

# $q$ -Probability: I. Basic Discrete Distributions

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## Abstract

For basic discrete probability distributions, – Bernoulli, Pascal, Poisson, hypergeometric, contagious, and uniform, –  $q$ -analogs are proposed.

## 1 Introduction

$q$ -analogs of classical formulae go back to Euler,  $q$ -binomial coefficients were defined by Gauss, and  $q$ -hypergeometric series were found by E. Heine in 1846. The  $q$ -analysis was developed by F. Jackson at the beginning of the 20<sup>th</sup> century, and the modern point of view subsumes most of the old developments into the subjects of Quantum Groups and Combinatorial Enumeration.

The general philosophy of  $q$ -analogs is that of a deformation, with the deformation parameter  $q$  being thought of as close to 1. This point of view is certainly not all-encompassing; for example, representations of Quantum groups when  $q$  is a root of unity are of independent interests; more importantly, the  $q$ -pictures sometimes possess properties *singular* in  $(q - 1)$  or otherwise not regularly dependent on  $(q - 1)$ ; regularization of divergent/infinite ( $q = 1$ )–quantities is another useful feature of  $q$ -analogs... the list goes on.

The typical example is

$$\lim_{x \rightarrow \infty} [x] = \frac{1}{1 - q}, \quad |q| < 1, \quad (1.1)$$

where

$$[x] = [x]_q = \frac{q^x - 1}{q - 1} \quad (1.2)$$

is the  $q$ -analog of a number (or object)  $x$ . (A quick introduction to the  $q$ -calculus is available from many sources, e.g. Chapter 2 in [6].)

More examples of such sort will be found below in this paper, the 1<sup>st</sup> one in a series devoted to a  $q$ -probabilities. In the next 6 sections we look at  $q$ -analogs of Bernoulli, Pascal, Poisson, hypergeometric, contagious, and uniform distributions, respectively. It is

surprising how many new effects appear compared to the classical theory at  $q = 1$ . For example, even for the Bernoulli distribution, the profound classical differences between finite and infinite number of trials are mitigated when  $q$  enters the picture, so that one can write down the probability (formerly zero) of *many* individual events of infinite type, such as

$$(1 \dot{-} p)^\infty = \prod_{i=0}^{\infty} (1 - pq^i), \quad |q| < 1, \quad (1.3)$$

the probability of coming up with all “tails” during an infinite number of coin flips; the probability of coming all tails during  $n$  coin flips is

$$(1 \dot{-} p)^n = \prod_{i=0}^{n-1} (1 - pq^i). \quad (1.4)$$

More generally, the probability of coming up with precisely  $\kappa$  heads during an infinite number of coin flips is

$$\frac{p^\kappa}{(1-q)\dots(1-q^\kappa)} (1 \dot{-} p)^\infty, \quad \kappa \in \mathbf{N}. \quad (1.5)$$

The six probability distributions discussed in this paper are all discrete, univariate, basic, and relatively simple. More classical distributions can be found in [1, 3-5].

## 2 $q$ -binomial distributions

Suppose we have a random variable  $\zeta$  – “2-sided coin” – which takes two values: 1 with probability  $p$ , and 0 with probability

$$p' = 1 - p. \quad (2.1)$$

After  $n$  throws, the total sum accumulated,

$$\xi_n = \zeta_1 + \dots + \zeta_n, \quad (2.2)$$

obeys the Bernoulli distribution

$$Pr(\xi_n = \kappa) = \binom{n}{\kappa} p^\kappa p'^{n-\kappa}, \quad 0 \leq \kappa \leq n. \quad (2.3)$$

As a  $q$ -analog of this distribution we take (with  $0 < q < 1$ )

$$Pr(\bar{\xi}_n = [\kappa]) = \begin{bmatrix} n \\ \kappa \end{bmatrix} p^\kappa (1 \dot{-} p)^{n-\kappa}, \quad 0 \leq \kappa \leq n, \quad (2.4)$$

where

$$\begin{bmatrix} x \\ \kappa \end{bmatrix} = \begin{bmatrix} x \\ \kappa \end{bmatrix}_q = \frac{[x] \dots [x - \kappa + 1]}{[\kappa]!}, \quad \kappa \in \mathbf{N}, \quad (2.5a)$$

$$\begin{bmatrix} x \\ 0 \end{bmatrix} = 1, \quad \begin{bmatrix} x \\ s \end{bmatrix} = 0, \quad s \notin \mathbf{Z}_+, \quad (2.5b)$$

are the  $q$ -binomial coefficients, and

$$[0]! = 1; \quad [\kappa]! = [1]\dots[\kappa], \quad \kappa \in \mathbf{N}, \quad (2.6)$$

are the  $q$ -factorials.

To justify formula (2.4), we need to prove that

$$\sum_{\kappa=0}^n \begin{bmatrix} n \\ \kappa \end{bmatrix} p^\kappa (1 \dot{-} p)^{n-\kappa} = 1, \quad \forall p. \quad (2.7)$$

This formula follows from the following identity:

$$\sum_{\kappa=0}^n \begin{bmatrix} n \\ \kappa \end{bmatrix} p^\kappa (1 \dot{-} v)^{n-\kappa} = \sum_{\kappa=0}^n \begin{bmatrix} n \\ \kappa \end{bmatrix} (p \dot{-} v)^{n-\kappa}, \quad (2.8)$$

when  $v = p$ ; here

$$(a \dot{+} b)^n := \prod_{i=0}^{n-1} (a + q^i b), \quad n \in \mathbf{N}, \quad (a \dot{+} b)^0 := 1. \quad (2.9)$$

Formula (2.8) is, in turn, the  $b = 1$ -case of the general formula

$$\sum_{\kappa=0}^n \begin{bmatrix} n \\ \kappa \end{bmatrix} a^\kappa (b \dot{+} v)^{n-\kappa} = \sum_{\kappa=0}^n \begin{bmatrix} n \\ \kappa \end{bmatrix} b^\kappa (a \dot{+} v)^{n-\kappa}. \quad (2.10)$$

To prove formula (2.10), let us use the easily checked by induction on  $m$  Euler's formula

$$(x \dot{+} y)^m = \sum_{j=0}^m \begin{bmatrix} m \\ j \end{bmatrix} x^{m-j} y^j q^{\binom{j}{2}}. \quad (2.11)$$

Then the LHS of (2.10) can be rewritten as

$$\sum_{\kappa, \ell} \begin{bmatrix} n \\ \kappa \end{bmatrix} a^\kappa \begin{bmatrix} n - \kappa \\ \ell \end{bmatrix} b^{n-\kappa-\ell} v^\ell q^{\binom{\ell}{2}}, \quad (2.12L)$$

while the RHS of (2.10) can be similarly rewritten as

$$\sum_{s, \ell} \begin{bmatrix} n \\ s \end{bmatrix} b^s \begin{bmatrix} n - s \\ \ell \end{bmatrix} a^{n-s-\ell} v^\ell q^{\binom{\ell}{2}}, \quad (2.12R)$$

and these two double sums bijectively coincide, for each fixed  $\ell$ , when  $s$  is identified with  $n - \ell - \kappa$ .

Now, to calculate the expectation values of powers of  $\bar{\xi}_n$ , we use the easily proved by induction on  $m \in \mathbf{N}$  formula

$$\left( x \frac{d}{d_q x} \right)^m = \sum_{\kappa=1}^m \frac{1}{[\kappa-1]!} \left( \sum_{s=0}^{\kappa-1} \begin{bmatrix} \kappa-1 \\ s \end{bmatrix} (-1)^s q^{\binom{s}{2}} [\kappa-s]^{m-1} \right) x^\kappa \left( \frac{d}{d_q x} \right)^\kappa, \quad (2.13)$$

where

$$\frac{df}{d_q x} := \frac{f(qx) - f(x)}{(q-1)x} \quad (2.14)$$

is the  $q$ -derivative:

$$\frac{d}{d_q x}(x^s) = [s]x^{s-1}. \quad (2.15)$$

In particular, for  $m = 2$ , we get

$$\left(x \frac{d}{d_q x}\right)^2 = x \frac{d}{d_q x} + qx^2 \left(\frac{d}{d_q x}\right)^2. \quad (2.16)$$

Applying the operator  $\left(p \frac{d}{d_q p}\right)^s \Big|_{v=p}$ ,  $s = 1, 2$ , to formula (2.8), we obtain

$$\langle \bar{\xi}_n \rangle = E(\bar{\xi}_n) = \sum_{\kappa=0}^n [\kappa] \begin{bmatrix} n \\ \kappa \end{bmatrix} p^\kappa (1-p)^{n-\kappa} = [n]p, \quad (2.17)$$

$$\langle \bar{\xi}_n^2 \rangle = E(\bar{\xi}_n^2) = \sum_{\kappa=0}^n [\kappa]^2 \begin{bmatrix} n \\ \kappa \end{bmatrix} p^\kappa (1-p)^{n-\kappa} = [n]p + qp^2[n][n-1], \quad (2.18)$$

where we used the obvious relation

$$\frac{d}{d_q p}(p+v)^\ell = [\ell](p+v)^{\ell-1}. \quad (2.19)$$

From formulae (2.17) and (2.18) we find that

$$\text{Var}(\bar{\xi}_n) = \langle \bar{\xi}_n^2 \rangle - \langle \bar{\xi}_n \rangle^2 = [n]p(1-p). \quad (2.20)$$

Notice that formulae (2.4), (2.17), (2.18), (2.20) have a well-defined limit as  $n \rightarrow \infty$ :

$$\text{Pr}(\bar{\xi}_\infty = [\kappa]) = \frac{1}{[\kappa]!} \left(\frac{p}{1-q}\right)^\kappa (1-p)^\infty, \quad (2.21)$$

$$\langle \bar{\xi}_\infty \rangle = \frac{p}{1-q}, \quad (2.22)$$

$$\langle \bar{\xi}_\infty^2 \rangle = \frac{p}{1-q} + q \left(\frac{p}{1-q}\right)^2, \quad (2.23)$$

$$\text{Var}(\bar{\xi}_\infty) = \frac{p(1-p)}{1-q}. \quad (2.24)$$

So far, we have treated the random variable  $\bar{\xi}_n$  as an object in its own right. Let us now turn to the representation of  $\bar{\xi}_n$  as a sum of  $n$  ‘‘coin’’ throws, as reflected in the classical formula (2.2). This sum-formula (2.2) remains intact under  $q$ -deformation. However, for general  $q$ , the random variables  $\bar{\zeta}_i$ 's are *no longer independent or identically distributed*.

More precisely, let

$$\bar{\zeta}_1 = \begin{cases} 1, & \text{with probability } p, \\ 0, & \text{with probability } 1 - p. \end{cases} \quad (2.25)$$

For  $\kappa \in \mathbf{N}$ , suppose the random variables  $\bar{\zeta}_1, \dots, \bar{\zeta}_\kappa$  have already been defined. Denote by  $\alpha_\kappa = \alpha_\kappa(\bar{\zeta}_1, \dots, \bar{\zeta}_\kappa)$  the number of zeroes appearing among the values of the random variables  $\bar{\zeta}_1, \dots, \bar{\zeta}_\kappa$ . The random variable  $\bar{\zeta}_{\kappa+1}$  takes the values  $0, q^0, \dots, q^\kappa$ , with the conditional probabilities

$$Pr(\bar{\zeta}_{\kappa+1} = 0 | \alpha_\kappa = r) = 1 - q^r p, \quad 0 \leq r \leq \kappa, \quad (2.26a)$$

$$Pr(\bar{\zeta}_{\kappa+1} = q^\ell | \alpha_\kappa = r) = \delta_{\kappa-r}^\ell q^r p, \quad 0 \leq r \leq \kappa. \quad (2.26b)$$

For example,

$$Pr(\bar{\zeta}_2 = 0 | \bar{\zeta}_1 = 0) = 1 - qp, \quad (2.27a)$$

$$Pr(\bar{\zeta}_2 = 0 | \bar{\zeta}_1 = 1) = 1 - p, \quad (2.27b)$$

$$Pr(\bar{\zeta}_2 = 1 | \bar{\zeta}_1 = 0) = qp, \quad (2.27c)$$

$$Pr(\bar{\zeta}_2 = q | \bar{\zeta}_1 = 1) = p. \quad (2.27d)$$

By induction on  $\kappa$ , it is easily seen that if the last before  $\bar{\zeta}_{\kappa+1}$  non-zero value appearing among  $\bar{\zeta}_1, \dots, \bar{\zeta}_\kappa$  was  $q^\ell$ , then  $\bar{\zeta}_{\kappa+1}$  can take only the values 0 and  $q^{\ell+1}$  with non-zero probabilities; if all the  $\bar{\zeta}_1, \dots, \bar{\zeta}_\kappa$  took value 0, then  $\bar{\zeta}_{\kappa+1}$  takes only the values 0 and 1 with non-zero probabilities.

A better description of the same distribution is possible, if instead of conditional probabilities we work with joint ones. Denote by  $\mathbf{0}^r$  the event of  $r$  in a row appearances of zeroes,  $r \in \mathbf{Z}_+$ , and similarly by

$$\mathbf{0}^{a(0)} q^0 \mathbf{0}^{a(1)} q^1 \dots \mathbf{0}^{a(k-1)} q^{\kappa-1} \mathbf{0}^{a(\kappa)}, \quad a(\cdot) \in \mathbf{Z}_+, \quad (2.28)$$

the event of  $a(0)$  of zeroes followed by  $q^0 = 1$  followed by  $a(1)$  zeroes ... Now set

$$Pr(\mathbf{0}^{a(0)} q^0 \dots q^{\kappa-1} \mathbf{0}^{a(\kappa)}) := (1 \dot{-} p)^{\sum_0^\kappa a(s)} \prod_{i=1}^{\kappa-1} (pq^{\sum_0^i a(s)}); \quad (2.29)$$

for  $\kappa = 0$ , formula (2.29) is to be understood as

$$Pr(\mathbf{0}^a) = (1 \dot{-} p)^a, \quad a \in \mathbf{N}. \quad (2.29')$$

Let us now verify that the ‘‘microscopic’’ formulae (2.29) imply the ‘‘macroscopic’’ formula (2.4). Denote

$$|a(i)| = \sum_{s=0}^i a(s). \quad (2.30)$$

We have to verify that

$$\sum_{|a(\kappa)|=n-\kappa} (1 \dot{-} p)^{n-\kappa} p^\kappa \prod_{i=0}^{\kappa-1} q^{|a(i)|} = \begin{bmatrix} n \\ \kappa \end{bmatrix} p^\kappa (1 \dot{-} p)^{n-\kappa}, \quad (2.31)$$

which is equivalent to the  $q$ -counting formula

$$\sum_{|a(\kappa)|=n-\kappa} \prod_{i=0}^{\kappa-1} q^{|a(i)|} = \begin{bmatrix} n \\ \kappa \end{bmatrix}. \quad (2.32)$$

(For  $q = 1$ , we recover the classical result: the number of solutions in nonnegative integers of the equation  $a(0) + \dots + a(\kappa) = n - \kappa$  is  $\binom{n}{\kappa}$ .)

We shall prove formula (2.32) by the double induction on  $N := n - \kappa$  and  $\kappa$ , in the form

$$\sum_{|a(\kappa)|=N} \prod_{i=0}^{\kappa-1} q^{|a(i)|} = \begin{bmatrix} N + \kappa \\ \kappa \end{bmatrix}. \quad (2.33)$$

Now,

$$\prod_{i=0}^{\kappa-1} q^{|a(i)|} = q^{a(0)} q^{a(0)+a(1)} \dots q^{a(0)+\dots+a(\kappa-1)} = q^{\sum_0^{\kappa} (k-s)a(s)} \quad (2.34a)$$

$$= q^{\kappa|a(\kappa)|} q^{-\sum_0^{\kappa} sa(s)}. \quad (2.34b)$$

Therefore, the identity (2.33) becomes

$$\sum_{|a(\kappa)|=N} q^{-\sum_0^{\kappa} sa(s)} = q^{-N\kappa} \begin{bmatrix} N + \kappa \\ \kappa \end{bmatrix}. \quad (2.35)$$

For  $N = 0$ , formula (2.35) becomes  $1 = 1$  no matter what  $\kappa$  is. For  $\kappa = 0$ , formula (2.35) is true by definition (2.29'); for  $\kappa = 1$ , formula (2.35) becomes

$$\sum_{a(0)+a(1)=N} q^{-a(1)} = q^{-N} \begin{bmatrix} N + 1 \\ 1 \end{bmatrix},$$

which is obviously true for all  $N$ . Suppose formula (2.35) is true for the pairs  $(\kappa, N = n)$  and  $(\kappa - 1, N = n + 1)$ . Consider the pair  $(\kappa, N = n + 1)$ . Let's divide the set of the  $a$ 's with  $|a(\kappa)| = n + 1$  into two groups: those with  $a(\kappa) > 0$  and those with  $a(\kappa) = 0$ . For the 1<sup>st</sup> group, the set  $\bar{a}(0) = a(0), \dots, \bar{a}(\kappa - 1) = a(\kappa - 1), \bar{a}(\kappa) = a(\kappa) - 1$ , satisfies  $|\bar{a}(\kappa)| = n$ , so that, by the induction assumption,

$$\sum q^{-\sum_0^{\kappa} sa(s)} = \sum_{|\bar{a}(\kappa)|=n} q^{-\sum_0^{\kappa} s\bar{a}(s)-\kappa} = q^{-\kappa} q^{-n\kappa} \begin{bmatrix} n + \kappa \\ \kappa \end{bmatrix}. \quad (2.36)$$

The 2<sup>nd</sup> group has effectively  $\kappa - 1$   $a$ 's, so that again, by the induction assumption,

$$\sum q^{-\sum_0^{\kappa-1} sa(s)} = q^{-(n+1)(\kappa-1)} \begin{bmatrix} n + \kappa \\ \kappa - 1 \end{bmatrix}. \quad (2.37)$$

We thus have to check that

$$q^{-\kappa(n+1)} \begin{bmatrix} n + \kappa \\ \kappa \end{bmatrix} + q^{-(n+1)(\kappa-1)} \begin{bmatrix} n + \kappa \\ \kappa - 1 \end{bmatrix} = q^{-(n+1)\kappa} \begin{bmatrix} n + 1 + \kappa \\ \kappa \end{bmatrix}, \quad (2.38)$$

which is equivalent to

$$\begin{bmatrix} n + \kappa \\ \kappa \end{bmatrix} + \begin{bmatrix} n + \kappa \\ \kappa - 1 \end{bmatrix} q^{n+1} = \begin{bmatrix} n + 1 + \kappa \\ \kappa \end{bmatrix}, \quad (2.39)$$

which is obviously true.

Notice that for  $a(\kappa) = \infty$ , formulae (2.29) become

$$Pr(\mathbf{0}^{a(0)} q^0 \dots q^{\kappa-1} \mathbf{0}^\infty) = p^\kappa (1 \dot{-} p)^\infty q^{\sum_0^{\kappa-1} (\kappa-s)a(s)}, \quad (2.40)$$

$$Pr(\mathbf{0}^\infty) = (1 \dot{-} p)^\infty. \quad (2.40')$$

As in the classical theory (cf [10] p. 59), we can calculate the probability of observing  $\leq \kappa$  ( $< n$ ) zeroes in  $n$  trials:

$$\begin{aligned} Pr(\alpha_n \leq \kappa) &= \sum_{i=0}^{\kappa} \begin{bmatrix} n \\ i \end{bmatrix} p^{n-i} (1 \dot{-} p)^i = [n] \begin{bmatrix} n-1 \\ \kappa \end{bmatrix} \int_0^p x^{n-1-\kappa} (1 \dot{-} qx)^\kappa d_q x \\ &= \left( \int_0^p x^{n-1-\kappa} (1 \dot{-} qx)^\kappa d_q x \right) / \left( \int_0^1 x^{n-1-\kappa} (1 \dot{-} qx)^\kappa d_q x \right). \end{aligned} \quad (2.41)$$

Similarly, the probability of  $\leq \ell$  ( $< n$ ) non-zeroes in  $n$  trials is

$$Pr(\alpha_n > n - \ell) = \sum_{i=0}^{\ell} \begin{bmatrix} n \\ i \end{bmatrix} p^i (1 \dot{-} p)^{n-i} = 1 - [n] \begin{bmatrix} n-1 \\ \ell \end{bmatrix} \int_0^p x^\ell (1 \dot{-} qx)^{n-1-\ell} d_q x. \quad (2.42)$$

(Formulae (2.41) and (2.42) can be easily proven upon  $q$ -differentiation with respect to  $p$  and using the obvious relation

$$\frac{d}{d_q p} (1 \dot{-} p)^n = -[n] (1 \dot{-} qp)^{n-1}. \quad (2.43)$$

In the limit  $n \rightarrow \infty$ , this probability becomes

$$\sum_{i=0}^{\ell} \frac{1}{[i]!} \left( \frac{p}{1-q} \right)^i (1 \dot{-} p)^\infty = 1 - \frac{1}{1-q} \int_0^p \frac{1}{[\ell]!} \left( \frac{x}{1-q} \right)^\ell (1 \dot{-} qx)^\infty d_q x. \quad (2.44)$$

Many, if not all, classical formulae in probability have  $q$ -analogs. Let's take a look at a few of such formulae involving higher moments for the Bernoulli distribution.

First, applying the operator  $p^r \left( \frac{d}{d_q p} \right)^r \Big|_{v=p}$  to formula (2.8) and using the relation

$$x = [\kappa] \Rightarrow [\kappa - i] = q^{-i} (x - [i]), \quad (2.45)$$

we get

$$E(\prod_{i=0}^{\kappa-1} q^{-i} (\bar{\xi}_n - [i])) = p^r \prod_{i=0}^{r-1} [n - i], \quad (2.46)$$

a  $q$ -analog of the familiar formula

$$\langle \xi_n (\xi_n - 1) \dots (\xi_n - r + 1) \rangle = p^r n \dots (n - r + 1). \quad (2.47)$$

Second, let

$$\mu'_r = \langle \xi_n^r \rangle, \quad \mu_r = \langle (\xi_n - \langle \xi_n \rangle)^r \rangle, \quad r \in \mathbf{Z}_+, \quad (2.48)$$

be the moments, around zero and  $\langle \xi_n \rangle$  respectively, considered as functions of  $p, n, r$ . Romanovsky [9] has proved that

$$\mu'_{r+1} = (np + p(1-p) \frac{d}{dp})(\mu'_r), \quad (2.49)$$

$$\mu_{r+1} = p(1-p)(nr\mu_{r-1} + \frac{d\mu_r}{dp}). \quad (2.50)$$

Formula (2.49) has the following  $q$ -analog:

$$\mu'_{r+1} = ([n]p + p(1-p) \frac{d}{d_q p})(\mu'_r). \quad (2.51)$$

Formula (2.50) has no clear  $q$ -analog, in part because the notion of *higher central moments* is not unique in  $q$ -probability. Certainly, the classical definition

$$\mu_r = E((\xi - \langle \xi \rangle)^r) \quad (2.52)$$

is not useful, as the objects

$$\frac{d}{d_q x} ((\alpha + \beta x)^r) \quad (2.53)$$

lie outside compact formulae of  $q$ -analysis. Two other possible definitions are

$$\mu_r = E((\xi - \langle \xi \rangle / q)^r), \quad (2.54)$$

where [7]

$$(a+b/q)^n = \sum_{\kappa=0}^n \begin{bmatrix} n \\ \kappa \end{bmatrix} a^\kappa b^{n-\kappa}, \quad n \in \mathbf{Z}_+, \quad (2.55)$$

and

$$\mu_r(s) = E((\xi - q^s \langle \xi \rangle)^r). \quad (2.56)$$

Using the definition (2.56) for the  $q$ -Bernoulli distribution (2.4), we find

$$\mu_{r+1}(0; p) = p(1-p)(q[n][r]\mu_{r-1}(1; pq) + \frac{d}{d_q p}(\mu_r(1; p))), \quad (2.57)$$

$$\mu_{r+1}(-r; p) = p(1-p)(q^{-r}[n][r]\mu_{r-1}(-r; pq) + \frac{d}{d_q p}(\mu_r(-r; p))). \quad (2.58)$$

These formulae indicate that the central limit theorem for the  $q$ -Bernoulli distribution may not exist, at least in the classical sense.



### 3 $q$ -analogs of negative binomial distributions

The negative binomial distribution, also called Pascal distribution, can be arrived at via many different routes. Perhaps the simplest one is as the waiting time in a succession of Bernoulli trials until the appearance of  $r^{th}$  non-zero for the first time.

Let's first consider the case  $r = 1$ . Let  $W$  be the random variable, waiting time until first non-zero. By formula (2.29),

$$Pr(W = j) = Pr(\mathbf{0}^{j-1}q^0) = (1 \dot{-} p)^{j-1}q^{j-1}p, \quad j \in \mathbf{N}. \quad (3.1)$$

We further  $q$ -modify this formula by setting

$$Pr(\bar{W} = [j]) = (1 \dot{-} p)^{j-1}q^{j-1}p, \quad j \in \mathbf{N}. \quad (3.2)$$

This is our  $q$ -analog of the geometric distribution. Since

$$\sum_{j=1}^{\infty} (1 \dot{-} p)^{j-1}q^{j-1}p = 1 - Pr(\mathbf{0}^{\infty}) = 1 - (1 \dot{-} p)^{\infty}, \quad (3.3)$$

we get a  $q$ -analog of the formula for the sum of a geometric progression:

$$\sum_{s=0}^{\infty} (1 \dot{-} p)^s q^s = \frac{1 - (1 \dot{-} p)^{\infty}}{p}. \quad (3.4)$$

**Remark 3.5.** Most of the formulae appearing in this paper remain true when  $q$  is considered as a formal variable, or as a complex one (with occasional restrictions of the type  $|q| < 1$ ,  $|q| > 1$ , etc.) It is only for the sake of probability interpretations that  $q$  is considered to be a real number between 0 and 1.

Formula (3.4) has a finite counterpart.

$$\sum_{s=0}^N (1 \dot{-} x)^s q^s = \frac{1 - (1 \dot{-} x)^{N+1}}{x}. \quad (3.5)$$

This relation is easily checked by induction on  $N$ .

For general  $r \in \mathbf{N}$ , the probability that the  $r^{th}$  non-zero occurs at exactly the  $j^{th}$  trial,  $j \geq r$ , is, by formulae (2.29), (2.30), (2.33):

$$\begin{aligned} \sum_{|a^{(r-1)}|=j-r} Pr(\mathbf{0}^{a_0}q^0 \dots \mathbf{0}^{a^{(r-1)}}q^{r-1}) &= \sum (1 \dot{-} p)^{j-r} p^r q^{r(j-r)} q^{-\sum_0^{r-1} sa(s)} \\ &= (1 \dot{-} p)^{j-r} p^r q^{r(j-r)} q^{-(j-r)(r-1)} \begin{bmatrix} j - r + r - 1 \\ r - 1 \end{bmatrix} \\ &= (1 \dot{-} p)^{j-r} q^{j-r} p^r \begin{bmatrix} j - 1 \\ r - 1 \end{bmatrix}. \end{aligned} \quad (3.6)$$

Since the probability of having exactly  $\ell$  non-zeroes during an infinite number of trials is, by formula (2.21),

$$\frac{1}{[\ell]!} \left( \frac{p}{1 - q} \right)^{\ell} (1 \dot{-} p)^{\infty}, \quad (3.7)$$

the probability of having  $< r$  non-zeroes is, therefore,

$$\left( \sum_{\ell=1}^{r-1} \frac{1}{[\ell]!} \left( \frac{p}{1-q} \right)^\ell \right) (1 \dot{-} p)^\infty. \quad (3.8)$$

Thus,

$$\sum_{s=0}^{\infty} (1 \dot{-} p)^s q^s \begin{bmatrix} r-1+s \\ s \end{bmatrix} = p^{-r} \left( 1 - (1 \dot{-} p)^\infty \sum_{\ell=0}^{r-1} \frac{1}{[\ell]!} \left( \frac{p}{1-q} \right)^\ell \right). \quad (3.9)$$

This identity can be gotten directly from formula (3.4) by applying the operator  $\left( \frac{d}{d_q p} \right)^{r-1}$  to it.

Notice that formulae (3.3) and (3.9) show that our  $q$ -distributions do not sum up to 1 and therefore have to be re-scaled. For example, formula (3.2) becomes

$$Pr(\bar{W}_q = [j]) = \frac{1}{1 - (1 \dot{-} p)^\infty} p (1 \dot{-} p)^{j-1} q^{j-1}, \quad j-1 \in \mathbf{Z}_+. \quad (3.10)$$

**Remark 3.11.** Formula (3.6) could have been arrived at via one of the standard routes as the conditional probability of having  $r-1$  non-zeroes at the  $1^{st}$   $j-1$  throws, with probability  $\begin{bmatrix} j-1 \\ r-1 \end{bmatrix} p^{r-1} (1 \dot{-} p)^{j-r}$  by formula (2.4), followed by a non-zero appearing at the  $j^{th}$  throw, with probability  $q^{j-r} p$  by formula (2.26b).

We conclude this section by re-visiting the “problème de parties”, one of the first problems in probability discussed and solved by Fermat and Pascal in their correspondence. In essence, we want to find the probability that  $a$  non-zeroes appear before  $b$  zeroes in the Bernoulli trial of  $a+b-1$  throws. This can happen in either one of the  $b$  ways, when the  $a^{th}$  non-zero appears at the  $(a+\ell)^{th}$  trial,  $0 \leq \ell \leq b-1$ . By formula (3.6), the probability of this is ( $j = a + \ell$ ,  $r = a$ ):

$$(1 \dot{-} p)^\ell q^\ell p^a \begin{bmatrix} a + \ell - 1 \\ a - 1 \end{bmatrix}, \quad (3.12)$$

so that the total probability is

$$P_1 = p^a \sum_{\ell=0}^{b-1} \begin{bmatrix} a + \ell - 1 \\ a - 1 \end{bmatrix} (1 \dot{-} p)^\ell q^\ell. \quad (3.13)$$

Similarly, the event that  $b$  zeroes appear before  $a$  non-zeroes can happen in one of the  $a$  ways, when the  $b^{th}$  zero appears at the  $(b+\ell)^{th}$  trial,  $0 \leq \ell \leq a-1$ . The probability of this event is, by formula (2.26a):

$$\begin{bmatrix} b + \ell - 1 \\ b - 1 \end{bmatrix} p^\ell (1 \dot{-} p)^{b-1} (1 - q^{b-1} p) = \begin{bmatrix} b + \ell - 1 \\ b - 1 \end{bmatrix} p^\ell (1 \dot{-} p)^b. \quad (3.14)$$

Thus, the total probability of  $b$  zeroes appearing before  $a$  non-zeroes is

$$P_2 = (1 \dot{-} p)^b \sum_{\ell=0}^{a-1} \begin{bmatrix} b + \ell - 1 \\ b - 1 \end{bmatrix} p^\ell. \quad (3.15)$$

Since  $P_1 + P_2 = 1$ , we find

$$p^a \sum_{\ell=1}^{b-1} \begin{bmatrix} a + \ell - 1 \\ a - 1 \end{bmatrix} (1 \dot{-} p)^\ell q^\ell + (1 \dot{-} p)^b \sum_{\ell=0}^{a-1} \begin{bmatrix} b + \ell - 1 \\ b - 1 \end{bmatrix} p^\ell = 1, \quad \forall a, b \in \mathbf{N}, \quad (3.16)$$

an identity which is not immediately obvious.

The same probability  $P_1$  can be calculated differently, as the outcome, out of  $a + b - 1$  trials, of  $a + s$  non-zeroes,  $0 \leq s \leq b - 1$ . Thus,

$$P_1 = \sum_{s=0}^{b-1} \begin{bmatrix} a + b - 1 \\ a + s \end{bmatrix} p^{a+s} (1 \dot{-} p)^{b-1-s}, \quad (3.17)$$

and we arrive at another nonobvious (even for  $q = 1$ ) identity

$$p^a \sum_{\ell=0}^{b-1} \begin{bmatrix} a + \ell - 1 \\ \ell \end{bmatrix} (1 \dot{-} p)^\ell q^\ell = \sum_{s=0}^{b-1} \begin{bmatrix} a + b - 1 \\ a + s \end{bmatrix} p^{a+s} (1 \dot{-} p)^{b-1-s}. \quad (3.18)$$

## 4 $q$ -Poisson distribution

One of the shortest derivations of the Poisson distribution consists of considering, as Poisson originally did, the limit

$$n \rightarrow \infty, \quad pn \rightarrow \lambda \quad (4.1)$$

in the Bernoulli distribution:

$$Pr(\xi = \kappa) = \binom{n}{\kappa} p^\kappa (1 - p)^{n-\kappa} = \frac{n \dots (n - \kappa + 1)}{\kappa!} \frac{\lambda^\kappa}{n^\kappa} \left(1 - \frac{\lambda}{n}\right)^{n-\kappa} \rightarrow \frac{\lambda^\kappa}{\kappa!} e^{-\lambda}. \quad (4.2)$$

The  $q$ -picture is more interesting. First, for  $|q| < 1$ , the expression  $Pr(\bar{\xi}_n = [\kappa])$  (2.4) has the  $n \rightarrow \infty$  - limit (2.21):

$$\begin{bmatrix} n \\ \kappa \end{bmatrix} p^\kappa (1 \dot{-} p)^{n-\kappa} \rightarrow \frac{1}{[k]!} \left(\frac{p}{1 - q}\right)^\kappa (1 \dot{-} p)^\infty. \quad (4.3)$$

We can get a  $q$ -Poisson distribution from this by setting

$$p = \lim_{n \rightarrow \infty} \frac{\lambda}{[n]} = \lambda(1 - q), \quad (4.4)$$

so that

$$Pr(X = [\kappa]) = \frac{\lambda^\kappa}{[\kappa]!} (1 \dot{-} \lambda(1 - q))^\infty, \quad (4.5)$$

and therefore

$$E_0(\lambda) = E_0(\lambda; q) = \sum_{\kappa=0}^{\infty} \frac{\lambda^\kappa}{[\kappa]!} = \frac{1}{(1 \dot{-} \lambda(1 - q))^\infty}, \quad (4.6)$$

the well-known formula. Here

$$E_\mu(\lambda) = \sum_{\kappa=0}^{\infty} \frac{\lambda^\kappa}{[\kappa]!} q^{\mu \binom{\kappa}{2}} \quad (4.7)$$

is the  $q$ -family of exponentials:

$$\frac{dE_\mu(\lambda)}{d_q \lambda} = E_\mu(q^\mu \lambda), \quad E_\mu(0) = 1. \quad (4.8)$$

By Euler's formula (2.11),

$$(1 \dot{-} p)^\infty = \sum_{j=0}^{\infty} \left( -\frac{p}{1-q} \right)^j \frac{1}{[j]!} q^{\binom{j}{2}} = E_1 \left( -\frac{p}{1-q} \right), \quad (4.9)$$

so that

$$E_0(\lambda) E_1(-\lambda) = 1; \quad (4.10)$$

the latter 2 formulae are of course classic.

Formula (4.10) can be generalized, as follows. Consider the probability generating function for the  $q$ -Bernoulli distribution:

$$\begin{aligned} F_{B;n}(z) &= \sum_{k=0}^{\infty} z^k Pr(\bar{\xi}_n = [k]) = \sum_{\kappa=0}^n \binom{n}{\kappa} z^\kappa p^\kappa (1 \dot{-} p)^{n-\kappa} \\ &= \sum_{\kappa=0}^n \binom{n}{\kappa} (zp)^\kappa (1 \dot{-} p)^{n-\kappa} \quad [\text{by (2.8)}] = \sum_{\kappa=0}^n \binom{n}{\kappa} (zp \dot{-} p)^\kappa \\ &= \sum_{\kappa=0}^n \binom{n}{\kappa} p^\kappa (z \dot{-} 1)^\kappa. \end{aligned} \quad (4.11)$$

As  $n \rightarrow \infty$ , this generating function becomes

$$F_{B;\infty} = \sum_{\kappa=0}^{\infty} \frac{1}{[\kappa]!} \left( \frac{p}{1-q} \right)^\kappa (z \dot{-} 1)^\kappa. \quad (4.12)$$

On the other hand, since

$$(1 \dot{-} x)^{\alpha+\beta} = (1 \dot{-} x)^\alpha (1 \dot{-} q^\alpha x)^\beta, \quad (4.13a)$$

we have:

$$(1 \dot{-} x)^{-\beta} = \frac{1}{(1 \dot{-} q^{-\beta} x)^\beta}, \quad (4.13b)$$

and therefore

$$\begin{aligned} (1 \dot{-} p)^{n-\kappa} &= (1 \dot{-} p)^n (1 \dot{-} q^n p)^{-\kappa} = \frac{(1 \dot{-} p)^n}{(1 \dot{-} q^{n-\kappa} p)^\kappa} \\ \xrightarrow{n \rightarrow \infty} (1 \dot{-} p)^\infty &= E_1 \left( -\frac{p}{1-q} \right) = \frac{1}{E_0(\lambda)}, \end{aligned}$$

so that

$$F_{B;\infty} = \lim_{n \rightarrow \infty} \sum z^\kappa \begin{bmatrix} n \\ \kappa \end{bmatrix} p^\kappa (1 \dot{-} p)^{n-\kappa} = \sum \left( \frac{zp}{1-q} \right)^\kappa \frac{1}{[\kappa]!} (1 \dot{-} p)^\infty = \frac{E_0(\lambda z)}{E_0(\lambda)}. \quad (4.14)$$

Thus,

$$\frac{E_0(\lambda z)}{E_0(z)} = \sum_{\kappa=0}^{\infty} \frac{\lambda^\kappa}{[\kappa]!} (z \dot{-} 1)^\kappa, \quad (4.15)$$

which can be equivalently rewritten as

$$E_0(b)E_1(-a) = \sum_{\kappa=0}^{\infty} \frac{(b \dot{-} a)^\kappa}{[\kappa]!}. \quad (4.16)$$

Taking the limit  $n \rightarrow \infty$  in formulae (2.17), (2.20), we find:

$$\langle X \rangle = \lambda, \quad (4.17)$$

$$Var(X) = \lambda(1 - (1 - q)\lambda). \quad (4.18)$$

Let us now consider the case when  $|q| > 1$ , so that we are walking outside the traditional probability theories. Again, set

$$p = \frac{\lambda}{[n]}. \quad (4.19)$$

Then, as  $n \rightarrow \infty$ ,

$$\begin{bmatrix} n \\ \kappa \end{bmatrix} p^\kappa = \frac{[n] \dots [n - \kappa + 1]}{[\kappa]!} \frac{\lambda^\kappa}{[n]^\kappa} \rightarrow \frac{\lambda^\kappa}{[\kappa]!} q^{-\binom{\kappa}{2}}, \quad (4.20)$$

since, as  $n \rightarrow \infty$ ,

$$\frac{[n - \ell]}{[n]} = \frac{q^{n-\ell} - 1}{q^n - 1} \rightarrow q^{-\ell}. \quad (4.21)$$

Next, by formula (4.13b),

$$(1 \dot{-} p)^{n-\kappa} = \left(1 \dot{-} \frac{\lambda}{[n]}\right)^{n-\kappa} = \frac{(1 \dot{-} \frac{\lambda}{[n]})^n}{(1 \dot{-} q^{n-\kappa} \frac{\lambda}{[n]})^\kappa} \rightarrow \lim_{n \rightarrow \infty} (1 \dot{-} \frac{\lambda}{[n]})^n (1 \dot{-} \lambda)^{-\kappa}. \quad (4.22)$$

Thus,

$$Pr(X = [\kappa]) = \frac{q^{-\binom{\kappa}{2}}}{[\kappa]!} \lambda^\kappa (1 \dot{-} \lambda)^{-\kappa} \lim_{n \rightarrow \infty} (1 \dot{-} \frac{\lambda}{[n]})^n, \quad |q| > 1. \quad (4.23)$$

In particular,

$$\lim_{n \rightarrow \infty} (1 \dot{-} \frac{\lambda}{[n]})^n = \left( \sum_{\kappa=0}^{\infty} \frac{q^{-\binom{\kappa}{2}}}{[\kappa]!} \lambda^\kappa (1 \dot{-} \lambda)^{-\kappa} \right)^{-1}, \quad |q| > 1; \quad (4.24)$$

this formula is not a  $q$ -analog of anything classical.

## 5 $q$ -hypergeometric distribution

Imagine that we have an urn consisting of two types of balls:  $m$  marked ‘1’ and  $u$  marked ‘0’. We pick out at random one ball, record its value and *leave it outside the urn*; then proceed again, for a total of  $n$  draws. Had we returned each picked ball back into the urn, we would have the Bernoulli trials; since we don’t return the balls, we get something different, called the hypergeometric distribution: the probability of ending up with  $\kappa$  ‘1’ balls out of  $n$  draws is

$$Pr(\xi_n = \kappa) = \binom{m}{\kappa} \binom{u}{n-\kappa} / \binom{N}{n}, \quad N = m + u, \quad (5.1)$$

see [4, 5].

As a  $q$ -analogue of this distribution we set

$$Pr(\bar{\xi}_n = [\kappa]) = \begin{bmatrix} m \\ \kappa \end{bmatrix} \begin{bmatrix} u \\ n-\kappa \end{bmatrix} q^{(m-\kappa)(n-\kappa)} / \begin{bmatrix} N \\ n \end{bmatrix}. \quad (5.2)$$

To justify this definition we have to verify that

$$\sum_{\kappa} \begin{bmatrix} m \\ \kappa \end{bmatrix} \begin{bmatrix} u \\ n-\kappa \end{bmatrix} q^{(m-\kappa)(n-\kappa)} = \begin{bmatrix} m+u \\ n \end{bmatrix}. \quad (5.3)$$

This identity results by picking the  $x^n$ -coefficient in formula (4.13a):

$$(1-x)^m (1-q^m x)^u = (1-x)^{m+u}, \quad (5.4)$$

and using the Euler formula (2.11):

$$\begin{aligned} & \sum_{\kappa} \begin{bmatrix} m \\ \kappa \end{bmatrix} (-x)^{\kappa} q^{\binom{\kappa}{2}} \sum_{\ell} \begin{bmatrix} u \\ \ell \end{bmatrix} (-x)^{\ell} q^{m\ell} q^{\binom{\ell}{2}} \\ &= \sum_n (-x)^n q^{\binom{n}{2}} \sum_{\kappa+\ell=n} \begin{bmatrix} m \\ \kappa \end{bmatrix} \begin{bmatrix} u \\ \ell \end{bmatrix} q^{\binom{\kappa}{2} + \binom{\ell}{2} - \binom{\kappa+\ell}{2}} q^{m\ell} \\ &= \sum_n (-x)^n q^{\binom{n}{2}} \sum_{\kappa} \begin{bmatrix} m \\ \kappa \end{bmatrix} \begin{bmatrix} u \\ n-\kappa \end{bmatrix} q^{(m-\kappa)(n-\kappa)} = \sum_n \begin{bmatrix} m+u \\ n \end{bmatrix} (-x)^n q^{\binom{n}{2}}, \end{aligned} \quad (5.5)$$

where we used the obvious relation

$$\binom{\kappa+\ell}{2} = \binom{\kappa}{2} + \binom{\ell}{2} + \kappa\ell. \quad (5.6)$$

Similar to the Bernoulli case, we can treat the  $q$ -hypergeometric distribution (5.2) as a macroscopic object and inquire about its microscopic representation. The latter can be guessed from the relations

$$P(\bar{\xi}_n = [n]) = \begin{bmatrix} m \\ n \end{bmatrix} / \begin{bmatrix} N \\ n \end{bmatrix} = \frac{[m]}{[n]} \frac{[m-1]}{[N-1]} \cdots \frac{[m-n+1]}{[N-n+1]}, \quad (5.7a)$$

$$Pr(\bar{\xi}_n = 0) = \begin{bmatrix} u \\ n \end{bmatrix} q^{mn} / \begin{bmatrix} N \\ n \end{bmatrix} = \frac{[u]}{[N]} q^m \frac{[u-1]}{[N-1]} q^m \cdots \frac{[u-n+1]}{[N-n+1]} q^m, \quad (5.7b)$$

which suggest that in the representation

$$\bar{\xi}_n = \bar{\zeta}_1 + \dots + \bar{\zeta}_n \tag{5.8}$$

of  $n$  successive draws, we should set

$$Pr(\mathbf{0}^{a(0)} q^0 \dots \mathbf{0}^{a(k-1)} q^{k-1} \mathbf{0}^{a(\kappa)}) = q^{\sum_0^\kappa (m-s)a(s)} \begin{bmatrix} m \\ \kappa \end{bmatrix} \begin{bmatrix} u \\ n - \kappa \end{bmatrix} / \begin{bmatrix} N \\ n \end{bmatrix} \begin{bmatrix} n \\ \kappa \end{bmatrix} \tag{5.9a}$$

$$= q^{\sum_0^\kappa (m-s)a(s)} \prod_{i=0}^{\kappa-1} [m - i] \prod_{j=0}^{n-\kappa-1} [u - j] / \prod_{\gamma=0}^{n-1} [N - \gamma]. \tag{5.9b}$$

$$Pr(\mathbf{0}^a) = q^{ma} \begin{bmatrix} u \\ a \end{bmatrix} / \begin{bmatrix} N \\ a \end{bmatrix}. \tag{5.9c}$$

(In the  $\bar{\zeta}$ -language, we have

$$Pr(\bar{\zeta} \neq 0) = \frac{[m]}{[N]}, \quad Pr(\bar{\zeta} = 0) = q^m \frac{[u]}{[N]} \tag{5.10}$$

at each ball pick-out when the number of marked ‘nonzero’ balls in the urn is  $m$ , and the number of those marked ‘zero’ is  $u = N - m$ .)

To prove that microscopic formula (5.9a) implies the macroscopic formula (5.2), we need to verify that

$$\sum_{|a(\kappa)|=n-\kappa} q^{\sum_0^\kappa (m-s)a(s)} = q^{(m-\kappa)(n-\kappa)} \begin{bmatrix} n \\ \kappa \end{bmatrix}, \tag{5.11}$$

and this equality follows from the already proven formula (2.35):

$$\begin{aligned} \sum_{|a(\kappa)|=n-\kappa} q^{\sum_0^\kappa (m-s)a(s)} &= \sum q^{m \sum a(s)} q^{-\sum sa(s)} = q^{m(n-\kappa)} q^{-(n-\kappa)\kappa} \begin{bmatrix} n - \kappa + \kappa \\ \kappa \end{bmatrix} \\ &= q^{(m-\kappa)(n-\kappa)} \begin{bmatrix} n \\ \kappa \end{bmatrix}. \end{aligned}$$

In the classical case  $q = 1$ , we can rewrite formula (5.1) as

$$Pr(\xi_n = \kappa) = \binom{n}{\kappa} \prod_{s=0}^{\kappa-1} \frac{m-s}{N-s} \prod_{\ell=0}^{n-\kappa+1} \frac{u-\ell}{N-\kappa-\ell}, \tag{5.12}$$

so that in the limit

$$N \rightarrow \infty, \quad \frac{m}{N} \rightarrow p, \quad \frac{u}{N} \rightarrow 1 - p, \tag{5.13}$$

formula (5.12) becomes the Bernoulli one; the classical explanation is that when  $m$  and  $u$  are large, it makes little difference whether the picked-out balls are returned back to the urn or not.

For  $q \neq 1$ , the situation is more interesting. Certainly formulae (5.13) are not the correct ones. We proceed as follows.

Let  $|q| > 1$ . Set

$$N \rightarrow \infty, \quad \frac{[m]}{[N]} \rightarrow p, \tag{5.14}$$

and re-write formula (5.2) in the form

$$Pr(\bar{\xi}_n = [\kappa]) = \begin{bmatrix} n \\ \kappa \end{bmatrix} \prod_{s=0}^{\kappa-1} \left( \frac{[m-s]q^{\kappa-n}}{[N-(n-\kappa)-s]} \right) \prod_{\ell=0}^{n-\kappa-1} \left( \frac{[u-\ell]q^m}{[N-\ell]} \right). \quad (5.15)$$

Now,

$$\begin{aligned} \frac{[m-s]q^{\kappa-n}}{[N-(n-\kappa)-s]} &= q^{\kappa-n} \frac{[-s] + q^{-s}[m]}{[\kappa-s-n] + q^{\kappa-s-n}[N]} \quad [\text{by (5.14)}] \\ &\rightarrow q^{\kappa-n} \frac{q^{-s}}{q^{\kappa-s-n}} p = p, \end{aligned} \quad (5.16a)$$

$$\frac{[u-\ell]q^m}{[N-\ell]} = \frac{[N-m-\ell]q^m}{[N-\ell]} = \frac{[N-\ell] - [m]}{[N-\ell]} = 1 - \frac{[m]}{[-\ell] + q^{-\ell}[N]} \rightarrow 1 - q^\ell p, \quad (5.16b)$$

so that

$$Pr(\bar{\xi}_n = [\kappa]) \rightarrow \begin{bmatrix} n \\ \kappa \end{bmatrix} p^\kappa (1-p)^{n-\kappa}, \quad (5.17)$$

as desired.

Suppose now that  $|q| < 1$ . Denote that  $q$  by  $Q$ . Set

$$q = Q^{-1}, \quad |q| > 1. \quad (5.18)$$

Since

$$[x]_Q = q^{1-x} [x]_q, \quad (5.19a)$$

$$[k]!_Q = q^{-\binom{\kappa}{2}} [k]!_q, \quad (5.19b)$$

$$\begin{bmatrix} a \\ b \end{bmatrix}_Q = q^{b(b-a)} \begin{bmatrix} a \\ b \end{bmatrix}_q, \quad (5.19c)$$

we have

$$\begin{aligned} Pr(\bar{\xi}_n = [\kappa]_q) &= Q^{(m-\kappa)(n-\kappa)} \begin{bmatrix} m \\ \kappa \end{bmatrix}_Q \begin{bmatrix} u \\ n-\kappa \end{bmatrix}_Q / \begin{bmatrix} N \\ n \end{bmatrix}_Q \\ &= q^{(\kappa-\kappa')(n-\kappa')} \begin{bmatrix} u \\ \kappa' \end{bmatrix}_q \begin{bmatrix} m \\ n-\kappa' \end{bmatrix}_q / \begin{bmatrix} N \\ n \end{bmatrix}_q, \end{aligned} \quad (5.20)$$

where

$$\kappa' = n - \kappa. \quad (5.21)$$

We see that our formula (5.2) in the form

$$Pr(\tilde{\xi}_n = \kappa) = \begin{bmatrix} m \\ \kappa \end{bmatrix} \begin{bmatrix} u \\ n-\kappa \end{bmatrix} q^{(m-\kappa)(n-\kappa)} / \begin{bmatrix} N \\ n \end{bmatrix} \quad (5.22)$$

allows the symmetry

$$m \rightarrow u, \quad u \rightarrow m, \quad \kappa \rightarrow n - \kappa, \quad q \rightarrow q^{-1}. \quad (5.23)$$



Setting

$$N \rightarrow \infty, \frac{[u]}{[N]} \rightarrow p' (= 1 - p), \quad (5.24)$$

we find, in the same way as formula (5.17) was gotten, that

$$Pr(\tilde{\xi}_n = [\kappa]) = \begin{bmatrix} n \\ \kappa \end{bmatrix} (1 - p')^\kappa p'^{n-\kappa}, \quad (5.25)$$

a different  $q$ -version of the classical Bernoulli distribution.

## 6 $q$ -contagious distribution

Suppose we again, like in the preceding section, have an urn with  $m$  marked and  $u$  unmarked balls. We pick out one ball at random, record its value, and then return to the urn  $s + 1$  balls identical to the one we just picked out. If  $s = 0$ , we return the ball itself, and this is the Bernoulli scheme; if  $s = -1$ , we return nothing, and this is the hypergeometric scheme. For general  $s$ , the probability to pick out  $k$  marked (by ‘1’) balls out of  $n$  draws is

$$Pr(\xi_n = \kappa) = \binom{n}{\kappa} \prod_{\alpha=0}^{\kappa-1} (m + \alpha s) \prod_{\beta=0}^{n-\kappa-1} (u + \beta s) / \prod_{\gamma=0}^{n-1} (m + u + \gamma s). \quad (6.1)$$

This particular member of the family of the so-called “contagious distributions” was discovered by Eggenberger and Pólya in 1923 [2, 8].

As a  $q$ -analog of this distribution, we set

$$\begin{aligned} Pr(\bar{\xi}_n = [\kappa]) &= \begin{bmatrix} n \\ \kappa \end{bmatrix}_{q^{-s}} q^{(m+s\kappa)(n-\kappa)} \prod_{\alpha=0}^{\kappa-1} [m + \alpha s] \prod_{\beta=0}^{n-\kappa-1} [u + \beta s] / \prod_{\gamma=0}^{n-1} [m + u + \gamma s]. \end{aligned} \quad (6.2)$$

Obviously, for the case  $s = -1$  we recover the hypergeometric formula (5.2). For the case  $s = 0$ , we recover the *classical* Bernoulli formula (2.3) with  $p = [m]/[m + u]$ , *not* the  $q$ -Bernoulli formula (2.4).

To justify formula (6.2), we need to check that

$$\sum_{\kappa=0}^n \begin{bmatrix} n \\ \kappa \end{bmatrix}_{q^{-s}} q^{(m+s\kappa)(n-\kappa)} \prod_{\alpha=0}^{\kappa-1} [m + \alpha s] \prod_{\beta=0}^{n-\kappa-1} [u + \beta s] = \prod_{\gamma=0}^{n-1} [m + u + \gamma s]. \quad (6.3)$$

To do that, we assume that  $s \neq 0$  and start with the formula (5.3) in the base  $Q = q^{-s}$ :

$$\sum_{\kappa=0}^n \begin{bmatrix} M \\ \kappa \end{bmatrix}_Q \begin{bmatrix} U \\ n - \kappa \end{bmatrix}_Q Q^{(M-\kappa)(n-\kappa)} = \begin{bmatrix} M + U \\ n \end{bmatrix}_Q, \quad Q = q^{-s}, \quad (6.4)$$

which we rewrite as

$$\sum_{\kappa=0}^n \begin{bmatrix} n \\ \kappa \end{bmatrix}_Q Q^{(M-\kappa)(n-\kappa)} \prod_{\alpha=0}^{\kappa-1} [M-\alpha]_Q \prod_{\beta=0}^{n-\kappa-1} [U-\beta]_Q = \prod_{\gamma=0}^{n-1} [M+U-\gamma]_Q. \quad (6.5)$$

Multiplying both sides by  $([-r]_q)^n$ , setting

$$M = -m/s, \quad U = -u/s, \quad (6.6)$$

and noticing that

$$Q^{(M-\kappa)(n-\kappa)} = q^{-s(-\kappa-m/s)(n-\kappa)} = q^{(m+\kappa s)(n-\kappa)}, \quad (6.7)$$

we arrive at formula (6.3). The latter formula can be considered as a new  $q$ -analog of Newton's binomial.

Similar to the hypergeometric case, we can arrive at the  $q$ -contagious distribution (6.2) as a macroscopic object,

$$\bar{\xi}_n = \bar{\zeta}_1 + \dots + \bar{\zeta}_n \quad (6.8)$$

from the microscopic formulae

$$\begin{aligned} Pr(\mathbf{0}^{a(0)} q^0 \dots \mathbf{0}^{a(k-1)} q^{k-1} \mathbf{0}^{a(k)}) \\ = q^{\sum_{i=0}^{\kappa} (m+si)a(i)} \prod_{\alpha=0}^{\kappa-1} [m+\alpha s] \prod_{\beta=0}^{n-\kappa-1} [u+\beta s] / \prod_{\gamma=0}^{n-1} [m+u+\gamma s]; \end{aligned} \quad (6.9)$$

the latter formulae are suggested by the extreme cases  $k = n$  and  $k = 0$  of formula (6.2):

$$Pr(\bar{\xi}_n = [n]) = \prod_{\alpha=0}^{n-1} \frac{[m+\alpha s]}{[N+\alpha s]}, \quad N = u + m, \quad (6.10a)$$

$$Pr(\bar{\xi}_n = 0) = \prod_{\beta=0}^{n-1} \left( \frac{[u+\beta s]}{[N+\beta s]} q^m \right). \quad (6.10b)$$

To verify that microscopic formulae (6.9) imply the macroscopic formula (6.2), we need to check that

$$\sum_{|a(\kappa)|=n-\kappa} q^{\sum_{i=0}^{\kappa} (m+si)a(i)} = q^{(m+s\kappa)(n-\kappa)} \begin{bmatrix} n \\ \kappa \end{bmatrix}_{q^{-s}}. \quad (6.11)$$

Now, for the LHS of formula (6.11) we get:

$$\begin{aligned} \sum_{|a(\kappa)|=n-\kappa} q^{\sum_{i=0}^{\kappa} (m+si)a(i)} &= q^{m(n-\kappa)} \sum (q^{-s})^{-\sum ia(i)} \quad [\text{by (2.35)}] \\ &= q^{m(n-\kappa)} (q^{-s})^{-(n-\kappa)\kappa} \begin{bmatrix} n \\ \kappa \end{bmatrix}_{q^{-s}} = q^{(m+s\kappa)(n-\kappa)} \begin{bmatrix} n \\ \kappa \end{bmatrix}_{q^{-s}}, \end{aligned} \quad (6.12)$$

and this is exactly the RHS of formula (6.11).

## 7 $q$ -uniform distribution

A classical random variable  $X$  taking  $M + 1$  discrete values  $v_0 < \dots < v_M$ , each with equal probability  $1/(M + 1)$ , represents a discrete uniform distribution. The values  $v'_i$ s are immaterial and can be taken as  $v_i = i$ , or  $v_i = a + hi$ , or  $v_i = [i]$ , ...

As a  $q$ -analog of this distribution, we set

$$Pr(\tilde{X} = i) = q^i/[M + 1], \quad 0 \leq i \leq M, \quad (7.1)$$

or

$$Pr(\bar{X} = [i]) = q^i/[M + 1], \quad 0 \leq i \leq M, \quad (7.2)$$

so that

$$\langle \bar{X} \rangle = \frac{q[M]}{[2]}, \quad (7.3)$$

$$\langle \bar{X}^2 \rangle = \frac{q[M][M + 1](q[2][M] + 1)}{[2][3]}, \quad (7.4)$$

$$Var(\bar{X}) = \frac{q[M](q^2[M] + [2])}{[2]^2[3]}. \quad (7.5)$$

Consider  $n$  independent identically distributed, via the discrete uniform distribution, random variables  $X_1, \dots, X_n$ . The *range* of these variables is the quantity

$$r_n = \max_i(X_i) - \min_i(X_i), \quad 0 \leq r \leq M. \quad (7.6)$$

The random variable  $r_n$  has the following distribution ([4] p. 240):

$$Pr(r_n = 0) = 1/(M + 1)^{n-1}, \quad (7.7a)$$

$$Pr(r_n = \ell) = ((\ell + 1)^n - 2\ell^n + (\ell - 1)^n)(M + 1 - \ell)/(M + 1)^n, \quad 1 \leq \ell \leq M. \quad (7.7b)$$

This distribution is certainly different from those appearing in the preceding sections.

Let us calculate the  $q$ -analog of the distribution (7.7). Taking as our basic definition formula (7.1), we have:

$$Pr(r_n = 0) = \sum_{i=0}^M (Pr(\tilde{X} = i))^n = [M + 1]q^n / [M + 1]^n; \quad (7.8)$$

$$\begin{aligned} Pr(r_n = 1) &= \sum_{i=0}^{M-1} \sum_{k \neq 0, n} \binom{n}{\kappa} \left( \frac{q^i}{[M + 1]} \right)^\kappa \left( \frac{q^{i+1}}{[M + 1]} \right)^{n-\kappa} \\ &= \frac{1}{[M + 1]^n} \sum_{i=0}^{M-1} \left( \sum_{\kappa=0}^n \binom{n}{\kappa} (q^i)^\kappa (q^{i+1})^{n-\kappa} - (q^{i+1})^n - (q^i)^n \right) \\ &= \frac{1}{[M + 1]^n} \sum_{i=0}^{M-1} ((q^i + q^{i+1})^n - q^{in} q^n - q^i) \\ &= \frac{1}{[M + 1]^n} \sum_{i=0}^{M-1} (q^i)^n ([2]^n - [2]q^n) = \frac{[M]q^n([2]^n - [2]q^n)}{[M + 1]^n}; \end{aligned} \quad (7.9)$$

finally, for  $\ell \geq 2$ ,

$$\begin{aligned}
Pr(r_n = \ell) &= \sum_{i=0}^{M-\ell} \sum_{\kappa(0), \kappa(\ell) \neq 0} \frac{n!}{\kappa(0)! \dots \kappa(\ell)!} \left( \frac{q^i}{[M+1]} \right)^{\kappa(0)} \dots \left( \frac{q^{i+\ell}}{[M+1]} \right)^{\kappa(\ell)} \\
&= \frac{1}{[M+1]^n} \sum_{i=0}^{M-\ell} \left( \sum_{\text{all } \kappa's} - \sum_{\kappa(0)=0} - \sum_{\kappa(\ell)=0} + \sum_{\kappa(0)=\kappa(\ell)=0} \right) \\
&= \frac{1}{[M+1]^n} \sum_{i=0}^{M-\ell} \left( (q^i + \dots + q^{i+\ell})^n - (q^{i+1} + \dots + q^{i+\ell})^n \right. \\
&\quad \left. - (q^i + \dots + q^{i+\ell-1})^n + (q^{i+1} + \dots + q^{i+\ell-1})^n \right) \\
&= \frac{1}{[M+1]^n} \sum_{i=0}^{M-\ell} q^{in} \left( [\ell+1]^n - q^n [\ell]^n - [\ell]^n + q^n [\ell-1]^n \right) \\
&= \frac{[M+1-\ell]_q^n}{[M+1]^n} \left( [\ell+1]^n - [2]_q^n [\ell]^n + q^n [\ell-1]^n \right). \tag{7.10}
\end{aligned}$$

For  $\ell = 1$ , formula (7.10) reproduces formula (7.8). Thus, formulae

$$Pr(r_n = 0) = \frac{[M+1]_q^n}{[M+1]^n}, \tag{7.11a}$$

$$Pr(r_n = \ell) = \frac{[M+1-\ell]_q^n}{[M+1]^n} \left( [\ell+1]^n - [2]_q^n [\ell]^n + q^n [\ell-1]^n \right), \quad 1 \leq \ell \leq M, \tag{7.11b}$$

are  $q$ -analogs of formulae (7.7). Notice, that

$$Pr(r_1 > 0) = 0, \quad \forall q. \tag{7.12}$$

Although not immediately apparent, formulae (7.11) are not the *only*  $q$ -analogs of the classical formulae (7.7). For example, for  $n = 2$ , formulae

$$Pr(r_2 = 0) = \frac{1}{[M+1]}, \tag{7.13a}$$

$$Pr(r_2 = \ell) = \frac{[2]}{[M+1]} \left( 1 - \frac{[\ell]}{[M+1]} \right) q^{M+1-2\ell}, \quad 1 \leq \ell \leq M, \tag{7.13b}$$

are different from formulae (7.11)  $\Big|_{n=2}$ .

## 8 Concluding remarks

The approach to  $q$ -Probability taken in this paper leaves the *rules* of classical probability intact and only  $q$ -deforms some basic probability distributions. It is quite likely that one can develop some new/bizarre rules of  $q$ -probability which contradict the comfortably familiar intuition, similar to what Quantum-mechanical interpretations appear to a Classical-mechanical disciple. The most direct route to such new rules probably goes through basic *continuous* probability distributions when integral  $\int(\cdot)dx$  is replaced by the  $q$ -integral  $\int(\cdot)d_qx$ .

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