}S_{1,x} - \frac{1}{m}\sigma_{,x}\sigma_{1,x} + \frac{3}{2m}\sigma_{,xx} = 0, \tag{30}$$

$$\sigma_{1,t} + \frac{1}{m}S_{,x}\sigma_{1,x} + \frac{1}{m}\sigma_{,x}S_{1,x} - \frac{1}{2}\left[\frac{1}{m}S_{,xx} + \frac{2}{\sigma_{,x}}\sigma_{,xt} + \frac{2}{m}\frac{S_{,x}}{\sigma_{,x}}\sigma_{,xx}\right] = 0. \tag{31}$$

To construct solution (27) and (28), let us look for a special solution of equation (29) in the form

$$S = \alpha_1 t + \alpha_2 x + \varphi_0, \qquad \sigma = \beta_1 t + \beta_2(x - x_0), \tag{32}$$

where α_1, α_2, β_1, β_2, φ_0, and x_0 are real constants. Substitution (32) into (29) gives

$$\alpha_1 = \frac{1}{2m}(\beta_2^2 - \alpha_2^2), \quad \beta_1 = -\frac{1}{m}\alpha_2\beta_2.$$

Substituting (32) into (30) and (31), we easily obtain the following complex equation for the functions S_1 and σ_1:

$$(S_1 + i\sigma_1)_{,t} + \frac{\alpha_2 + i\beta_2}{m}(S_1 + i\sigma_1)_{,x} = 0. \tag{33}$$

Equation (33) is a complex wave equation the solution of which can be written as

$$w(x,t) \equiv S_1(x,t) + i\sigma_1(x,t) = f(x - at). \tag{34}$$

Here, $a \equiv (\alpha_2+i\beta_2)/m$ and $f(\zeta)$ are analytical functions of the complex variable $\zeta = x-at$. Denote $\beta_2 = 2\eta$ and $\alpha_2 = 2\xi$, then solution (27) takes the form

$$\begin{aligned}\Psi = &-\frac{2\eta}{\varkappa\sqrt{2m}}\ \frac{1}{\cosh\left[\frac{2\eta}{\hbar}\left(x-x_0-\frac{2\xi}{m}t\right)+\mathrm{Im}f(x-at)\right]}\\ &\times\exp\left[\frac{i}{\hbar}(2\xi x-\frac{2}{m}(\xi^2-\eta^2)t+\varphi_0+\hbar\mathrm{Re}\,f(x-at))\right].\end{aligned} \tag{35}$$

If $f(x-at)=0$ in (34), then (35) takes the form of the exact one-soliton solution of the nonlinear Schrödinger equation (26) which is reduced to the standard form [8] when $\hbar = m = 1$, $\varkappa^2 = 1/2$.

If $f(x-at)\neq 0$, we have an asymptotic solution similar to the one-soliton solution.

4.2 NLSE with a separated potential

Consider equation (26) with a potential $V(x,t)$:

$$\left[i\hbar\frac{\partial}{\partial t}+\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+2\varkappa^2|\Psi(x,t,\hbar)|^2-V(x,t)\right]\Psi(x,t,\hbar)=0. \tag{36}$$

Solution (27) and (28) is determined by (30), (31), and

$$(S+i\sigma)_{,t}+\frac{1}{2m}(S+i\sigma)^2_{,x}+V=0. \tag{37}$$

For the potential V of the separated form,

$$V(x,t)=v_0(t)+v_1(x),$$

we can easily find for the system (37), (30), and (31) two classes of separated solutions determining (27) and (28).

The first class is described by the expressions:

$$S(x,t)=c_1t-\int v_0(t)dt,\qquad \sigma(x,t)=\int\sqrt{2m(c_1+v_1(x))}dx,$$

$$S_1(x,t)=c_2t+c_3,\qquad \sigma_1=(3/2)\log|\sigma_{,x}|+mc_2\int\sigma_{,x}^{-1}dx+c_4,$$

where $c_1,\dots,c_4=\mathrm{const}$. The second class is presented by

$$S(x,t)=c_3t-\int v_0(t)dt+c_4,\qquad \sigma(x,t)=c_1\left[t-m\int\frac{dx}{p'(x)}\right]+c_2,$$

$$S_1(x,t)=a_1t+a_3+f(x),\qquad \sigma_1=a_2t+a_4+g(x),$$

Here, $c_1,\dots,c_4; a_1,\dots,a_4=\mathrm{const}$ and the functions $p(x)$, $f(x)$, $g(x)$ are determined in quadratures from the following equations:

$$\frac{1}{m}(p'(x))^2=-(v_1(x)+c_3)+\sqrt{(v_1(x)+c_3)^2+c_1^2},$$

$$g'(x)=\frac{m}{p'(x)}\left[\frac{c_1}{p'(x)}f'(x)-\frac{1}{2m}p''(x)-a_2\right],$$

$$[(p'(x))^4+c_1^2m^2)f'(x)=c_1m^2a_2p'(x)-ma_1[p'(x)]^3-c_1mp'(x)p''(x).$$

The above solutions show the potentialities of separation of variables as applied to the solution of the general system of equations (16), (22), and (23).

4.3 Cylindrical coordinates

The self-focusing effect of a beam of high power optical radiation propagating along the z-axis is described by a nonlinear Schrödinger equation of the form [22]:

$$\left[i\hbar\frac{\partial}{\partial t}+\frac{\hbar^2}{2m}\Delta_\perp+2\varkappa^2|\Psi(x,y,t,\hbar)|^2\right]\Psi(x,y,t,\hbar)=0. \tag{38}$$

Here, t is the time coordinate in the coordinate system moving with a group velocity along the direction of the radiation propagation. $\Delta_\perp=\partial^2/\partial x^2+\partial^2/\partial y^2$ is the Laplace operator in a plane orthogonal to the radiation direction. A function $V(t,x,y)$ simulates some nonstationarity and heterogeneity of the medium where the pulse is propagating. In the stationary statement of the problem, the variable z takes the place of t.

Let us find asymptotic solutions of the form (20) for equation (38) in the cylindrical coordinates

$$x=r\cos\varphi,\qquad y=r\sin\varphi. \tag{39}$$

The leading term of the asymptotic solution (20) takes the form

$$\Psi(r,\varphi,t,\hbar)=\rho(r,\varphi,t,\hbar)\exp\left\{\frac{i}{\hbar}[S(r,\varphi,t)+\hbar S_1(r,\varphi,t)]\right\}, \tag{40}$$

where

$$\rho(r,\varphi,t,\hbar)=\sqrt{\frac{(\nabla_\perp\sigma)^2}{2m\varkappa^2}}\,\frac{1}{\cosh\left(\frac{1}{\hbar}\sigma(r,\varphi,t)+\sigma_1(r,\varphi,t)\right)}, \tag{41}$$

$\nabla_\perp$ is the gradient operator with respect to the variables r and φ (39).

The special solution, illustrating a transverse heterogeneity of the optical pulse, can be written as follows:

$$\begin{aligned}\Psi(r,\varphi,t,\hbar)&=\frac{c_1}{\varkappa\sqrt{2m}}\cosh^{-1}\left[\left(\frac{c_1}{\hbar}+a_2\right)r+c_1b_1t+\frac{1}{2}\log r+\frac{a_1}{\hbar}+a_3\right]\\&\times\exp\left\{i\left[\left(\frac{c_1^2}{2m\hbar}+\frac{a_2c_1}{m}\right)t-mb_1r+\frac{c_2}{\hbar}+c_3\right]\right\},\end{aligned}$$

where $c_1,c_2,c_3,a_1,a_2,a_3,b_1=\text{const}$; $c_1\neq0$.

5 Conclusion

An outcome of the work are the asymptotic solutions of the solitary wave type for NLSE (1) obtained for a finite time interval $0<t<T$. The question of the validity of long-time asymptotics requires a special consideration, since, even for the linear case, this problem has received rather much attention in the literature [23].

Solitary wave solutions are considered here similar to quantum wavepackets. This permits one to investigate the particle-like properties of the SWs in terms of the Definition 1 based on the Ehrenfest theorem.

The equations of motion for the centroid of an SW are found to be independent of the nonlinearity factor r in (1), while the solution itself depends essentially on r. In such an approach, the centroid of the solitary wave solution moves like a classical particle in the external field described by the potentials $(V, \vec{\mathcal{A}})$ in (1). This completely corresponds to the well-known soliton properties for the one-dimensional case [12]. Such a situation takes place only when the nonlinearity factor is constant. If $r = r(\vec{x}, t)$, the field $\Psi(\vec{x}, t, \hbar)$ results in the appearance of additional classical variables, and the correspondent equations of motion become much more complicated.

The problems discussed are beyond the scope of this work and requires a special investigation.

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