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Psi-Series Solutions of the Cubic Hénon-Heiles System and Their Convergence

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Abstract

The cubic Hénon-Heiles system contains parameters, for most values of which, the system is not integrable. In such parameter regimes, the general solution is expressible in formal expansions about arbitrary movable branch points, the so-called psi-series expansions. In this paper, the convergence of known, as well as new, psi-series solutions on real time intervals is proved, thereby establishing that the formal solutions are actual solutions.

1. Introduction

The generalized Hénon-Heiles Hamiltonian is

$$H = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + Ax^2 + By^2\right) + Dx^2y^{m-2} - \frac{C}{m}y^m, \tag{1.1}$$

where $\dot{x} = \frac{dx}{dt}$, $\dot{y} = \frac{dy}{dt}$ and A, B, C and D are nonzero parameters. It corresponds to the original form given by Hénon and Heiles [1] in their studies of astronomical problems when $A = B = C = D = 1$ and $m = 3$. Here, we consider the cubic form ($m = 3$) of (1.1), with the corresponding equations of motion

$$\begin{aligned} &\ddot{x} + Ax + 2Dxy = 0, \\ &\ddot{y} + By + Dx^2 - Cy^2 = 0. \end{aligned} \tag{1.2}$$

Several studies have been made regarding the integrability properties and singularity structures of various versions of (1.2); for example, Chang, Tabor and Weiss [2] have analysed (1.2) with $A = B = 1$, whereas Chang, Greene, Tabor and Weiss [3] have considered the same system for general m. In keeping with their notation, we rescale the variables,

$$x \to \frac{x}{C}, \qquad y \to \frac{y}{C}, \qquad \lambda = \frac{D}{C}, \tag{1.3}$$

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to express (1.2) as

$$\begin{aligned} &\ddot{x} + Ax + 2\lambda xy = 0, \\ &\ddot{y} + By + \lambda x^2 - y^2 = 0. \end{aligned} \tag{1.4}$$

The leading-order behaviours of solutions about an arbitrary singularity, as well as their resonance structures (in the sense of the ARS algorithm, Ablowitz, Ramani and Segur [4, 5]), have been derived (see [2, 3]; Chang, Tabor, Weiss and Corliss [6]; Bountis, Segur and Vivaldi [7]; Weiss [8]; Tabor [9, p. 337]).

We summarize the results briefly, in order to fix notation and to single out the present objectives. Setting $\tau = t - t_0$, where t_0 is an arbitrary constant (the location of the singularity), and

$$x \sim a\tau^{\alpha}, \qquad y \sim b\tau^{\beta}, \qquad \tau \to 0, \tag{1.5}$$

two types of leading-order singular behaviours are found:

$$\begin{aligned} &\text{(i)} \quad \alpha = \beta = -2, \\ &\text{(ii)} \quad \beta = -2, \quad Re(\alpha) > \beta. \end{aligned} \tag{1.6}$$

Carrying the analysis further by writing

$$x \sim a\tau^{\alpha} + p\tau^{\alpha+k_r}, \qquad y \sim b\tau^{\alpha} + q\tau^{\beta+k_r}, \tag{1.7}$$

and determining those values of k_r for which p or q is undetermined, one finds the corresponding resonances. It turns out that

$$\begin{aligned} &\text{(i)} \quad \alpha = \beta = -2, \quad k_r = -1, 6, r, \bar{r}, \\ &\qquad \text{where} \quad r, \bar{r} = \frac{5}{2} \pm \frac{1}{2}\sqrt{1 - 24(\frac{1}{\lambda} + 1)}; \\ &\text{(ii)} \quad Re(\alpha) > -2, \quad \beta = -2, \quad \alpha, \bar{\alpha} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 48\lambda}, \quad k_r = -1, 6, 0, r, \bar{r}, \\ &\qquad \text{where} \quad r, \bar{r} = \mp\sqrt{1 - 48\lambda}. \end{aligned} \tag{1.8}$$

In case (ii), α and $\bar{\alpha}$ are two possible leading orders, with corresponding resonances r and $\bar{r}$. This case occurs only for $\lambda > -\frac{1}{2}$, since $Re(\alpha) > -2$.

The nature of the resonances for each value of λ is summarized as follows:

Case(i)

$\lambda < -\frac{24}{23}$	:	r and $\bar{r}$ are complex
$\lambda = -\frac{24}{23}$	:	$r = \bar{r} = \frac{5}{2}$
$-\frac{24}{23} < \lambda < -\frac{1}{2}$	:	$r > 0, \quad \bar{r} > 0, \quad r \neq \bar{r}$
$\lambda = -\frac{1}{2}$	:	$r = 5, \quad \bar{r} = 0$
$-\frac{1}{2} < \lambda < 0$	:	$r > 0, \quad \bar{r} < 0$
$\lambda > 0$	:	r and $\bar{r}$ are complex

Case (ii)

$$
\begin{array}{lll}
-\dfrac{1}{2} < \lambda < 0 & : & r < 0, \quad \bar{r} > 0 \\[2ex]
0 < \lambda < \dfrac{1}{48} & : & r < 0, \quad \bar{r} > 0 \\[2ex]
\lambda = \dfrac{1}{48} & : & r = \bar{r} = 0 \\[2ex]
\lambda > \dfrac{1}{48} & : & r \quad \text{and} \quad \bar{r} \quad \text{are pure imaginary}
\end{array}
$$

Those cases where a negative resonance is present (in addition to -1) likely correspond to singular, rather than general, solutions (see, e.g., [9, p. 339]). The significance of repeated resonances (case (i), $\lambda = -\frac{24}{23}$, and case (ii), $\lambda = \frac{1}{48}$) is, at present, unknown. The case $\lambda = -\frac{1}{2}$ is somewhat anomalous [9, p. 340] and we shall comment upon it later. In those cases where there exist four distinct resonances, the usual -1 and three others with nonnegative real parts, general solutions may be written down in series expansions about the arbitrary singularity t_0. These may be Laurent series, if the system passes the Painlevé test [5] or, otherwise, psi-series (Hille [10, p. 249]).

It has been found [2] that the system is integrable when $\lambda = -1$ and $A = B$, when $\lambda = -\frac{1}{6}$, and when $\lambda = -\frac{1}{16}$ and $B = 16A$. In the latter case, although the system is integrable, it passes only the "weak" Painlevé test in that it admits a rational movable branch point. Outside of these three cases, the system possesses solutions with movable branch points (logarithmic, irrational or complex), and a psi-series expansion is required to represent the general solution about an arbitrary branch point. Such expansions constitute formal general solutions containing four arbitrary constants, the remaining coefficients being well defined by a self-consistent recursion relation.

In this paper, we prove the absolute convergence of such series on real intervals of the form $0 < \tau < R$, $R > 0$, thus establishing that the formal solutions are actual solutions. Similar results have been obtained by Melkonian and Zypchen [11] for the Lorenz system (see, e.g., [9, p. 344] and Sparrow [12]), where only logarithmic psi-series occur. A study of the convergence problem for a class of second- and third-order ordinary differential equations has been made by Hemmi and Melkonian [13]. Some of the key ideas within the proofs may be extracted from a paper by Hille [14] regarding second-order quadratic systems. Studies of general systems of partial differential equations which admit WTC (Weiss, Tabor and Carnevale [15]) expansions containing logarithms have been made by Kichenassamy and Srinivan [16] and Kichenassamy and Littman [17], who have proved convergence by entirely different methods. Logarithmic psi-series occur for the Cubic Hénon Heiles system (1.4) when $\lambda = -1$ and $A \neq B$, or when $\lambda = -\frac{1}{16}$ and $B \neq 16A$, and these may be dealt with by the methods discussed in [11, 13, 16, 17].

Here, we restrict our attention to the non-logarithmic cases. Sections 2 and 3 deal with the case(i)-leading orders, involving a series with complex exponents (Section 2) and one with irrational exponents (Section 3). The necessary Lemmas are relegated to Appendix A. The series which we employ in Sections 2 and 3 are not the same as the ones given, e.g., in [3], and we confirm the validity (self-consistency) of our version in Appendix B. Section 4 concerns the case(ii)-leading orders where, once again, we employ a new series, the validity of which is confirmed in Appendix C. Concluding remarks are made in Section 5, including a discussion of the case $\lambda = -\frac{1}{2}$.

2. Case(i)-leading orders: $\alpha = \beta = -2$, $k_r = -1, 6, \frac{5}{2} \pm \frac{1}{2}\sqrt{1 - 24\left(\frac{1}{\lambda} + 1\right)}$ complex

Here, the resonances $r, \bar{r} = \frac{5}{2} \pm \frac{1}{2}\sqrt{1 - 24(\frac{1}{\lambda} + 1)}$ are complex, with $\lambda < -\frac{24}{23}$ or $\lambda > 0$. The general solution may be taken in the form

$$\begin{aligned} x &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} a_{klm}\tau^{k-2+rl+\bar{r}m}, \\ y &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} b_{klm}\tau^{k-2+rl+\bar{r}m}. \end{aligned} \tag{2.1}$$

The self-consistency of (2.1), as well as the presence of four arbitrary constants, is shown in Appendix B. It is also shown therein that (2.1) is equivalent to

$$\begin{aligned} x &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \left[a_{kl}\tau^{k-2+rl} + \bar{a}_{kl}\tau^{k-2+\bar{r}l}\right], \\ y &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \left[b_{kl}\tau^{k-2+rl} + \bar{b}_{kl}\tau^{k-2+\bar{r}l}\right], \end{aligned} \tag{2.2}$$

and that (2.2) is equivalent to the solution given in [3]. Either (2.1) or (2.2) may be employed to prove convergence. However, the advantage of starting with (2.1) is that it is much easier to deal with a single, triply-indexed series such as (2.1) than a sum of two, doubly-indexed ones such as (2.2), vis-à-vis the confirmation of its validity.

The convergence proof consists of resumming (2.2) into more tractable forms, and showing that the latter are majorized by series which converge on an interval, by the ratio test. Thus, let $\varepsilon = \frac{1}{2}\sqrt{24(\frac{1}{\lambda} + 1) - 1} > 0$, note that $r - 1, \bar{r} - 1 = \frac{3}{2} \pm i\varepsilon$, and write

$$\begin{aligned} x &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \left[a_{kl}\tau^{k-2+l+\left(\frac{3}{2}+i\varepsilon\right)l} + \bar{a}_{kl}\tau^{k-2+l+\left(\frac{3}{2}-i\varepsilon\right)l}\right] \\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \left[(a_{kl} + \bar{a}_{kl})\cos(\varepsilon l z) + i(a_{kl} - \bar{a}_{kl})\sin(\varepsilon l z)\right] e^{\frac{3}{2}lz}\tau^{k-2+l}, \end{aligned} \tag{2.3}$$

where $z = \ln(\tau)$, $\tau > 0$. It follows that

$$x = \sum_{\gamma=0}^{\infty} f_{\gamma}(z)\tau^{\gamma-2}, \tag{2.4}$$

where

$$f_{\gamma}(z) = \sum_{k+l=\gamma} \left[(a_{kl} + \bar{a}_{kl})\cos(\varepsilon l z) + i(a_{kl} - \bar{a}_{kl})\sin(\varepsilon l z)\right] e^{\frac{3}{2}lz} \tag{2.5}$$

is a polynomial in $e^{\frac{3}{2}z}$ of degree γ, with coefficients that are bounded functions of z. Similarly,

$$y = \sum_{\gamma=0}^{\infty} h_\gamma(z)\tau^{\gamma-2}, \tag{2.6}$$

where

$$h_\gamma(z) = \sum_{k+l=\gamma} \left[(b_{kl} + \bar{b}_{kl})\cos(\varepsilon l z) + i(b_{kl} - \bar{b}_{kl})\sin(\varepsilon l z)\right] e^{\frac{3}{2}lz} \tag{2.7}$$

is a polynomial of the same type as $f_\gamma(z)$.

Let $u(t) = \dot{x}(t) \equiv \frac{dx}{dt}$ and $v(t) = \dot{y}(t) \equiv \frac{dy}{dt}$, then

$$\begin{aligned} u = \dot{x} &= \sum_{\gamma=0}^{\infty} g_\gamma(z)\tau^{\gamma-3} = \sum_{\gamma=0}^{\infty} \left[(\gamma-2)f_\gamma + f'_\gamma\right]\tau^{\gamma-3}, \\ v = \dot{y} &= \sum_{\gamma=0}^{\infty} k_\gamma(z)\tau^{\gamma-3} = \sum_{\gamma=0}^{\infty} \left[(\gamma-2)h_\gamma + h'_\gamma\right]\tau^{\gamma-3}, \end{aligned} \tag{2.8}$$

where $f'_\gamma \equiv \frac{df_\gamma}{dz}, h'_\gamma \equiv \frac{dh_\gamma}{dz}$, and g_γ and k_γ are also polynomials of the same type as f_γ and h_γ.

Express (1.4) as a system of four first-order equations in x, u, y, v to obtain

$$\begin{aligned} \dot{x} &= u, \\ \dot{u} &= -Ax - 2\lambda xy, \\ \dot{y} &= v, \\ \dot{v} &= -By - \lambda x^2 + y^2. \end{aligned} \tag{2.9}$$

Substitution of (2.4), (2.6) and (2.8) into (2.9) gives, for $\gamma \geq 0$,

$$\begin{aligned} & f'_\gamma + (\gamma-2)f_\gamma - g_\gamma = 0, \\ & g'_\gamma + (\gamma-3)g_\gamma + Af_{\gamma-2} + 2\lambda\sum_{\mu=0}^{\gamma} f_{\gamma-\mu}h_\mu = 0, \\ & h'_\gamma + (\gamma-2)h_\gamma - k_\gamma = 0, \\ & k'_\gamma + (\gamma-3)k_\gamma + Bh_{\gamma-2} + \lambda\sum_{\mu=0}^{\gamma} f_{\gamma-\mu}f_\mu - \sum_{\mu=0}^{\gamma} h_{\gamma-\mu}h_\mu = 0. \end{aligned} \tag{2.10}$$

For $\gamma = 0$, we find

$$f_0 = \pm\frac{3}{\lambda}\sqrt{\frac{1}{\lambda}+2}, \qquad g_0 = -2f_0, \qquad h_0 = -\frac{3}{\lambda}, \qquad k_0 = \frac{6}{\lambda}. \tag{2.11}$$

For $\gamma > 0$, (2.10) is expressible as

$$\vec{f}'_\gamma + \boldsymbol{A}_\gamma \vec{f}_\gamma = \vec{F}_\gamma, \tag{2.12}$$

where

$$\vec{f}_\gamma = \begin{pmatrix} f_\gamma \\ g_\gamma \\ h_\gamma \\ k_\gamma \end{pmatrix}, \qquad \boldsymbol{A}_\gamma = \begin{pmatrix} \gamma-2 & -1 & 0 & 0 \\ -6 & \gamma-3 & 2\lambda f_0 & 0 \\ 0 & 0 & \gamma-2 & -1 \\ 2\lambda f_0 & 0 & \dfrac{6}{\lambda} & \gamma-3 \end{pmatrix},$$

$$\vec{F}_\gamma = \begin{pmatrix} 0 \\ -Af_{\gamma-2} - 2\lambda \displaystyle\sum_{\mu=1}^{\gamma-1} f_{\gamma-\mu} h_\mu \\ 0 \\ -Bh_{\gamma-2} - \lambda \displaystyle\sum_{\mu=1}^{\gamma-1} f_{\gamma-\mu} f_\mu + \displaystyle\sum_{\mu=1}^{\gamma-1} h_{\gamma-\mu} h_\mu \end{pmatrix} \equiv \begin{pmatrix} F_\gamma \\ G_\gamma \\ H_\gamma \\ K_\gamma \end{pmatrix}. \tag{2.13}$$

The eigenvalues of $\boldsymbol{A}_\gamma$ are $\gamma+1, \gamma-6$ and $\gamma - \left(\frac{5}{2} \pm i\varepsilon\right)$, precisely $\gamma - k_r$ where k_r is a resonance. The matrices $\boldsymbol{A}_\gamma$ are simultaneously diagonalizable by a matrix $\boldsymbol{P}$ independent of γ. $\boldsymbol{P}$ is the matrix of eigenvectors,

$$\boldsymbol{P} = \begin{pmatrix} -\lambda f_0 & -\lambda f_0 & \lambda f_0 & \lambda f_0 \\ 3\lambda f_0 & -4\lambda f_0 & \lambda f_0 \left(\dfrac{1}{2} + i\varepsilon\right) & \lambda f_0 \left(\dfrac{1}{2} - i\varepsilon\right) \\ 3 & 3 & 3\left(2 + \dfrac{1}{\lambda}\right) & 3\left(2 + \dfrac{1}{\lambda}\right) \\ -9 & 12 & 3\left(2 + \dfrac{1}{\lambda}\right)\left(\dfrac{1}{2} + i\varepsilon\right) & 3\left(2 + \dfrac{1}{\lambda}\right)\left(\dfrac{1}{2} - i\varepsilon\right) \end{pmatrix}, \tag{2.14}$$

and $\boldsymbol{P}^{-1}\boldsymbol{A}_\gamma\boldsymbol{P} = \boldsymbol{D}_\gamma$, where

$$\boldsymbol{D}_\gamma = \begin{pmatrix} \gamma+1 & 0 & 0 & 0 \\ 0 & \gamma-6 & 0 & 0 \\ 0 & 0 & \gamma - \left(\dfrac{5}{2} + i\varepsilon\right) & 0 \\ 0 & 0 & 0 & \gamma - \left(\dfrac{5}{2} - i\varepsilon\right) \end{pmatrix}. \tag{2.15}$$

For $\gamma > 6$ (in which case all eigenvalues of $\boldsymbol{A}_\gamma$ have positive real parts), the solution of the matrix differential equation (2.12) may be expressed as

$$\vec{f}_\gamma(z) = \int_{-\infty}^{z} e^{\boldsymbol{A}_\gamma(x-z)} \vec{F}_\gamma(x)dx = \int_{-\infty}^{z} \boldsymbol{P} e^{\boldsymbol{D}_\gamma(x-z)} \boldsymbol{P}^{-1} \vec{F}_\gamma(x)dx, \tag{2.16}$$

where the exponential within the integrals denotes the exponential of a matrix.

Given an $m \times n$ matrix $\boldsymbol{A} = (a_{ij})$, define its norm as

$$\|\boldsymbol{A}\| = \max_{1 \le i \le m} \left(\sum_{j=1}^{n} |a_{ij}| \right). \tag{2.17}$$

It follows from (2.16) that

$$\|\vec{f}_\gamma(z)\| \le \|\boldsymbol{P}\| \|\boldsymbol{P}^{-1}\| \int_{-\infty}^{z} e^{(\gamma-6)(x-z)} \|\vec{F}_\gamma(x)\| dx, \tag{2.18}$$

since $\|e^{\boldsymbol{D}_\gamma(x-z)}\| = e^{(\gamma-6)(x-z)}$ for $x \le z$. $\boldsymbol{P}$ depends upon λ, so that $\|\boldsymbol{P}\|\|\boldsymbol{P}^{-1}\| \le M(\lambda)$, a constant (independent of γ). Thus,

$$\|\vec{f}_\gamma(z)\| \le M(\lambda) \int_{-\infty}^{z} e^{(\gamma-6)(x-z)} \|\vec{F}_\gamma(x)\| dx. \tag{2.19}$$

Theorem 2.1. *There exists $K > 0$ such that for all $z < 0$ and $\gamma \ge 1$,*

$$\|\vec{f}_\gamma(z)\| \le \frac{(2K)^\gamma}{\sqrt{\gamma+1}}. \tag{2.20}$$

Proof. Since $f_\gamma(z)$ is a polynomial in $X = e^{\frac{3}{2}z}$ of degree γ, with coefficients that are bounded functions of z (linear combinations of $\cos(\varepsilon l z)$ and $\sin(\varepsilon l z)$),

$$|f_\gamma(z)| \le \sum_{m=0}^{\gamma} a_m^{(\gamma)} e^{\frac{3}{2}mz} \equiv P_\gamma(X), \tag{2.21}$$

with $a_m^{(\gamma)} \ge 0$ for $0 \le m \le \gamma$. By Lemma A1 of Appendix A (with $q = 1,\ n_\gamma = \gamma,\ M = 1$ and $p = 1$), given $N > 1$, there exists $K_1 > 0$ such that

$$|f_\gamma(z)| \le P_\gamma(X) \le \frac{(K_1 + KX)^\gamma}{\sqrt{\gamma+1}} \qquad \text{for} \quad 1 \le \gamma \le N-1. \tag{2.22}$$

Similarly, there exist positive constants K_2, K_3 and K_4 corresponding to the polynomials g_γ, h_γ and k_γ, repectively. With $K = \max\limits_{1 \le i \le 4}\{K_i\}$, we obtain

$$\|\vec{f}_\gamma(z)\| \le \frac{(K + Ke^{\frac{3}{2}z})^\gamma}{\sqrt{\gamma+1}} \le \frac{(2K)^\gamma}{\sqrt{\gamma+1}} \qquad \text{for} \quad 1 \le \gamma \le N-1,\ z < 0, \tag{2.23}$$

since $0 < e^{\frac{3}{2}z} < 1$. The inequality (2.23) constitutes our inductive hypothesis, and we shall show that (2.20) holds for $\gamma = N$, thus establishing that it holds for all $N > 1$.

By (2.13) and the inductive hypothesis (2.23), we find that

$$\begin{aligned} |G_N| &\le |A||f_{N-2}| + 2|\lambda| \sum_{\mu=1}^{N-1} |f_{N-\mu}||h_\mu| \\ &\le |A| \frac{(2K)^{N-2}}{\sqrt{N-1}} + 2|\lambda| \sum_{\mu=1}^{N-1} \frac{(2K)^N}{\sqrt{N-\mu+1}\sqrt{\mu+1}} \\ &\le \left\{ \frac{|A|(2K)^{-2}}{\sqrt{N-1}} + 2|\lambda|\pi \right\} (2K)^N \end{aligned} \tag{2.24}$$

by Lemma A2 of Appendix A. Similarly,

$$|K_N| \leq \left\{ \frac{|B|(2K)^{-1}}{\sqrt{N-1}} + (|\lambda|+1)\pi \right\} (2K)^N. \tag{2.25}$$

Thus,

$$\|\vec{F}_N\| \leq E(2K)^N, \tag{2.26}$$

where

$$E \leq \max\{|A| + 2|\lambda|\pi, |B| + (|\lambda|+1)\pi\}. \tag{2.27}$$

Thus, from (2.19) and (2.26), we obtain

$$\begin{aligned} \|\vec{f}_N(z)\| &\leq M(\lambda)E \int_{-\infty}^{z} e^{(N-6)(x-z)}(2K)^N dx \\ &= \left\{ \frac{M(\lambda)E\sqrt{N+1}}{N-6} \right\} \frac{(2K)^N}{\sqrt{N+1}} \qquad \text{for} \quad N > 6. \end{aligned} \tag{2.28}$$

For all sufficiently large N, $\frac{M(\lambda)E\sqrt{N+1}}{N-6} < 1$, so beginning with such an $N > 6$, we obtain

$$\|\vec{f}_N(z)\| \leq \frac{(2K)^N}{\sqrt{N+1}}, \tag{2.29}$$

which completes the proof.

By (2.4),

$$x = \sum_{\gamma=0}^{\infty} f_\gamma(z)\tau^{\gamma-2}, \tag{2.30}$$

and by (2.20),

$$|f_\gamma(z)\tau^{\gamma-2}| \leq \frac{(2K)^\gamma}{\sqrt{\gamma+1}}\tau^{\gamma-2}, \tag{2.31}$$

and the series

$$\sum_{\gamma=0}^{\infty} \frac{(2K)^\gamma}{\sqrt{\gamma+1}}\tau^{\gamma-2} \tag{2.32}$$

converges absolutely by the ratio test for $\tau < \frac{1}{2K}$. Thus, the series (2.30) for x converges absolutely for $0 < \tau < R$, where R is at least $\frac{1}{2K}$. Similarly, the series for $\dot{x}$, y and $\dot{y}$ converge absolutely for $0 < \tau < R$.

3. Case(i)-leading orders: $\alpha = \beta = -2$, $k_r = -1, 6, \frac{5}{2} \pm \frac{1}{2}\sqrt{1 - 24\left(\frac{1}{\lambda} + 1\right)}$ irrational

In the present case, the resonances $r, \bar{r} = \frac{5}{2} \pm \frac{1}{2}\sqrt{1 - 24\left(\frac{1}{\lambda} + 1\right)}$ are irrational, positive and distinct, with $-\frac{24}{23} < \lambda < -\frac{1}{2}$. The general solution is given by (2.2) (or (2.1)) as in the complex-resonance case of Section 2, but the details of the resummation are slightly different since, in the present case, $r < 1$ and/or $\bar{r} < 1$ is possible.

Let n be the least positive integer such that $\mu_1 = r - \frac{1}{n} > 0$ and $\mu_2 = \bar{r} - \frac{1}{n} > 0$. (Any positive integer satisfying the latter two conditions will also do.) Then the solution x in (2.2) may be expressed as

$$x = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left[a_{kl} \tau^{k-2+\frac{1}{n}l+\mu_1 l} + \bar{a}_{kl} \tau^{k-2+\frac{1}{n}l+\mu_2 l} \right]. \tag{3.1}$$

Let $w = \tau^{1/n}$ and $z = \ln(\tau)$, $\tau > 0$, then

$$\begin{aligned} x &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left[a_{kl} w^{nk-2n+l} e^{\mu_1 l z} + \bar{a}_{kl} w^{nk-2n+l} e^{\mu_2 l z} \right] \\ &= \sum_{\gamma=0}^{\infty} \sum_{nk+l=\gamma} \left(a_{kl} e^{\mu_1 l z} + \bar{a}_{kl} e^{\mu_2 l z} \right) w^{\gamma-2n} = \sum_{\gamma=0}^{\infty} f_\gamma(z) w^{\gamma-2n}, \end{aligned} \tag{3.2}$$

where

$$f_\gamma(z) = \sum_{nk+l=\gamma} \left(a_{kl} e^{\mu_1 l z} + \bar{a}_{kl} e^{\mu_2 l z} \right) \tag{3.3}$$

is a polynomial of degree γ in the two variables $e^{\mu_1 z}$ and $e^{\mu_2 z}$. Similarly,

$$y = \sum_{\gamma=0}^{\infty} h_\gamma(z) w^{\gamma-2n}, \tag{3.4}$$

where

$$h_\gamma(z) = \sum_{nk+l=\gamma} \left(b_{kl} e^{\mu_1 l z} + \bar{b}_{kl} e^{\mu_2 l z} \right) \tag{3.5}$$

is a polynomial of the same type as $f_\gamma(z)$.

Let

$$\begin{aligned} u = \dot{x} &= \sum_{\gamma=0}^{\infty} g_\gamma(z) w^{\gamma-3n} = \sum_{\gamma=0}^{\infty} \left[\left(\frac{\gamma}{n} - 2 \right) f_\gamma + f'_\gamma \right] w^{\gamma-3n}, \\ v = \dot{y} &= \sum_{\gamma=0}^{\infty} k_\gamma(z) w^{\gamma-3n} = \sum_{\gamma=0}^{\infty} \left[\left(\frac{\gamma}{n} - 2 \right) h_\gamma + h'_\gamma \right] w^{\gamma-3n}, \end{aligned} \tag{3.6}$$

where, as before, $\dot{x} = \frac{dx}{dt}$, $f'_\gamma = \frac{df_\gamma}{dz}$, etc., and g_γ and k_γ are polynomials of the same type as f_γ and h_γ. Substitution of (3.2), (3.4) and (3.6) into the system (2.9) gives, for $\gamma = 0$,

$$f_0 = \pm\frac{3}{\lambda}\sqrt{\frac{1}{\lambda}+2}, \qquad g_0 = -2f_0, \qquad h_0 = -\frac{3}{\lambda}, \qquad k_0 = \frac{6}{\lambda}, \tag{3.7}$$

(as (2.11)) and, for $\gamma > 0$,

$$\vec{f}'_\gamma + \boldsymbol{A}_\gamma \vec{f}_\gamma = \vec{F}_\gamma \tag{3.8}$$

(as (2.12)), where $\vec{F}_\gamma$ is again given by (2.13), but now,

$$\boldsymbol{A}_\gamma = \begin{pmatrix} \frac{\gamma}{n}-2 & -1 & 0 & 0 \\ -6 & \frac{\gamma}{n}-3 & 2\lambda f_0 & 0 \\ 0 & 0 & \frac{\gamma}{n}-2 & -1 \\ 2\lambda f_0 & 0 & \frac{6}{\lambda} & \frac{\gamma}{n}-3 \end{pmatrix}, \tag{3.9}$$

and coincides with the $\boldsymbol{A}_\gamma$ of (2.13) if $n = 1$. The eigenvalues of $\boldsymbol{A}_\gamma$ in the present case are $\frac{\gamma}{n} - k_r$, where $k_r = -1, 6, r, \bar{r}$ are the resonances.

The matrix $\boldsymbol{P}$ which diagonalizes all of the $\boldsymbol{A}_\gamma$ is the same as (2.14) which, for the present situation, is expressed as

$$\boldsymbol{P} = \begin{pmatrix} -\lambda f_0 & -\lambda f_0 & \lambda f_0 & \lambda f_0 \\ 3\lambda f_0 & -4\lambda f_0 & \lambda f_0(r-2) & \lambda f_0(\bar{r}-2) \\ 3 & 3 & 3\left(2+\frac{1}{\lambda}\right) & 3\left(2+\frac{1}{\lambda}\right) \\ -9 & 12 & 3\left(2+\frac{1}{\lambda}\right)(r-2) & 3\left(2+\frac{1}{\lambda}\right)(\bar{r}-2) \end{pmatrix}, \tag{3.10}$$

with $\boldsymbol{P}^{-1}\boldsymbol{A}_\gamma\boldsymbol{P} = \boldsymbol{D}_\gamma$ and

$$\boldsymbol{D}_\gamma = \begin{pmatrix} \frac{\gamma}{n}+1 & 0 & 0 & 0 \\ 0 & \frac{\gamma}{n}-6 & 0 & 0 \\ 0 & 0 & \frac{\gamma}{n}-r & 0 \\ 0 & 0 & 0 & \frac{\gamma}{n}-\bar{r} \end{pmatrix}. \tag{3.11}$$

As before, $\|\boldsymbol{P}\|\,\|\boldsymbol{P}^{-1}\|$ is bounded by a constant $M(\lambda)$, the solution of the system (3.8) is expressible as

$$\vec{f}_\gamma(z) = \int_{-\infty}^{z} \boldsymbol{P} e^{\boldsymbol{D}_\gamma(x-z)} \boldsymbol{P}^{-1} \vec{F}_\gamma(x)\,dx \tag{3.12}$$

for $\frac{\gamma}{n} > 6$ (noting that within the range of λ presently under consideration, $0 < r, \bar{r} < 6$, so that all eigenvalues of $\boldsymbol{A}_\gamma$ are positive for $\frac{\gamma}{n} > 6$), and

$$\|\vec{f}_\gamma(z)\| \le M(\lambda) \int_{-\infty}^{z} e^{\left(\frac{\gamma}{n}-6\right)(x-z)} \|\vec{F}_\gamma(x)\| dx. \tag{3.13}$$

Theorem 3.1. *There exists $K > 0$ such that for all $z < 0$ and $\gamma \ge 1$,*

$$\|\vec{f}_\gamma(z)\| \le \frac{(3K)^\gamma}{\sqrt{\gamma+1}}. \tag{3.14}$$

Proof. Since $f_\gamma(z)$ is a polynomial of degree γ in $X = e^{\mu_1 z}$ and $Y = e^{\mu_2 z}$, Lemma A3 of Appendix A (with $M = 1$ and $p = 1$) implies that given $N > 1$, there exists $K_1 > 0$ such that

$$|f_\gamma(z)| \le \frac{(K_1 + K_1 e^{\mu_1 z} + K_1 e^{\mu_2 z})^\gamma}{\sqrt{\gamma+1}} \qquad \text{for} \quad 1 \le \gamma \le N-1. \tag{3.15}$$

Similarly, there exist positive constants K_2, K_3 and K_4 corresponding to the polynomials g_γ, h_γ and k_γ, respectively. With $K = \max\limits_{1\le i\le 4}\{K_i\}$, we obtain

$$\|\vec{f}_\gamma(z)\| \le \frac{(K + K e^{\mu_1 z} + K e^{\mu_2 z})^\gamma}{\sqrt{\gamma+1}} \qquad \text{for} \quad 1 \le \gamma \le N-1. \tag{3.16}$$

Since $0 < e^{\mu_i z} < 1$ for $i = 1, 2$ and for all $z < 0$, we have

$$\|\vec{f}_\gamma(z)\| \le \frac{(3K)^\gamma}{\sqrt{\gamma+1}} \qquad \text{for} \quad 1 \le \gamma \le N-1. \tag{3.17}$$

The inequality (3.17) constitutes our inductive hypothesis, and we shall show that (3.14) holds for $\gamma = N$, thereby establishing that it holds for all $N > 1$.

As in Section 2, we find that

$$\|\vec{F}_N\| \le E(3K)^N, \tag{3.18}$$

where E is bounded as in (2.27). Thus, from (3.13) and (3.18), we obtain

$$\begin{aligned} \|\vec{f}_N(z)\| &\le M(\lambda) E \int_{-\infty}^{z} e^{\left(\frac{N}{n}-6\right)(x-z)} (3K)^N dx \\ &= \left\{ \frac{M(\lambda) E \sqrt{N+1}}{\frac{N}{n} - 6} \right\} \frac{(3K)^N}{\sqrt{N+1}} \qquad \text{for} \quad \frac{N}{n} > 6. \end{aligned} \tag{3.19}$$

For all sufficiently large N (and any $n \ge 1$), $\frac{M(\lambda)E\sqrt{N+1}}{\frac{N}{n}-6} < 1$, so beginning with such an $N > 6n$, we obtain

$$\|\vec{f}_N(z)\| \le \frac{(3K)^N}{\sqrt{N+1}}, \tag{3.20}$$

which completes the proof.

By (3.2),

$$x = \sum_{\gamma=0}^{\infty} f_\gamma(z) w^{\gamma-2n}, \tag{3.21}$$

and by (3.14),

$$|f_\gamma(z)w^{\gamma-2n}| \le \frac{(3K)^\gamma}{\sqrt{\gamma+1}} w^{\gamma-2n}, \tag{3.22}$$

and the series

$$\sum_{\gamma=0}^{\infty} \frac{(3K)^\gamma}{\sqrt{\gamma+1}} w^{\gamma-2n} \tag{3.23}$$

converges absolutely by the ratio test for $w < \frac{1}{3K}$, i.e., for $\tau < \frac{1}{(3K)^n}$. Thus, the series (3.21) for x converges absolutely for $0 < \tau < R$, where R is at least $\frac{1}{(3K)^n}$. Similarly, the series for $\dot{x}$, y and $\dot{y}$ converge absolutely for $0 < \tau < R$.

4. Case(ii)-leading orders: $Re(\alpha) > -2$, $\beta = -2$

As mentioned in Section 1, if $Re(\alpha) > -2$ and $\lambda > -\frac{1}{2}$, two types of leading-order and resonance structures are possible, namely,

$$\begin{aligned} &\text{(a) Leading order} \quad \alpha = \frac{1}{2} + \frac{1}{2}\sqrt{1-48\lambda}, \ \text{resonances} \quad k_r = -1, 6, 0, r; \\ &\text{(b) Leading order} \quad \bar{\alpha} = \frac{1}{2} - \frac{1}{2}\sqrt{1-48\lambda}, \ \text{resonances} \quad k_r = -1, 6, 0, \bar{r}, \end{aligned} \tag{4.1}$$

where $r, \bar{r} = \mp\sqrt{1-48\lambda}$. For $\lambda < \frac{1}{48}$, $r < 0$, so that (a) does not correspond to a general solution, and will therefore not be considered. (See Conte, Fordy and Pickering [18] for a discussion of negative resonances and the appropriate series expansions.) In case (b), $\bar{r} > 0$ for $-\frac{1}{2} < \lambda < 0$ and $0 < \lambda < \frac{1}{48}$, and $\bar{r}$ is pure imaginary for $\lambda > \frac{1}{48}$. The general solution takes the form

$$\begin{aligned} x &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} a_{klm}\tau^{k+\bar{\alpha}+(2+\bar{\alpha})m+\bar{r}l}, \\ y &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} b_{klm}\tau^{k-2+(2+\bar{\alpha})m+\bar{r}l}, \end{aligned} \tag{4.2}$$

discussion of the validity of which, as well as other relevant issues, is relegated to Appendix C. If $\bar{r}$ is pure imaginary ($\lambda > \frac{1}{48}$), then the methods employed earlier do not apply to (4.2) (see Section 5), and we have not determined whether the series converge or not. For $\bar{r} > 0$ ($\lambda < \frac{1}{48}$), we proceed as in the preceding sections.

Let n be the least positive integer such that $\bar{\beta} = \bar{r} - \frac{1}{n} > 0$, let $w = \tau^{1/n}$ and $z = \ln(\tau)$, $\tau > 0$, in order to express x in (4.2) as

$$\begin{aligned} x &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} a_{klm}\tau^{k+2m+\frac{1}{n}l+\bar{\alpha}+\bar{\alpha}m+\bar{\beta}l} \\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} a_{klm}w^{nk+2nm+l+\bar{\alpha}n}\tau^{\bar{\alpha}m+\bar{\beta}l} = \sum_{\gamma=0}^{\infty} f_\gamma(z)w^{\gamma+\bar{\alpha}n}, \end{aligned} \tag{4.3}$$

where

$$f_\gamma(z) = \sum_{nk+2nm+l=\gamma} a_{klm} e^{\bar{\alpha}mz+\bar{\beta}lz} \tag{4.4}$$

is a polynomial of degree γ in the two variables $e^{\bar{\alpha}z}$ and $e^{\bar{\beta}z}$. Similarly,

$$y = \sum_{\gamma=0}^{\infty} h_\gamma(z) w^{\gamma-2n}, \tag{4.5}$$

where

$$h_\gamma(z) = \sum_{nk+2nm+l=\gamma} b_{klm} e^{\bar{\alpha}mz+\bar{\beta}lz} \tag{4.6}$$

is a polynomial of the same type as $f_\gamma(z)$.

Let

$$\begin{aligned} u = \dot{x} &= \sum_{\gamma=0}^{\infty} g_\gamma(z) w^{\gamma+\bar{\alpha}n-n} = \sum_{\gamma=0}^{\infty} \left[\left(\frac{\gamma}{n} + \bar{\alpha}\right) f_\gamma + f'_\gamma\right] w^{\gamma+\bar{\alpha}n-n}, \\ v = \dot{y} &= \sum_{\gamma=0}^{\infty} k_\gamma(z) w^{\gamma-3n} = \sum_{\gamma=0}^{\infty} \left[\left(\frac{\gamma}{n} - 2\right) h_\gamma + h'_\gamma\right] w^{\gamma-3n}, \end{aligned} \tag{4.7}$$

and substitute (4.3), (4.5) and (4.7) into the system (2.9) to obtain, for $\gamma = 0$,

$$f_0 \text{ arbitrary}, \qquad g_0 = \bar{\alpha} f_0, \qquad h_0 = 6, \qquad k_0 = -12, \tag{4.8}$$

and for $\gamma > 0$, the system

$$\vec{f}'_\gamma + \boldsymbol{A}_\gamma \vec{f}_\gamma = \vec{F}_\gamma, \tag{4.9}$$

where, as before,

$$\vec{f}_\gamma = \begin{pmatrix} f_\gamma \\ g_\gamma \\ h_\gamma \\ k_\gamma \end{pmatrix}, \qquad \vec{F}_\gamma = \begin{pmatrix} F_\gamma \\ G_\gamma \\ H_\gamma \\ K_\gamma \end{pmatrix}, \qquad F_\gamma = H_\gamma = 0, \tag{4.10}$$

and here,

$$\begin{aligned} G_\gamma &= -Af_{\gamma-2n} - 2\lambda \sum_{\mu=1}^{\gamma-1} f_{\gamma-\mu} h_\mu, \\ K_\gamma &= -Bh_{\gamma-2n} - \lambda \sum_{\mu=0}^{\gamma-4n} e^{2\bar{\alpha}z} f_{\gamma-4n-\mu} f_\mu + \sum_{\mu=1}^{\gamma-1} h_{\gamma-\mu} h_\mu, \end{aligned} \tag{4.11}$$

and

$$\boldsymbol{A}_\gamma = \begin{pmatrix} \dfrac{\gamma}{n}+\bar{\alpha} & -1 & 0 & 0 \\ 12\lambda & \dfrac{\gamma}{n}+\bar{\alpha}-1 & 2\lambda f_0 & 0 \\ 0 & 0 & \dfrac{\gamma}{n}-2 & -1 \\ 0 & 0 & -12 & \dfrac{\gamma}{n}-3 \end{pmatrix}. \tag{4.12}$$

As expected, the eigenvalues of $\boldsymbol{A}_\gamma$ are $\frac{\gamma}{n}-k_r$, where $k_r=-1,0,6,\bar{r}$ are the resonances. The diagonalizing matrix for $\boldsymbol{A}_\gamma$ is

$$\boldsymbol{P} = \begin{pmatrix} -\dfrac{\bar{\alpha}}{12}f_0 & 1 & -\lambda f_0 & 1 \\ \lambda f_0 & \bar{\alpha} & -\lambda f_0(\bar{\alpha}+6) & 1-\bar{\alpha} \\ 1 & 0 & 3(2\bar{\alpha}+5) & 0 \\ -3 & 0 & 12(2\bar{\alpha}+5) & 0 \end{pmatrix}, \tag{4.13}$$

with $\boldsymbol{D}_\gamma = \boldsymbol{P}^{-1}\boldsymbol{A}_\gamma\boldsymbol{P}$, and $\|\boldsymbol{P}\|\,\|\boldsymbol{P}^{-1}\| \le M(\lambda, f_0)$, a constant independent of γ, but depending upon λ and the arbitrary constant f_0. Since $-\frac{1}{2}<\lambda<0$ or $0<\lambda<\frac{1}{48}$, we have $0<\bar{r}<5<6$, so that $k_r=6$ is again the largest resonance, and all eigenvalues of $\boldsymbol{A}_\gamma$ are positive for $\frac{\gamma}{n}>6$. Thus, for $\frac{\gamma}{n}>6$, the solution of (4.9) may be expressed as

$$\vec{f}_\gamma(z) = \int_{-\infty}^{z} \boldsymbol{P}e^{\boldsymbol{D}_\gamma(x-z)}\boldsymbol{P}^{-1}\vec{F}_\gamma(x)dx, \tag{4.14}$$

with

$$\|\vec{f}_\gamma(z)\| \le M(\lambda, f_0)\int_{-\infty}^{z} e^{\left(\frac{\gamma}{n}-6\right)(x-z)}\|\vec{F}_\gamma(x)\|dx. \tag{4.15}$$

Theorem 4.1. *There exists $K>0$ such that for all $z<0$ and $\gamma\ge 1$,*

$$\|\vec{f}_\gamma(z)\| \le \frac{(3K)^\gamma}{\sqrt{\gamma+1}}. \tag{4.16}$$

As the proof is practically identical to the one of Theorem 3.1, we omit the details.

By (4.3),

$$x = \sum_{\gamma=0}^{\infty} f_\gamma(z)w^{\gamma+\bar{\alpha}n} \tag{4.17}$$

and by (4.16),

$$\left|f_\gamma(z)w^{\gamma+\bar{\alpha}n}\right| \le \frac{(3K)^\gamma}{\sqrt{\gamma+1}}w^{\gamma+\bar{\alpha}n}, \tag{4.18}$$

and the series

$$\sum_{\gamma=0}^{\infty}\frac{(3K)^\gamma}{\sqrt{\gamma+1}}w^{\gamma+\bar{\alpha}n} \tag{4.19}$$

converges absolutely by the ratio test for $w<\frac{1}{3K}$, i.e., for $\tau<\frac{1}{(3K)^n}$. Thus, the series (4.17) for x and, similarly, the series (4.5) for y and (4.7) for $\dot{x}$ and $\dot{y}$ converge absolutely for $0<\tau<R$, where R is at least $\frac{1}{(3K)^n}$.

5. Concluding Remarks

Three types of psi-series solutions of the cubic Hénon-Heiles system have been shown to be absolutely convergent on intervals of the form $0 < \tau < R$, $R > 0$. It is clear that the series for x, $\dot{x}$, y and $\dot{y}$ converge uniformly on compact subintervals of $(0, R)$, thereby justifying the termwise differentiation of the series.

The resummations that have been performed in order to proceed with the convergence proofs amount to the replacement of multiply-indexed series with constant coefficients by single series with polynomial coefficients. This procedure makes use of the fact that the resonances appearing as exponents within the psi-series have positive real parts. For example, to obtain (2.3) and (2.6) from (2.2), it is essential that $Re(r) > 0$ and $Re(\bar{r}) > 0$. Otherwise, as occurs in (4.2) when $\bar{r}$ is pure imaginary, a resummation would give rise to series containing $f_\gamma(z)$ which are not polynomials, but infinite series.

The situation when $\lambda = -\frac{1}{2}$ is very interesting. Here, $\alpha = -2$ (the leading order for x), $r = 5$ and $\bar{r} = 0$. But the leading coefficient for x is $\pm\frac{3}{\lambda}\sqrt{2 + \frac{1}{\lambda}} = 0$, contradicting the fact that $x = O(\tau^{-2})$. The correct leading-order behaviours have been given in [7] and [9, p. 340]. In our notation,

$$x \sim (-30)^{1/2}\tau^{-2}(\ln(\tau))^{-\frac{1}{2}},$$

$$y \sim 6\tau^{-2} + \frac{5}{2}\tau^{-2}(\ln(\tau))^{-1}.$$

A full series expansion of the form

$$x = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} a_{kl}\tau^{k-2}(\ln(\tau))^{-l-\frac{1}{2}},$$

$$y = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} b_{kl}\tau^{k-2}(\ln(\tau))^{-l},$$

fails, due to an incompatible resonance at $(k, l) = (0, 2)$. We are not aware of the correct series in this case.

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Appendix A

This section contains the Lemmas necessary for the convergence proofs.

Lemma A1. *Let $X \neq 0$ have constant sign, let $q \geq 1$ be an integer, and for $\gamma \geq 1$, let*

$$P_\gamma(X) = \sum_{m=0}^{n_\gamma} c_m^{(\gamma)} X^m$$

be a sequence of polynomials of degree $n_\gamma = [\frac{\gamma}{q}]$*. Given an integer* $N > q$, $0 < M \le 1$ *and* $p > 0$*, there exists* K *with* $\operatorname{sign}(K) = \operatorname{sign}(X)$ *and* $p|K| > 1$ *such that*

$$|P_\gamma(X)| \le M \frac{(p|K| + KX)^{\gamma/q}}{\sqrt{\gamma+1}} \qquad \textit{for} \quad 1 \le \gamma \le N-1.$$

This is Lemma 4.4 of [13], to which we refer the reader for a proof.

Lemma A2.

$$\lim_{\gamma\to\infty} \sum_{\mu=0}^{\gamma-1} \frac{1}{\sqrt{\mu+1}\,\sqrt{\gamma-\mu}} = \pi.$$

The result follows easily by the proof of the integral test for convergence of series. For the details, we refer to Hemmi [19, p. 64].

Lemma A3. *Let* $X, Y > 0$ *and for* $\gamma \ge 1$*, let* $P_\gamma(X,Y)$ *be a sequence of polynomials of degree* γ *in* X *and* Y*, i.e.,*

$$P_\gamma(X,Y) = \sum_{\beta=0}^{\gamma} \sum_{\mu=0}^{\beta} c_{\mu\beta}^{(\gamma)} X^\mu Y^{\beta-\mu}.$$

Given an integer $N > 1$, $0 < M \le 1$ *and* $p > 0$*, there exists* $K > 0$ *such that*

$$|P_\gamma(X,Y)| \le M \frac{(pK + KX + KY)^\gamma}{\sqrt{\gamma+1}} \qquad \textit{for} \quad 1 \le \gamma \le N-1.$$

Proof. Let

$$K = \max_{\substack{1 \le \gamma \le N-1 \\ 0 \le \beta \le \gamma \\ 0 \le \mu \le \beta}} \left\{ \frac{|c_{\mu\beta}^{(\gamma)}|\sqrt{\gamma+1}}{\begin{pmatrix} \gamma \\ \beta \end{pmatrix} \begin{pmatrix} \beta \\ \mu \end{pmatrix} M p^{\gamma-\beta}} \right\}^{1/\gamma},$$

then for $1 \le \gamma \le N-1$, $0 \le \beta \le \gamma$, $0 \le \mu \le \beta$, we have

$$|c_{\mu\beta}^{(\gamma)}| \le \frac{M}{\sqrt{\gamma+1}} \begin{pmatrix} \gamma \\ \beta \end{pmatrix} \begin{pmatrix} \beta \\ \mu \end{pmatrix} K^\gamma p^{\gamma-\beta},$$

which implies that

$$\begin{aligned} |P_\gamma(X,Y)| &\le \sum_{\beta=0}^{\gamma} \sum_{\mu=0}^{\beta} |c_{\mu\beta}^{(\gamma)}| X^\mu Y^{\beta-\mu} \\ &\le \frac{M}{\sqrt{\gamma+1}} \sum_{\beta=0}^{\gamma} \sum_{\mu=0}^{\beta} \begin{pmatrix} \gamma \\ \beta \end{pmatrix} \begin{pmatrix} \beta \\ \mu \end{pmatrix} K^\gamma p^{\gamma-\beta} X^\mu Y^{\beta-\mu} \\ &= \frac{M}{\sqrt{\gamma+1}} \sum_{\beta=0}^{\gamma} \sum_{\mu=0}^{\beta} \begin{pmatrix} \gamma \\ \beta \end{pmatrix} \begin{pmatrix} \beta \\ \mu \end{pmatrix} (pK)^{\gamma-\beta} (KX)^\mu (KY)^{\beta-\mu} \\ &= M \frac{(pK + KX + KY)^\gamma}{\sqrt{\gamma+1}}, \end{aligned}$$

by the binomial theorem.

Appendix B

First, we show that the solution (2.1), namely,

$$
\begin{aligned}
x &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} a_{klm}\tau^{k-2+rl+\bar{r}m},\\
y &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} b_{klm}\tau^{k-2+rl+\bar{r}m},
\end{aligned}
\tag{B1}
$$

of the system (1.4), corresponding to the case(i)-leading orders and non-integral resonances, is self-consistent and contains four arbitrary constants.

Substitution of (B1) into (1.4) gives

$$
a_{000} = \pm\frac{3}{\lambda}\sqrt{2+\frac{1}{\lambda}}, \qquad b_{000} = -\frac{3}{\lambda}, \tag{B2}
$$

and for $(k,l,m) \neq (0,0,0)$, the coefficient recursion relations

$$
\begin{aligned}
&(k+rl+\bar{r}m)(k+rl+\bar{r}m-5)a_{klm} + 2\lambda a_{000}b_{klm}\\
&\quad = -Aa_{k-2,l,m} - 2\lambda \sum_{(0,0,0)\prec(p,q,s)\prec(k,l,m)} a_{pqs}b_{k-p,l-q,m-s},\\
&2\lambda a_{000}a_{klm} + \left[(k+rl+\bar{r}m-2)(k+rl+\bar{r}m-3)+\frac{6}{\lambda}\right]b_{klm}\\
&\quad = -Bb_{k-2,l,m} + \sum_{(0,0,0)\prec(p,q,s)\prec(k,l,m)} [-\lambda a_{pqs}a_{k-p,l-q,m-s} + b_{pqs}b_{k-p,l-q,m-s}],
\end{aligned}
\tag{B3}
$$

where we have employed the notation $(0,0,0) \prec (p,q,s) \prec (k,l,m)$ to mean that $0 \le p \le k$, $0 \le q \le l$, $0 \le s \le m$, with $(p,q,s) \neq (0,0,0)$ and $(p,q,s) \neq (k,l,m)$. Expressed in matrix form, the left-hand sides become

$$
\begin{bmatrix} (k+rl+\bar{r}m)(k+rl+\bar{r}m-5) & 2\lambda a_{000} \\ 2\lambda a_{000} & (k+rl+\bar{r}m-2)(k+rl+\bar{r}m-3)+\frac{6}{\lambda} \end{bmatrix} \begin{bmatrix} a_{klm} \\ b_{klm} \end{bmatrix}, \tag{B4}
$$

and the coefficient matrix is singular precisely when $k+rl+\bar{r}m$ has one of the values $-1, 6, r, \bar{r}$, i.e., when

$$
(k,l,m) = (-1,0,0), (6,0,0), (0,1,0), (0,0,1). \tag{B5}
$$

These are the resonances of (B1), and compatibility must be checked at these values, confirming both the self-consistency of (B1) and the presence of four arbitrary constants.

The resonance at $(k,l,m) = (-1,0,0)$ is trivially compatible, and corresponds to the arbitrariness of t_0. At $(k,l,m) = (0,1,0)$, we find that

$$
a_{010} \quad \text{is arbitrary,} \qquad b_{010} = -\frac{r(r-5)}{2\lambda a_{000}}a_{010}, \tag{B6}
$$

and at $(k,l,m) = (0,0,1)$, that

$$
a_{001} \quad \text{is arbitrary,} \qquad b_{001} = -\frac{\bar{r}(\bar{r}-5)}{2\lambda a_{000}}a_{001}. \tag{B7}
$$

At $(k,l,m)=(6,0,0)$, the compatibility condition is

$$\lambda a_{000}a_{400}A+2\lambda^2 a_{000}(a_{200}b_{400}+a_{400}b_{200})-3b_{400}B-6(\lambda a_{200}a_{400}-b_{200}b_{400})=0. \tag{B8}$$

The coefficients which affect (B8) are

$$\begin{aligned}
&a_{000}=\pm\frac{3}{\lambda}\sqrt{2+\frac{1}{\lambda}}, \qquad b_{000}=-\frac{3}{\lambda},\\
&a_{100}=a_{300}=a_{500}=0, \qquad b_{100}=b_{300}=b_{500}=0,\\
&a_{200}=\frac{a_{000}\left(\frac{A}{\lambda}+B\right)}{12\left(1+\frac{1}{\lambda}\right)}, \qquad b_{200}=\frac{B-A\left(2+\frac{1}{\lambda}\right)}{4\lambda\left(1+\frac{1}{\lambda}\right)},\\
&a_{400}=\frac{a_{200}\left(1+\frac{3}{\lambda}\right)(A+2\lambda b_{200})-\lambda a_{000}\left(Bb_{200}+\lambda a_{200}^2-b_{200}^2\right)}{10\left(4+\frac{3}{\lambda}\right)},\\
&b_{400}=-\frac{\lambda a_{000}a_{200}(A+2\lambda b_{200})-2\left(Bb_{200}+\lambda a_{200}^2-b_{200}^2\right)}{10\left(4+\frac{3}{\lambda}\right)},
\end{aligned} \tag{B9}$$

from which (B8) is confirmed. The system governing a_{600} and b_{600} is

$$\begin{pmatrix} 6 & 2\lambda a_{000} \\ 2\lambda a_{000} & 12+\frac{6}{\lambda}\end{pmatrix}\begin{pmatrix} a_{600} \\ b_{600}\end{pmatrix}=\begin{pmatrix} -Aa_{400}-2\lambda(a_{200}b_{400}+a_{400}b_{200}) \\ -Bb_{400}-2(\lambda a_{200}a_{400}-b_{200}b_{400})\end{pmatrix}, \tag{B10}$$

and reduces to

$$6a_{600}+2\lambda a_{000}b_{600}=-Aa_{400}-2\lambda(a_{200}b_{400}+a_{400}b_{200}), \tag{B11}$$

showing that a_{600} is arbitrary, and b_{600} is defined in terms of a_{600}.

Next, we demonstrate that (2.1) is equivalent to (2.2). The series which defines x in (2.1) may be broken up into three parts: $m=l$, $m<l$, and $m>l$. Thus,

$$\begin{aligned}
x=&\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}a_{kll}\tau^{k-2+5l}+\sum_{k=0}^{\infty}\sum_{l=1}^{\infty}\sum_{m=0}^{l-1}a_{klm}\tau^{k-2+5m+(l-m)r}\\
&+\sum_{k=0}^{\infty}\sum_{m=1}^{\infty}\sum_{l=0}^{m-1}a_{klm}\tau^{k-2+5l+(m-l)\bar{r}},
\end{aligned} \tag{B12}$$

where use has been made of the fact that $r+\bar{r}=5$. The first sum in (B12) is expressible as

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}a_{kll}\tau^{k-2+5l}=\sum_{i=0}^{\infty}\sum_{k+5l=i}a_{kll}\tau^{i-2}=\sum_{i=0}^{\infty}a_{i0}\tau^{i-2}, \tag{B13}$$

where

$$a_{i0}=\sum_{k+5l=i}a_{kll}. \tag{B14}$$

The second sum in (B12) may be manipulated as follows:

$$\begin{aligned}
&\sum_{k=0}^{\infty}\sum_{l=1}^{\infty}\sum_{m=0}^{l-1} a_{klm}\tau^{k-2+5m+(l-m)r} \\
&\quad= \sum_{k=0}^{\infty}\sum_{l=1}^{\infty}\sum_{j=1}^{l} a_{k,l,l-j}\tau^{k-2+5(l-j)+jr} \qquad (\text{letting} \quad j = l-m) \\
&\quad= \sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\sum_{l=j}^{\infty} a_{k,l,l-j}\tau^{k-2+5(l-j)+jr} \qquad (\text{reversing the order of summation}) \\
&\quad= \sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\sum_{l=0}^{\infty} a_{k,l+j,l}\tau^{k-2+5l+jr} \qquad (\text{replacing } l \text{ by } l+j) \\
&\quad= \sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\sum_{k+5l=i} a_{k,l+j,l}\tau^{i-2+jr} \\
&\quad= \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} a_{ij}\tau^{i-2+jr},
\end{aligned} \tag{B15}$$

where

$$a_{ij} = \sum_{k+5l=i} a_{k,l+j,l}. \tag{B16}$$

Similarly, the third term in (B12) becomes

$$\sum_{i=0}^{\infty}\sum_{j=1}^{\infty} \bar{a}_{ij}\tau^{i-2+j\bar{r}}, \tag{B17}$$

where

$$\bar{a}_{ij} = \sum_{k+5m=i} a_{k,m,m+j}. \tag{B18}$$

Combining (B13), (B15) and (B17), we obtain

$$x = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}\tau^{i-2+jr} + \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} \bar{a}_{ij}\tau^{i-2+j\bar{r}}. \tag{B19}$$

Similarly,

$$y = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} b_{ij}\tau^{i-2+jr} + \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} \bar{b}_{ij}\tau^{i-2+j\bar{r}}. \tag{B20}$$

Defining $\bar{a}_{i0} = \bar{b}_{i0} = 0$ for all $i \geq 0$, (B19), (B20) coincide with (2.2).

Finally, the solution given in [3] is

$$x = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} A_{ji}\tau^{i-2+j(r-2)} + \sum_{j=1}^{\infty}\sum_{i=0}^{\infty} \bar{A}_{ji}\tau^{i-2+j(\bar{r}-2)} \tag{B21}$$

(their equation (3.5a), in our notation), with a similar expression for y. To obtain the form (B21) from (B19), write

$$\begin{aligned} x &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}\tau^{i-2+jr} + \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} \bar{a}_{ij}\tau^{i-2+j\bar{r}} \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}\tau^{i+2j-2+j(r-2)} + \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} \bar{a}_{ij}\tau^{i+2j-2+j(\bar{r}-2)} \\ &= \sum_{j=0}^{\infty}\sum_{i=2j}^{\infty} a_{i-2j,j}\tau^{i-2+j(r-2)} + \sum_{j=1}^{\infty}\sum_{i=2j}^{\infty} \bar{a}_{i-2j,j}\tau^{i-2+j(\bar{r}-2)} \\ &= \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} A_{ji}\tau^{i-2+j(r-2)} + \sum_{j=1}^{\infty}\sum_{i=0}^{\infty} \bar{A}_{ji}\tau^{i-2+j(\bar{r}-2)}, \end{aligned} \tag{B22}$$

where

$$A_{ji} = \left\{ \begin{array}{ll} a_{i-2j,j}, & \text{if } \ i \geq 2j \\ 0, & \text{otherwise} \end{array} \right\}, \qquad \bar{A}_{ji} = \left\{ \begin{array}{ll} \bar{a}_{i-2j,j}, & \text{if } \ i \geq 2j \\ 0, & \text{otherwise} \end{array} \right\}, \tag{B23}$$

and similarly for y.

Appendix C

First, we confirm that the solution (4.2), namely,

$$\begin{aligned} x &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} a_{klm}\tau^{k+\bar{\alpha}+(2+\bar{\alpha})m+\bar{r}l}, \\ y &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} b_{klm}\tau^{k-2+(2+\bar{\alpha})m+\bar{r}l}, \end{aligned} \tag{C1}$$

of the system (1.4), corresponding to the case(ii)-singularities, is self-consistent and contains four arbitrary constants.

Substitution of (C1) into (1.4) gives

$$a_{000} \quad \text{is arbitrary}, \qquad b_{000} = 6, \tag{C2}$$

and for $(k,l,m) \neq (0,0,0)$, the coefficient recursion relations

$$\begin{aligned} &[k+(2+\bar{\alpha})m+\bar{r}l][k+(2+\bar{\alpha})m+\bar{r}(l-1)]a_{klm} + 2\lambda a_{000}b_{klm} \\ &\quad = -Aa_{k-2,l,m} - 2\lambda \sum_{(0,0,0)\prec(p,q,r)\prec(k,l,m)} a_{k-p,l-q,m-r}b_{pqr}, \\ &[k+(2+\bar{\alpha})m+\bar{r}l+1][k+(2+\bar{\alpha})m+\bar{r}l-6]b_{klm} \\ &\quad = -Bb_{k-2,l,m} - \lambda\sum_{p=0}^{k}\sum_{q=0}^{l}\sum_{r=0}^{m} a_{k-p,l-q,m-r-2}a_{pqr} \\ &\quad + \sum_{(0,0,0)\prec(p,q,r)\prec(k,l,m)} b_{k-p,l-q,m-r}b_{pqr}, \end{aligned} \tag{C3}$$

where we have defined the "$\prec$" symbol as in Appendix B. Expressed in matrix form, the left-hand sides become

$$\begin{bmatrix} [k+(2+\bar{\alpha})m+\bar{r}l][k+(2+\bar{\alpha})m+\bar{r}(l-1)] & 2\lambda a_{000} \\ 0 & [k+(2+\bar{\alpha})m+\bar{r}l+1][k+(2+\bar{\alpha})m+\bar{r}l-6] \end{bmatrix} \begin{bmatrix} a_{klm} \\ b_{klm} \end{bmatrix}, \tag{C4}$$

and the coefficient matrix is singular precisely when $k+(2+\bar{\alpha})m+\bar{r}l$ has one of the values $-1, 6, \bar{r}$, i.e., when

$$(k,l,m) = (-1,0,0), (6,0,0), (0,1,0). \tag{C5}$$

These, together with $(k,l,m) = (0,0,0)$, are the resonances of (C1). Compatibility at $(0,0,0)$ and the arbitrariness of a_{000} has already been confirmed. The compatibility of $(-1,0,0)$, reflecting the arbitrariness of t_0, is trivial, as usual. It remains to check the remaining two.

At $(k,l,m) = (0,1,0)$, we find that

$$b_{010} = 0, \qquad a_{010} \quad \text{is arbitrary.} \tag{C6}$$

The compatibility condition at $(k,l,m) = (6,0,0)$ is

$$-Bb_{400} + \sum_{p=1}^{5} b_{6-p,0,0} b_{p00} = 0. \tag{C7}$$

The coefficients which affect (C7) are

$$b_{100} = 0, \qquad b_{300} = 0, \qquad b_{500} = 0, \qquad b_{200} = \frac{B}{2}, \qquad b_{400} = \frac{B^2}{40}, \tag{C8}$$

from which (C7) is confirmed. The second equation in (C3) then shows that b_{600} is arbitrary, and the first gives a_{600} in terms of b_{600}.

Second, we note that a solution different from (C1) has been given in [3], but we find that the coefficient recursion relations (their equation (3.9)) do not follow from the solution (their equation (3.6)) without neglecting certain terms, precisely those terms which lead to anomalies when a resummation is performed in an attempt to prove convergence.

Finally, we remark upon the "derivation" of the form of (C1). The exponent $\bar{r}l$ is required in order to produce an arbitrary coefficient corresponding to the resonance at $\bar{r}$. This is usual. The exponent $\bar{\alpha}m$ is required in order that all terms in the system (1.4) can be balanced. But since $Re(\bar{\alpha}) > -2$, $\bar{\alpha}m$ must be replaced by $(2+\bar{\alpha})m$ in order to avoid exponents of τ with arbitrarily large negative real parts, which would render the series invalid.

References

[1] Hénon M. and Heiles C., *Astron. J.*, 1964, V.69, 73.

[2] Chang Y.F., Tabor M. and Weiss J., *J. Math. Phys.*, 1982, V.23, 531.

[3] Chang Y.F., Greene J.M., Tabor M. and Weiss J., *Physica D*, 1983, V.8, 183.

[4] Ablowitz M.J., Ramani A. and Segur H., *Lett. Nuovo Cimento*, 1978, V.23, 333.

[5] Ablowitz M.J., Ramani A. and Segur H., *J. Math. Phys.*, 1980, V.21, 715.

[6] Chang Y.F., Tabor M., Weiss J. and Corliss G., *Phys. Lett. A*, 1981, V.85, 211.

[7] Bountis T., Segur H. and Vivaldi F., *Phys. Rev. A*, 1982, V.25, 1257.

[8] Weiss J., *Phys. Lett. A*, 1984, V.105, 387.

[9] Tabor M., Chaos and Integrability in Nonlinear Dynamics, An Introduction, New York, Wiley-Interscience, 1989.

[10] Hille E., Ordinary Differential Equations in the Complex Domain, Mineola, Dover Publications, Inc., 1976.

[11] Melkonian S. and Zypchen A., *Nonlinearity*, 1995, V.8, 1143.

[12] Sparrow C., The Lorenz Equations: Bifurcations, Chaos and Strange Attractors, New York, Springer, 1982.

[13] Hemmi M.A. and Melkonian S., *Can. Appl. Math. Quart.*, 1995, V.3, 43.

[14] Hille E., *Proc. Roy. Soc. Edinburgh A*, 1974, V.72, 17.

[15] Weiss J., Tabor M. and Carnevale G., *J. Math. Phys.*, 1983, V.24, 522.

[16] Kichenassamy S. and Srinivasan K., *J. Phys. A: Math. Gen.*, 1995, V.28, 1977.

[17] Kichenassamy S. and Littman W., *Commun. PDE*, 1993, V.18, 431, 1869.

[18] Conte R., Fordy A.P. and Pickering A., *Physica D*, 1993, V.69, 33.

[19] Hemmi M.A., Series Solutions of Nonlinear Ordinary Differential Equations, Master's Thesis, Carleton University, Ottawa, Canada, 1994.