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Particles and Strings in a $2 + 1$-D Integrable Quantum Model

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Abstract

We give a review of some recent work on generalization of the Bethe ansatz in the case of $2 + 1$-dimensional models of quantum field theory. As such a model, we consider one associated with the tetrahedron equation, i.e. the $2+1$-dimensional generalization of the famous Yang–Baxter equation. We construct some eigenstates of the transfer matrix of that model. There arise, together with states composed of point-like particles analogous to those in the usual $1 + 1$-dimensional Bethe ansatz, new string-like states and string-particle hybrids.

Introduction

The Hamiltonians of $1 + 1$-dimensional ($1 + 1$-D) integrable quantum models, i.e. models with one-dimensional space and one-dimensional time, are diagonalized by means of the *Bethe ansatz* [1]. This means that their eigenvectors are searched for in the special form, namely as a superposition of several pointlike particles with definite momenta or "quasimomenta". The same Bethe ansatz is used for diagonalizing transfer matrices of two-dimensional statistical physics models. Clearly, the problem of extending the Bethe ansatz onto the $2 + 1$-D models, or maybe finding some other method for constructing their eigenvectors, is of great importance. In this paper, we are going to present some results obtained in this direction.

Let us recall the main features of the usual Bethe ansatz. We will have in mind a model with *discrete* space, like XXZ spin chain. In the usual Bethe ansatz, it is supposed that in the vector space of system states there are naturally distinguished "0-particle", "1-particle", etc. subspaces that are eigensubspaces of the Hamiltonian. As a rule, the 0-particle subspace is simply one-dimensional. Diagonalizing the Hamiltonian in the 1-particle particle space also is not hard work — it is sufficient to diagonalize the "quasimomentum" operator (commuting with the Hamiltonian), i.e. the operator of translation by one lattice unit, at least if we are considering a homogeneous finite chain with periodic boundary conditions or homogeneous infinite chain. In the latter

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case, we can obtain *formal* eigenstates if we allow for any quasimomentum eigenvalues, or *generalized* eigenstates (in the sense of functional analysis) if we require the module of eigenvalues be unity.

A remarkable property of integrable models is that we can construct a superposition of n one-particle states, lying in the n-particle subspace and being an eigenvector of the Hamiltonian. In the case of a finite closed chain, there arise severe constraints on quasimomenta — *Bethe ansatz equations*. The essence of those equations is that the "amplitude" corresponding to a given particle must not change after its having made a complete revolution around the chain, undergoing in its way the "scattering" on all other particles.

Algebraically, the problem is much easier for the infinite chain, because in such case revolutions around the chain don't exist. The conditions that single out generalized eigenvectors from the formal eigenvectors are also quite evident: the "amplitude" must not grow exponentially when the coordinate of any particle tends to the left or right infinity.

Now let us pass to the $2+1$-D models. Integrable $2+1$-D models are those related to the *tetrahedron equation* whose first solutions were found by Zamolodchikov [2]. The first solutions for the "vertex" modification of tetrahedron equation were found in paper [3] using some new structure called *tetrahedral Zamolodchikov algebra* [4, 5, 6]. As a further development, Zamolodchikov's solutions were generalized in papers [7, 8, 9, 10] by Bazhanov, Baxter, Mangazeev, Kashaev, and Stroganov. The discovery of the interrelation between original Zamolodchikov model and the vertex model of paper [3] is due to Sergeev, Boos, Mangazeev, and Stroganov [11]. Before that, J. Hietarinta [12] studied general possibilities for writing down modifications of tetrahedron equation, and found some new, with respect to paper [3], solutions of its vertex form.

An attempt to generalize the Bethe ansatz for such models runs at once into difficulties, because the known solutions of tetrahedron equation (out of which one can e.g. build a transfer matrix and search for its eigenvalues) do not allow to single out even a one-dimensional space. Certainly, in the $1+1$-D case there may also occur more complicated models than XXZ for which the above stated scheme of Bethe ansatz requires considerable modifications, but even the experience of studying those more complicated models did not yet yield any indications as to how to act in $2+1$ dimensions. Moreover, the experience of studying *classical* integrable equations like Kadomtsev–Petviashvili suggests that in $2+1$ dimensions we should expect greater variety of states than simply collections of pointlike particles.

In this paper, we will bring together the results of five preprints [14] by the author. We will present some eigenvectors for the transfer matrix corresponding to the simplest (and historically first [3]) nontrivial solution of the "vertex type" tetrahedron equation. They will be basically formal eigenvectors for the infinite lattice obeying, however, some conditions of "good behaviour" at the infinity. We will also present some eigenvectors for the finite closed lattice ("lattice on the torus").

Our starting point will be the fact that although we could not succeed in finding what might be called a one-dimensional space invariant under the action of transfer matrix, there exists a one-particle space whose *subspace is invariant* under it. This will allow us to construct *some* one-particle and then two-particle states.

The other starting point will be string-like states that arise naturally from the *tetrahedral Zamolodchikov algebra* — an algebraic structure using which the first ever solutions of the vertex type tetrahedron equation have been constructed. With all their simplicity, such strings are a new phenomenon specific for multidimensional models.

The culmination of this paper is achieved in uniting the particles and the strings. Here more general solutions of the tetrahedron equation due to Sergeev, Mangazeev and Stroganov [13] will be of use. We will provide examples showing how one can, with the help of strings, remove the obstacles on the way of constructing multi-particle states.

Before explaining the details, let us mention here some other interesting works whose relation with the subject of this paper is not yet completely clear. In paper [15], H. Boos studied some "strings" in a two-dimensional XY spin lattice. In papers [16, 17], Boos and Mangazeev obtain functional relations for the eigenvalues of a transfer matrix consisting of *three* plane layers.

Our model will be a model on the cubic lattice: the lattice vertices are points whose all three coordinates are integers. In each vertex there will be an "S-operator" (or "R-operator" in the notations of [13]. In the present paper, we are using the letter R for other purposes) acting in the tensor product of three linear spaces attached to the *links*. The transfer matrix we will be dealing with will be a "diagonal" one: it is cut out of the lattice by two planes perpendicular to the vector $(1, 1, 1)$ in such a way that it consist of separate, not linked to each other, vertices that can be imagined as placed on one of those planes like anti-tank hedgehogs. In each of the planes, the intersection with the cubic lattice yields a kagome lattice consisting, as is known, of triangles and hexagons. So it is not surprising that we will use as an important auxiliary tool one more transfer matrix, of the kagome form, whose image can be found in this paper as Fig. 6.

The S-operators are obtained as follows [5, 6]. We start from the usual (1 + 1-D) L-operators

$$L = \begin{pmatrix} a & & & d \\ & b & c & \\ & c & b & \\ d & & & a \end{pmatrix}$$

obeying the "free-fermion condition"

$$a^2 + b^2 = c^2 + d^2.$$

If L and M are two such operators with the same ratio cd/ab, then the usual Yang–Baxter equation

$$R^0 LM = MLR^0 \qquad (0.1)$$

holds with some operator R^0 of the same kind.

It is remarkable that, besides (0.1), one more similar relation holds:

$$(R^1)^{\mathrm{T}} LM = MLR^1,$$

where the superscript T means matrix transposing and $R^1 \neq R^0$.

Each of the operators R^0 and R^1 acts in the tensor product of two two-dimensional vector spaces, and thus can be represented by a matrix with two upper and two lower indices, or graphically — as a vertex where four edges meet. We now combine R^0 and R^1 in one object R^a, $a = 0, 1$, with one more edge ("fifth leg") bearing the index a.

Some special triple of R's enters in the following *defining relation of the tetrahedral Zamolodchikov algebra*, where we use the subscripts of R just to indicate the numbers of spaces where this R acts nontrivially:

$$R^a_{12}\tilde{R}^b_{13}\tilde{\tilde{R}}^c_{23} = \sum_{d,e,f} S^{abc}_{def}\tilde{\tilde{R}}^f_{23}\tilde{R}^e_{13}R^d_{12}. \tag{0.2}$$

This is illustrated by Fig. 1.

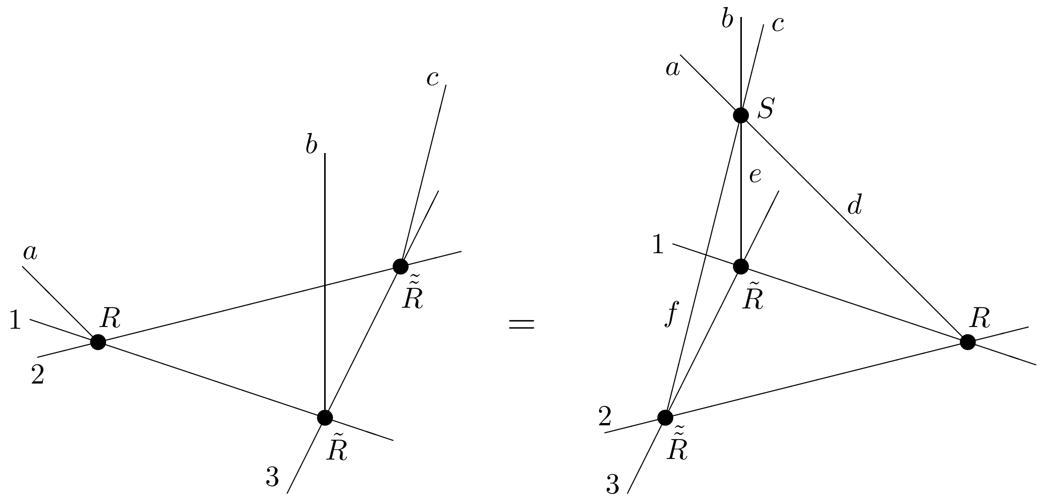


Figure 1:

Matrix S is exactly the matrix satisfying the vertex-type tetrahedron equation. It depends on (the differences of) three parameters, say φ_1, φ_2 and φ_3, and its matrix elements are [6]:

$$S^{000}_{000} = S^{011}_{011} = S^{101}_{101} = S^{110}_{110} = 1, \tag{0.3}$$

$$S^{001}_{010} = S^{010}_{001} = -S^{100}_{111} = -S^{111}_{100} = \sqrt{\coth(\varphi_1-\varphi_3)}\sqrt{\tanh(\varphi_2-\varphi_3)}, \tag{0.4}$$

$$S^{001}_{100} = S^{010}_{111} = -S^{100}_{001} = -S^{111}_{010} = \sqrt{\tanh(\varphi_1-\varphi_2)}\sqrt{\tanh(\varphi_2-\varphi_3)}, \tag{0.5}$$

$$S^{001}_{111} = S^{010}_{100} = S^{100}_{010} = S^{111}_{001} = \sqrt{\tanh(\varphi_1-\varphi_2)}\sqrt{\coth(\varphi_1-\varphi_3)}, \tag{0.6}$$

the other elements are zeroes, and the tetrahedron equation itself is

$$\begin{aligned} &S_{01,02,12}(\varphi_0,\varphi_1,\varphi_2)S_{01,03,13}(\varphi_0,\varphi_1,\varphi_3)S_{02,03,23}(\varphi_0,\varphi_2,\varphi_3)S_{12,13,23}(\varphi_1,\varphi_2,\varphi_3) \\ &= S_{12,13,23}(\varphi_1,\varphi_2,\varphi_3)S_{02,03,23}(\varphi_0,\varphi_2,\varphi_3)S_{01,03,13}(\varphi_0,\varphi_1,\varphi_3)S_{01,02,12}(\varphi_0,\varphi_1,\varphi_2). \end{aligned} \tag{0.7}$$

Here again the subscripts indicate the numbers of spaces where an operator acts nontrivially, e.g. in $S_{12,13,23} = (S^{abc}_{def})_{12,13,23}$ the indices a and d correspond to the space number 12, the indices b and e — to the space number 13, and the indices c and f — to the space number 23.

Note that in Section 4 we will be using more general solutions of the tetrahedron equation due to Sergeev, Mangazeev and Stroganov [13] together with their parameterization different from (0.3)–(0.6).

The contents of the remaining sections is as follows. In Sections 1 and 2 we construct some particle-like states; in Section 3 we construct the simplest strings; in Section 4 we construct a hybrid of a string and a particle; and in Section 5 we construct hybrids of several strings and particles. A Discussion completes the paper.

1 One- and two-particle states

As stated in the Introduction, our "hedgehog" transfer matrix — let us denote it T — is cut out of the cubic lattice by two planes perpendicular to the vector $(1,1,1)$. In each of those planes, the intersection with the cubic lattice yields a kagome lattice consisting, as known, of triangles and hexagons. We can group all vertices of the kagome lattice in triples — vertices of triangles — in such a way that the transfer matrix acts on each triangle separately, turning it inside out and making a linear transformation in the tensor product of three corresponding subspaces.

Consider one of such triangles. The tensor product of three subspaces corresponding to its vertices is comprised of the 0-, 1-, 2- and 3-particle sectors. According to papers [3, 6], the sectors with even and odd particle numbers do not mix together under the action of S and, moreover, in the even sector the S-operator acts as an identical unity. The 1- and 3-particle sectors do mix together, but it turns out that *there are two eigenvectors of the S-operator in the one-particle sector*, with eigenvalues 1 and -1. Their explicit form can be extracted out of the end of p. 94 and the beginning of p. 95 of [6]. Namely, denote as $(x,y,z)^{\mathrm{T}}$ a one-particle state describing the situation when the "amplitudes" for a particle to be in the 1st, 2nd and 3rd spaces are x, y and z. According to [6], and taking into account the fact that we are considering vectors without the 3-particle component, the two vectors $(x_\pm, y_\pm, z_\pm)^{\mathrm{T}}$ with eigenvalues ± 1 are "isotropic", i.e. such that $x_\pm^2 - y_\pm^2 + z_\pm^2 = 0$, and in fact it follows from [6] that we can restore the S-operator out of *any* two given vectors of this kind.

Any linear combination $(x,y,z)^{\mathrm{T}}$ of the vectors $(x_\pm, y_\pm, z_\pm)^{\mathrm{T}}$ satisfies, of course, some linear restriction of the form

$$y = ax + bz, \tag{1.1}$$

where a and b are easily expressed via $x_\pm$, $y_\pm$ and $z_\pm$.

1.1 One-particle states

Consider now the whole kagome lattice, which will be assumed infinite in all directions, unless the contrary is stated explicitly. For it, the one-particle space is the direct sum of one-particle spaces over all its vertices multiplied by vacuums in other places. To indicate a one-particle vector φ means to attach a number — amplitude φ_A — to each vertex A of the kagome lattice. In order to ensure that the vector never comes out of the one-particle space when we apply to it the transfer matrix any number of times, we must take it to be an *eigenvector* of the transfer matrix. We do not fix here the exact definition of transfer matrix (see Subsection 1.3 for three variants of it), but note that we must properly take into account the fact that the transfer matrix turns inside out half of the triangles of the kagome lattice, thus moving the lines.

So, let us write down the conditions for a vector in the one-particle space to be an eigenvector. Consider the following picture (Fig. 2) representing a fragment of the kagome lattice. Here the triangle DCE is going to be turned inside out, while the triangle BCA has been obtained by turning inside out a triangle on the previous step. So, the two conditions arise:

$$\varphi_C = a\varphi_D + b\varphi_E \tag{1.2}$$

and

$$\varphi_C = a\varphi_B + b\varphi_A \tag{1.3}$$

(compare (1.1)).

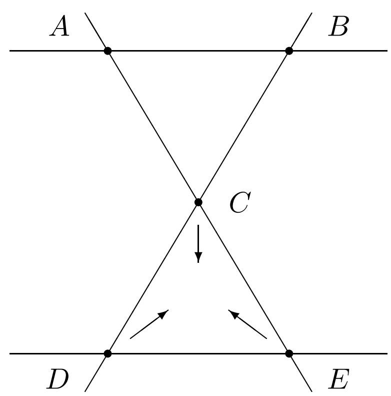


Figure 2:

When the triangle DCE is turned inside out, it yields a triangle $D'C'E'$ (Fig. 3), and the new "field" variables are expressed through the old ones as

$$\begin{pmatrix} \varphi_{D'} \\ \varphi_{E'} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \varphi_D \\ \varphi_E \end{pmatrix}, \tag{1.4}$$

where it can be derived from the above that

$$\alpha = -\delta, \qquad \alpha^2 + \beta\gamma = 1. \tag{1.5}$$

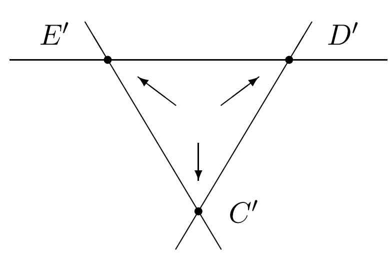


Figure 3:

The points E' and D' of the "new" lattice are analogs of the points A and B correspondingly belonging to the "old" lattice. Thus, in order to obtain on the new lattice a vector proportional to the vector on the old lattice, we must require that

$$\frac{\varphi_{E'}}{\varphi_{D'}} = \frac{\varphi_A}{\varphi_B}. \tag{1.6}$$

If this condition holds, one can extend both the old vector φ and the new "primed" vector periodically onto the whole lattice and in such way that the new one will be proportional to the (shifted) old one.

The condition (1.6) together with (1.2), (1.3), (1.4) is enough to obtain φ_A and φ_B (as well as φ_C, $\varphi_{D'}$ and $\varphi_{E'}$) out of given φ_D and φ_E. Thus, only one essential free parameter, e.g. φ_D/φ_E, remains for our construction of one-particle eigenvectors.

1.2 Two-particle states

How can the superposition of two one-particle states of Subsection 1.1 look like? The experience of studying the 2+1-dimensional *classical* integrable models hints that probably the "scattering" of two particles on one another must be trivial, i.e. it makes sense to assume for the "amplitude of the event that two particles are in two different points F and G of the lattice" the form

$$\Phi_{FG} = \varphi_F \psi_G + \varphi_G \psi_F, \tag{1.7}$$

where $\varphi_{\dots}$ and $\psi_{\dots}$ are one-particle amplitudes like those constructed in Subsection 1.1.

To see how Φ_{FG} transforms under the action of transfer matrix, let us decompose φ_F and ψ_F, considered as functions of F, in sums over triangles of the type DCE in Fig. 2, i.e. represent φ_F and ψ_F as sums of summands each of which equals zero if F lies beyond the corresponding triangle. In this way, Φ_{FG} naturally decomposes in a sum over (non-ordered) pairs of such triangles, including pairs of two coinciding triangles. We want that Φ_{FG} be transformed by the transfer matrix in an expression of the same form (1.7), with $\varphi_{\dots}$ and $\psi_{\dots}$ changed to their images with respect to this action.

It is easy to see that this holds automatically if F and G belong to *different* triangles. So, it remains to consider the case where F and G belong to the same triangle, say triangle DCE in Fig. 2. When this triangle is transformed by the transfer matrix in the triangle $D'C'E'$ of Fig. 3, the one-particle amplitudes are transformed according to (1.4):

$$\begin{pmatrix} \varphi_{D'} \\ \varphi_{E'} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \varphi_D \\ \varphi_E \end{pmatrix}, \qquad \begin{pmatrix} \psi_{D'} \\ \psi_{E'} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \psi_D \\ \psi_E \end{pmatrix}, \tag{1.8}$$

where one must add the conditions of type (1.2):

$$\begin{aligned} \varphi_C &= a\varphi_D + b\varphi_E, \qquad & \varphi_{C'} &= a\varphi_{D'} + b\varphi_{E'}, \\ \psi_C &= a\psi_D + b\psi_E, \qquad & \psi_{C'} &= a\psi_{D'} + b\psi_{E'}. \end{aligned} \tag{1.9}$$

On the other hand, the S-matrix (0.3)–(0.6) acts trivially, i.e. as a unity matrix, in the 2-particle sector. Thus, it must be

$$\Phi_{C'D'} = \Phi_{CD}, \qquad \Phi_{C'E'} = \Phi_{CE}, \qquad \Phi_{D'E'} = \Phi_{DE}. \tag{1.10}$$

Together, the formulae (1.9), (1.10) and (1.7) lead to the following conditions on the one-particle amplitudes $\varphi_{\dots}$ and $\psi_{\dots}$:

$$\varphi_{E'}\psi_{D'} + \varphi_{D'}\psi_{E'} = \varphi_E\psi_D + \varphi_D\psi_E, \tag{1.11}$$

$$\varphi_{D'}\psi_{D'} = \varphi_D\psi_D, \tag{1.12}$$

$$\varphi_{E'}\psi_{E'} = \varphi_E\psi_E. \tag{1.13}$$

The three conditions (1.11)–(1.13) together with (1.8) and (1.5) give, remarkably, just *one* condition

$$-\gamma\varphi_D\psi_D + \alpha(\varphi_D\psi_E + \varphi_E\psi_D) + \beta\varphi_E\psi_E = 0 \tag{1.14}$$

on $\varphi_{\dots}$ and $\psi_{\dots}$. Recall that, according to Subsection 1.1, each of the vectors φ and ψ is parametrized by one parameter (besides a trivial scalar factor). Together they are parametrized by two parameters, but condition (1.14) subtracts one parameter. Thus, the two-particle eigenstates constructed in this subsection depend on one significant parameter.

1.3 Dispersion relations

The constructed eigenvectors of transfer matrix T are of course also eigenvectors for translation operators through periods of kagome lattice. Let us consider here relations between the corresponding eigenvalues for the one-particle eigenstate.

It will be convenient to deform slightly the kagome lattice and imagine it as consisting of horizontal, oblique and vertical lines. Consider once again some triangle ABC of the kagome lattice, and its image $A'B'C'$ under the action of S-matrix-hedgehog, as in Fig. 4. Let us write out some relations of type (1.4), namely

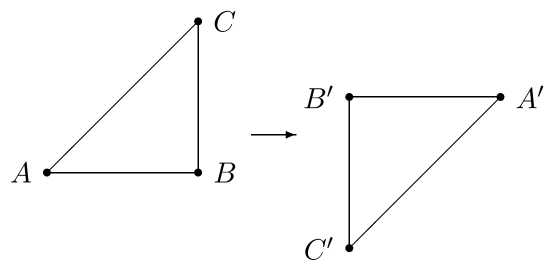


Figure 4:

$$\begin{pmatrix} \varphi_{A'} \\ \varphi_{B'} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \varphi_A \\ \varphi_B \end{pmatrix}, \tag{1.15}$$

$$\begin{pmatrix} \varphi_{B'} \\ \varphi_{C'} \end{pmatrix} = \begin{pmatrix} \tilde a & \tilde b \\ \tilde c & \tilde d \end{pmatrix} \begin{pmatrix} \varphi_B \\ \varphi_C \end{pmatrix}, \tag{1.16}$$

where $\varphi_{\dots}$ is any one-particle vector, and the numbers $a, \dots, \tilde d$ (now playing the role of Greek letters in (1.4)) satisfy conditions of type (1.5), i.e.

$$a = -d, \qquad ad - bc = -1,$$
$$\tilde a = -\tilde d, \qquad \tilde a\tilde d - \tilde b\tilde c = -1.$$

From (1.15) follows

$$\frac{\varphi_B}{\varphi_{B'}} = \frac{-a(\varphi_A/\varphi_{A'})+1}{(\varphi_A/\varphi_{A'})-a}, \tag{1.17}$$

and from (1.16) follows

$$\frac{\varphi_C}{\varphi_{C'}} = \frac{-\tilde{a}(\varphi_B/\varphi_{B'})+1}{(\varphi_B/\varphi_{B'})-\tilde{a}}.$$

Surely, the numbers a and $\tilde{a}$ depend on an S-operator-hedgehog. On the other hand, this latter is parameterized by exactly two parameters. So, in this Subsection we will take a and $\tilde{a}$ as those parameters.

We can take for eigenvalue of the hedgehog transfer matrix T either $\varphi_{A'}/\varphi_A$, or $\varphi_{B'}/\varphi_B$, or $\varphi_{C'}/\varphi_C$. These variants correspond, strictly speaking, to different definitions of T, but each of them is consistent with the requirement that the degrees of T must be represented graphically as "oblique layers" of cubic lattice (the difference being that, with the three different definitions, the action of transfer matrix T corresponds to the shifts through cubic lattice periods along three different axes). Our goal is to express the eigenvalues of translation operators acting *within* the kagome lattice for a given one-particle state through, say, $\varphi_{A'}/\varphi_A$.

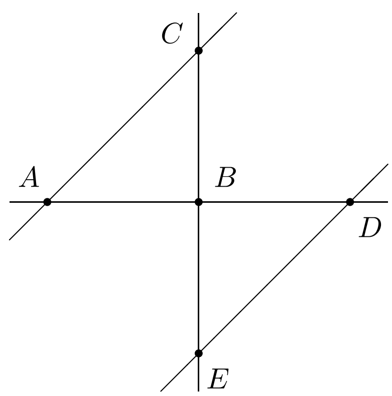


Figure 5:

If we speak about translation through one lattice period *to the right* in the sense of Figs. 4 and 5, then this eigenvalue is φ_D/φ_A. It is clear that

$$\frac{\varphi_D}{\varphi_B} = \frac{\varphi_{A'}}{\varphi_{B'}}$$

— the ratios of values $\varphi_{\dots}$ in the triangle DBE are the same as in $A'B'C'$. Thus,

$$\frac{\varphi_D}{\varphi_A} = \frac{\varphi_{A'}}{\varphi_{B'}}\frac{\varphi_B}{\varphi_A} = \frac{\varphi_{A'}}{\varphi_A}\cdot\frac{-a(\varphi_A/\varphi_{A'})+1}{(\varphi_A/\varphi_{A'})-a} \tag{1.18}$$

(we have used (1.17). A similar relation can be written out for the translation through one lattice period in *upward* direction in the sense of Figs. 4 and 5, namely

$$\frac{\varphi_C}{\varphi_E} = \frac{\varphi_{B'}}{\varphi_B}\cdot\frac{-\tilde{a}(\varphi_B/\varphi_{B'})+1}{(\varphi_B/\varphi_{B'})-\tilde{a}}, \tag{1.19}$$

where one has to substitute the expression (1.17) for $\varphi_B/\varphi_{B'}$.

2 Algebraization: a creation operator

In this section we will provide a more "algebraic" construction of one-particle states, resembling the $1+1$-dimensional algebraic Bethe ansatz.

Let us give some definitions and remarks. We will depict the operators graphically in such a way that each operator will have some number of "incoming edges" and the same number of "outgoing edges" (or "links"). To each edge corresponds its own copy of a two-dimensional complex linear space, and to several edges of the same (incoming or outgoing) kind together corresponds the tensor product of their spaces. Each of the mentioned two-dimensional spaces has a basis of a *0-particle* and *1-particle* vectors.

For any, maybe infinite, collection of edges, we will define this *collection's 0-particle vector*, or *vacuum*, as the tensor product of 0-particle vectors throughout the collection (in this paper, the meaning of infinite tensor products will be always clear). Further, we will identify a 1-particle vector in an edge with its tensor product with the 0-particle vectors in all the collection's other edges and define a *collection's 1-particle vector* as a formal sum over all its edges of the corresponding 1-particle vectors, with any complex coefficients. Then, we can define in an obvious way the 2-particle, 3-particle etc. states.

So, according to the above, we assume in this Section that an operator acts from the tensor product of "incoming" spaces to the tensor product of "outgoing", i.e. different, spaces. Still, sometimes we will assume that all the edges along one straight line represent *the same* two-dimensional space. This is convenient e.g. when we write out the tetrahedron equation, as in formula (0.7), and this will never lead to confusion.

We are going to present a one-parameter family of "creation operators". When applied to the "vacuum", these operators produce one-particle states — plane waves, described in Section 1. As we will see, the very construction of these operators presupposes that they act on vectors which don't differ much from the "vacuum". We will not try to make this statement more exact here. Instead, in this Section it will be enough for us that the domain of definition of those operators contains the one-dimensional space generated by vacuum.

2.1 Description of transfer matrices from which the creation operators are constructed

Creation operators will be transfer matrices on a kagome lattice with some special boundary conditions. Graphically, such a transfer matrix is depicted in Fig. 6. As we are considering the eigenstates of transfer matrix T made up of "hedgehogs", as in Section 1, the kagome transfer matrix must be such that it should be possible to bring the hedgehogs through it using the tetrahedron equation.

In the tetrahedron equation (0.7) a number 0, 1, 2, or 3 is attached to a *plane*, that is, to a face of the tetrahedron. An operator $S_{ij,ik,jk}$ acts in the tensor product of three linear spaces corresponding to the *lines* — intersections of those planes.

Let us assume that parameters $\varphi_1, \varphi_2, \varphi_3$ belong to the "hedgehogs" $S_{12,13,23}$ and are given. Then we will build the kagome transfer matrix out of matrices $S_{01,02,12}$, $S_{01,03,13}$ and $S_{02,03,23}$ in such way that with the help of (0.7) one could pass a hedgehog through the kagome lattice. Here the number 0 (and the corresponding parameter φ_0) is attached to the plane of the kagome lattice.

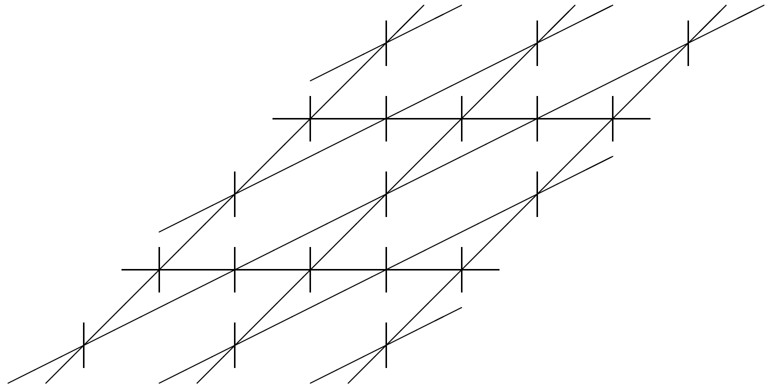


Figure 6:

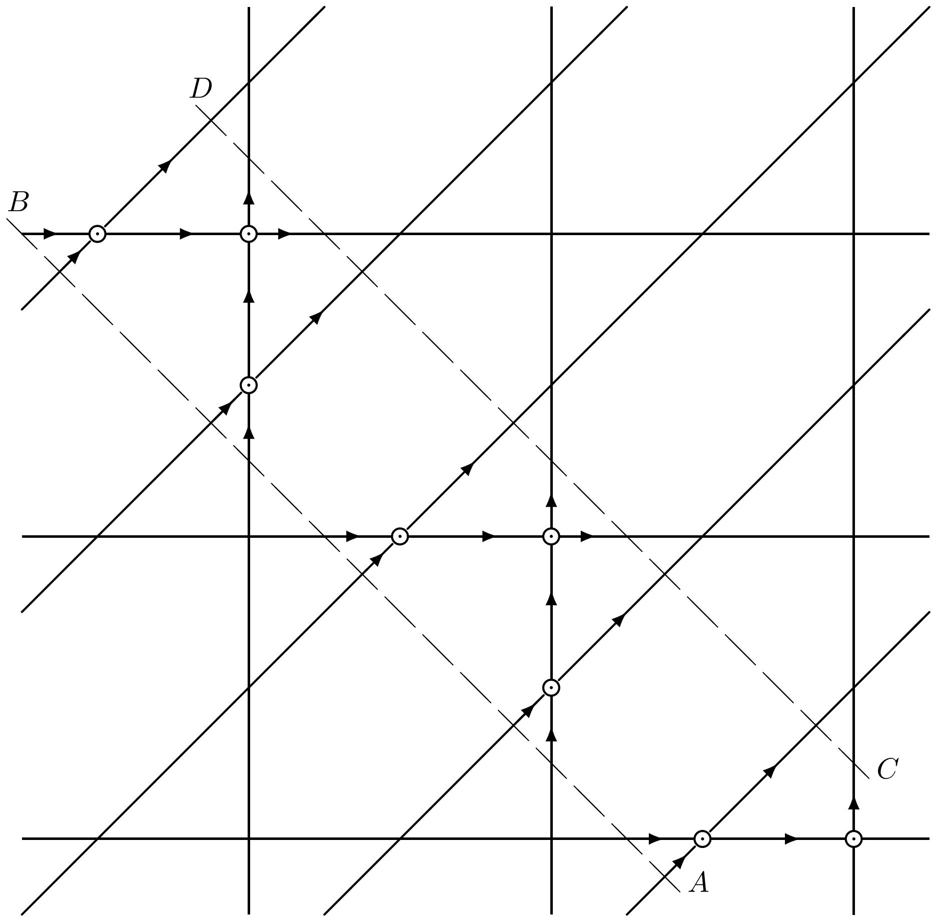


Figure 7:

2.2 Boundary conditions for creation operators

It will take some effort to describe the boundary conditions that we are going to impose on kagome transfer matrices of Subsection 2.1 to obtain out of them creation operators. The problem is that we are considering a kagome lattice *infinite* in all plane directions.

So, first, let us draw in Fig. 7 the lattice viewed from above. Then let us draw a dashed line AB and cut off for a while the part of the lattice lying to the left of that line (it will be explained in Subsection 2.3 that really there is much arbitrariness in choosing the line AB, but let it be for now as in Fig. 7). For the rest of transfer matrix, let us define the boundary condition along AB as follows. Consider all the lattice edges intersecting AB. They are *incoming edges* for the remaining part of transfer matrix. To define the boundary conditions, we must indicate some vector Σ_{AB} in the tensor product of corresponding spaces. Let us assume that Σ_{AB} is a *1-particle vector* as defined in the beginning of this Section, whose exact form is to be determined.

Consider the band — the part of transfer matrix lying between the lines AB and CD. This band represents an operator acting from the space corresponding to its incoming edges into the space corresponding to its outgoing edges, where the incoming edges are those intersecting AB and also those pointing from behind the kagome lattice plane into the vertices situated within the band, while the outgoing edges are those intersecting CD and those pointing from the vertices situated within the band at the reader. The latter vertices are marked $\odot$ in Fig. 7.

Let us require that our band operator — let us call it $\mathcal{B}$ — transform the tensor product $\Sigma_{AB} \otimes \Omega_\odot$, where $\Omega_\odot$ is the vacuum for the set of edges pointing into the "$\odot$" vertices, into the following sum:

$$\mathcal{B}\,\Sigma_{AB} \otimes \Omega_\odot = \kappa\,\Sigma_{CD} \otimes \Omega'_\odot + \Omega_{CD} \otimes \Psi_\odot, \tag{2.1}$$

where κ is a number; Σ_{CD} is the vector similar to Σ_{AB}, but corresponding to the edges situated one lattice period to the right, i.e. intersecting CD; $\Omega'_\odot$ is the vacuum for edges pointing from the "$\odot$" vertices to the reader (who can thus identify $\Omega'_\odot$ with $\Omega_\odot$ if desired); Ω_{CD} is the vacuum for the set of vectors intersecting CD; $\Psi_\odot$ is some vector lying in the same tensor product of spaces as $\Omega'_\odot$. It is remarkable that, for any φ_0, relation (2.1) can be satisfied with a proper choice of Σ_{AB}. The vector $\Psi_\odot$ will then be a 1-particle vector. Some details of calculations concerning relation (2.1) are explained in Subsection 2.3, while here we are going to *use* this relation.

The next band, lying to the right of CD, has vector $\kappa\,\Sigma_{CD}$ as its incoming vector. Thus, the whole situation is repeated up to the factor κ. On the other hand, we could have cut the lattice, instead of the line AB, along some other line lying, e.g., n lattice periods to the left. In that case, we should have taken for the incoming vector the vector Σ_{AB} shifted by n periods to the left and multiplied by κ^{-n}. Letting $n \to \infty$, we get a $\Psi_\odot$-like vector in *every* band of the sort depicted in Fig. 7. Summing up all those vectors, we get a 1-particle state of the same kind as in Section 1. Those states are now parameterized by the parameter φ_0.

In fact, one more boundary condition must be imposed at the "right infinity" of the lattice. This is explained in the end of Subsection 2.3.

2.3 Some technical details

Instead of the straight line AB in Fig. 7, we could use any (connected) curve l *intersecting each straight line of the kagome lattice exactly one time* in such way that a boundary condition is given in the tensor product corresponding to the edges that intersect l. Assuming that a 1-particle vector is given as the boundary condition, we will require that after any deformation of l such that it passes through *one* of the lattice vertices, the boundary condition remain to be 1-particle.

In other words, the mentioned vertex is added to or withdrawn from the considered part of the lattice. Let that vertex be, e.g., such as in Fig. 8. An incoming 1-particle vector for it is described by two amplitudes a and b, with a corresponding to the edge 01 (see Subsection 2.1) and b — to the edge 02. It is required that the result of transforming this incoming vector by the matrix S (0.3)–(0.6) contain no three-particle part. This leads at once to the condition

$$\frac{a}{b} = -\sqrt{\tanh(\varphi_0 - \varphi_1)}\,\sqrt{\tanh(\varphi_0 - \varphi_2)}. \tag{2.2}$$

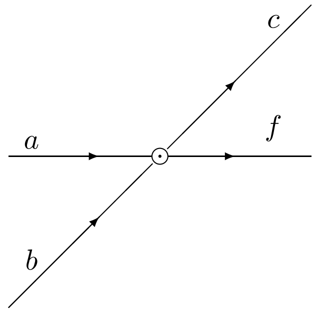
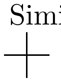
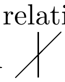


Figure 8:

The other ratios of the amplitudes written out in Fig. 8 are simply matrix elements of S:

$$\frac{c}{a} = \frac{f}{b} = \sqrt{\tanh(\varphi_0 - \varphi_1)}\,\sqrt{\coth(\varphi_0 - \varphi_2)}. \tag{2.3}$$

Similar relations can be written for the kagome lattice vertices of two other kinds, that is  and . It turns out that all those relations together determine the amplitudes at *all* lattice edges from a given one of them without contradiction. Thus, the amplitudes for the $\odot$-edges, that are incoming for the hedgehog transfer matrix, are determined correctly.

Those amplitudes give exactly its eigenstate for any fixed φ_0. This can be proved by a rather obvious reasoning: use the tetrahedron equation and the possibility to express the amplitudes at different edges through one another. The details are left for the reader.

There remains, however, another detail that is important: we must impose one more boundary condition, that is at the "right infinity" (see again Fig. 7). In order to obtain the 1-particle vector at the $\odot$-edges, and no *vacuum* component, let us take a straight line $C'D'$ — like CD, but somewhere far to the right — and require that the vector in the space corresponding to edges that intersect $C'D'$ have no 1-particle component. Then, of course, we let $C'D'$ tend to the right infinity, so this procedure does not change the 1-particle component of the $\odot$-vector, but the vacuum component vanishes.

3 Tetrahedral Zamolodchikov algebras and string-like states

In this section we show how to construct some string-like eigenstates using tetrahedral Zamolodchikov algebras. We will be using the trigonometrical tetrahedral Zamolodchikov algebras described in [4]. The idea is that sometimes we can control the evolution under the action of transfer matrix powers for the states arising from a (kagome) lattice of five-legged 'R-operators', with given boundary conditions.

Let us return to Fig. 1. Suppose we have fixed some boundary conditions in the tensor product of spaces denoted 1, 2 and 3 (which means, most generally, that we have taken the trace of a product of each side of (0.2) and some linear operator acting in the mentioned tensor product). This yields, in the l.h.s. of (0.2), some vector in the tensor product of spaces corresponding to indices a, b and c, and in the r.h.s. of (0.2) — the result of S-operator action upon a similar vector. For different boundary conditions, this provides enough (consistent) relations for S-operator to be determined uniquely.

We will look at this, however, from another point of view, using boundary conditions for R's as a means to define vectors in the space where S's act. Of course, we will take, instead of just three R-operators, a large lattice made up of them, whose fragment is depicted in Fig. 9, and apply a layer of S-operators — a hedgehog transfer matrix — to it. We will see that sometimes, for simple boundary conditions, this can be a reasonable way of describing vectors on which the transfer matrix acts, as well as results of such action.

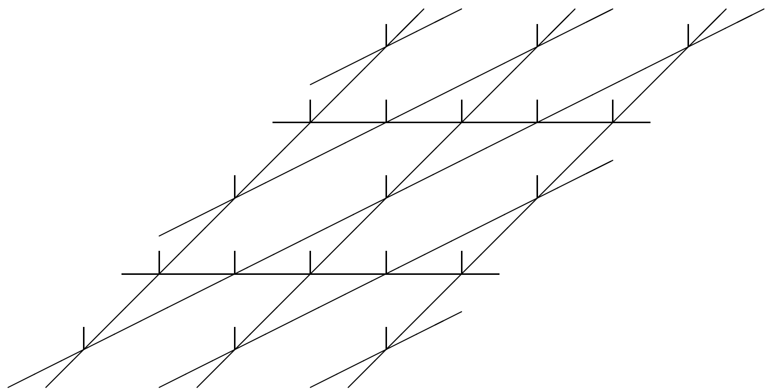


Figure 9:

As already stated, the R-operators we will be dealing with in this section are the simplest possible — trigonometrical — ones. In this connection, let us refer to Theorem 2.3 of the work [6] wherefrom it follows that to an S-matrix corresponds a *two-parameter* family of triples of R-operators. If we restrict ourselves to only trigonometrical R-matrices from [4], then there remains a *one-parameter* family of those. So, below it is implied that we are constructing one-parameter families of states for a given transfer matrix.

3.1 Two kinds of strings in a finite lattice

3.1.1 Eigenstates with eigenvalue 1 yielded by the lattice with a given "polarization"

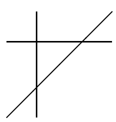
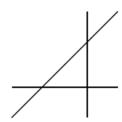
For a finite lattice on a torus, the "periodic" boundary conditions seem, at first sight, to be already fixed. However, trigonometrical R-operators of work [4] conserve the "number of particles", sometimes called also "polarization" (because they are very much like the usual 6-vertex model L-operators), and this provides more possibilities. Namely, the reader can easily verify that the following construction yields some states that are transformed into themselves by the hedgehog transfer matrix.

Let us declare some edges of the kagome lattice (Fig. 9) 'black' and the others 'white' in such a way that the number of black lines is conserved at each vertex (the incoming edges being situated below and to the left of the vertex, and the outgoing edges — above and to the right). It can be said that such a configuration of black edges — we will call it *permitted* configuration — forms a cycle belonging to some homology class of the torus. Let us say that vectors $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ correspond to white edges, and vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ — to black edges. This selects some matrix element for each R-operator, but as there are really two operators R^0 and R^1, this selects a pair of numbers forming a vector in the two-dimensional space corresponding to a *vertical* edge in Fig. 9. The tensor product of such vectors lies in the space where the transfer matrix acts. Now let us take a sum of those vectors over all black edges configurations belonging to the same homological class. It is quite straightforward to see that the transfer matrix transforms this sum into exactly the same sum.

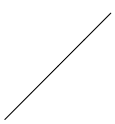
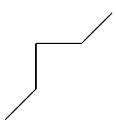
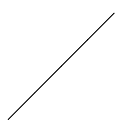
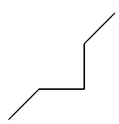
3.1.2 How moving strings arise from the lattice of *R*-operators

A slight modification of the above construction yields moving straight strings. Namely, fix arbitrarily some straight lines of the kagome lattice of Fig. 9 and paint black all the edges belonging to them. Then those lines will move under the action of transfer matrix, just because they are the intersection lines of the planes of cubic lattice and the moving plane orthogonal to vector $(1, 1, 1)$.

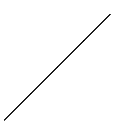
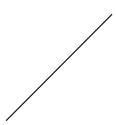
This is quite evident for horizontal and vertical lines, and the only slightly nontrivial consideration is required for *oblique* lines. To make this clear, let us draw some more pictures. First, let us interchange l.h.s. and r.h.s. in Fig. 1 and redraw it like the following formula:

$S \sum$ configurations of  $= \sum$ configurations of  ,

where "configurations" means "vectors corresponding to permitted configurations of three black edges within a triangle with given 'boundary condition' for six external edges". In these terms, Fig. 1 itself says only that

$S\Big($  $+$  $\Big) =$  $+$ 

(here only the black edges are depicted), and not that

S  $=$  . (3.1)

However, (3.1) is proved by direct calculation involving the explicit expressions for matrix elements of R-operators.

For horizontal and vertical lines, relations similar to (3.1) arise, of course, at once. Straight strings are further discussed in Subsection 3.2.

3.2 Another approach to straight strings

3.2.1 Straight strings from vacuum vectors

The matrix S, according to the work [6], has two families of vacuum vectors. Here we will restrict ourselves to considering the first family, i.e. the vacuum vectors transformed by S into themselves:

$$S(X(\zeta) \otimes Y(\zeta) \otimes Z(\zeta)) = X(\zeta) \otimes Y(\zeta) \otimes Z(\zeta), \tag{3.2}$$

ζ being a parameter taking values in an elliptic curve (compare with formula (1.12) from [6]. Strictly speaking, in [6] we were considering vacuum *covectors*, but this does not make much difference). What we are going to do in this Subsubsection can be done with the same success for the second family as well. Let us note that the particle-like excitations of Sections 1 and 2 have been built using *both* families of vacuum vectors.

The simplest eigenvectors $\Omega(\zeta)$ of the transfer matrix, with the eigenvalue 1, are built as follows: fix ζ and put in correspondence to each point of type A (Fig. 4) of the kagome lattice the vector $X(\zeta)$, to each point of type B — the vector $Y(\zeta)$, and of type C — the vector $Z(\zeta)$. Then take the tensor product of all those vectors. The formula (3.2) shows at once that this is indeed an eigenvector with eigenvalue 1.

A little bit more intricate eigenvectors, for which the eigenvalues in case of a *finite* lattice are roots of unity, can be constructed as follows. It is seen from formulae (2.13)–(2.15) of paper [6], where enter the values x, y and z — ratios of two coordinates of vectors X, Y and Z respectively, — that the triple X, Y, Z will remain vacuum if one makes one of the following changes:

$$(x, y, z) \to (x,\, 1/y,\, 1/z), \tag{3.3}$$

$$(x, y, z) \to (1/x,\, y,\, -1/z), \tag{3.4}$$

$$(x, y, z) \to (-1/x,\, -1/y,\, z). \tag{3.5}$$

It can be said that the changes (3.3), (3.4) and (3.5) affect respectively the sides BC, AC and AB of the triangle ABC in Fig. 4. Obviously, two such changes, if applied successively, commute with one another.

To construct a vector whose transformation under the action of transfer matrix is easy to trace, let us act like this: first, select arbitrarily some straight lines — *strings* — going along the edges of the kagome lattice. Then, take the vector $\Omega(\zeta)$ and change it as follows: make in each triangle of the type DCE the transformation(s) of type (3.3)–(3.5) if its corresponding side lies in a selected line.

The obtained vector — let us call it Θ — goes under the action of transfer matrix T into a vector of a similar form, but with the properly shifted lines (the latter, let us recall, result from the intersection of the cubic lattice faces with a plane perpendicular to the vector $(1, 1, 1)$, and move in that plane when the plane itself moves). An eigenvector of T can be now built in the form

$$\cdots + \omega^{-1}T^{-1}\Theta + \Theta + \omega T\Theta + \omega^2 T^2 \Theta + \cdots, \tag{3.6}$$

where in the case of a finite lattice the sum must be finite, and the number ω must be a root of unity of a proper degree, determined by the sizes of the lattice.

3.2.2 Straight strings as symmetries

Let us introduce the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and the unity matrix

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that the subscripts of these matrices have other meaning than the subscripts of S-matrices in equations like (0.7).

It follows from the explicit form of our S-matrix that it commutes with operators

$$\sigma_2 \otimes \sigma_2 \otimes \sigma_0, \qquad \sigma_1 \otimes \sigma_0 \otimes \sigma_2 \qquad \text{and} \qquad \sigma_0 \otimes \sigma_1 \otimes \sigma_1. \tag{3.7}$$

If now we select some set of the kagome lattice horizontal lines (to be exact, of those depicted in Fig. 7 as horizontal) and consider the tensor product of matrices σ_2 over all vertices belonging to those lines, then the hedgehog transfer matrix T will be permutable with that product up to the fact that the lines move in the lattice plane. This permutability follows immediately from the fact that S commutes with the first of operators (3.7). Similarly, it is not difficult to formulate the analogous statements for sets of oblique and vertical lines, using respectively the second and third of products (3.7).

Using the described symmetries of transfer matrix T, we can, starting from any state vector Θ_0 whose evolution under the action of degrees of T we can describe in this or that way, build many new states Θ whose evolution we will also be able to describe.

3.3 'Broken' infinite strings

Let us return to the kagome lattice made up of five-legged R-operators. In this subsection this lattice will be infinite. As in Subsection 3.1, we will paint black some edges. Namely, they will form two horizontal rays, one going to the right and one going to (or rather coming from) the left, as in Fig. 10, and those rays will be connected by some path going along the lattice edges and *permitted* in the sense of Subsubsection 3.1.1. Consider the vector — the formal infinite tensor product — corresponding to such black edge configuration, and take a sum over *all* the permitted paths linking the two rays (in particular, the ends A and B of the rays can move anywhere to the left and/or to the right).

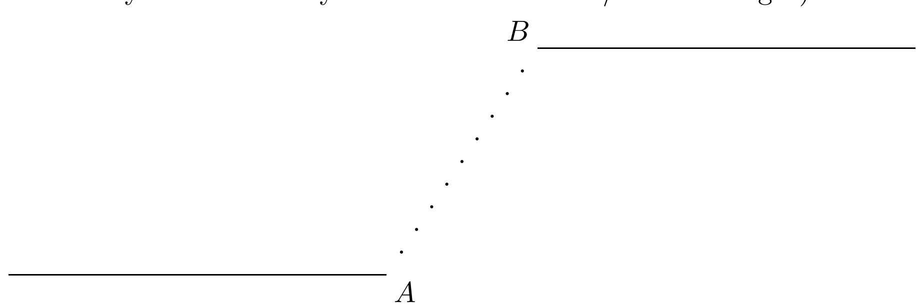


Figure 10:

Thus, the simplest 'broken' strings on the infinite lattice appear. As in Subsubsection 3.2.1, formal eigenvectors can be built out of translations of such strings.

4 String — particle marriage

In Section 3, we have been using a kagome lattice made up of five-legged 'R-operators' for string construction. The idea of this Section is that probably we will be able to construct more sophisticated strings using a kagome lattice made up of six-legged 'S-operators' instead. Indeed, it turns out that this is a right way to combine ideas of previous sections.

Here we will be considering the kagome lattice infinite in all plane directions. Let us imagine it again as in Fig. 6: situated in a horizontal plane, having an S-operator in each its vertex and using four of six S-operator legs as lattice edges. Two other legs of an S-operators are vertical. The S-operators form in such way some *auxiliary transfer matrix*. We will apply some vector — namely, a "one-particle" vector from Section 1 — to lower ends of S-operators, while a string will be posed, in a sense, within the plane, as in Section 3. The result will be new eigenstates for the "hedgehog" transfer matrix.

As is known, our transfer matrix (0.3)–(0.6) is made up of S-operators that can be described as the "static limit" of a more general construction due to Sergeev, Mangazeev and Stroganov (SMS), see paper [13] (operators are called there 'R' instead of 'S'). On the other hand, the auxiliary transfer matrix will be made of some *other* special case of SMS operators described in Subsections 4.1 and 4.2 below.

Then in Subsection 4.3 we show that a particle in absence of a string just vanishes, while their "marriage" produces in Subsection 4.4 new nontrivial strings.

4.1 Tetrahedron equation: a special case

As is known from the paper [13], the solution of the tetrahedron equation is parameterized by dihedral angles $\vartheta_1, \dots, \vartheta_6$ (of which five are independent) between four planes in the three-dimensional euclidean space. The parameterized equation looks like

$$\begin{aligned} &S_{123}(\vartheta_1,\vartheta_2,\vartheta_3)\,S_{145}(\vartheta_1,\vartheta_4,\vartheta_5)\,S_{246}(\pi-\vartheta_2,\vartheta_4,\vartheta_6)\,S_{356}(\vartheta_3,\pi-\vartheta_5,\vartheta_6) \\ &\quad = S_{356}(\vartheta_3,\pi-\vartheta_5,\vartheta_6)\,S_{246}(\pi-\vartheta_2,\vartheta_4,\vartheta_6)\,S_{145}(\vartheta_1,\vartheta_4,\vartheta_5)\,S_{123}(\vartheta_1,\vartheta_2,\vartheta_3), \end{aligned} \tag{4.1}$$

where the subscripts of each S number the spaces where it acts.

Let us imagine that those four planes pass through the center of a unit sphere. Then an alternative way of parameterizing, say, the operator S_{123} is by using the edges a_1, a_2 and a_3 of the spherical triangle instead of ϑ_1, ϑ_2 and ϑ_3. Each vertex of the triangle is, naturally, the intersection point of the sphere and a pair of planes belonging to the operator S_{123}.

We will assume that the operator S_{123} in the equation (4.1) is one of the hedgehogs of the "hedgehog transfer matrix", while the other S-operators belong to the auxiliary kagome transfer matrix. Recall that we are considering the hedgehogs corresponding to the *static limit* in terms of [13]. This means that the sides a_1, a_2 and a_3 of the spherical triangle are infinitely small. Let this infinitely small spherical triangle be situated at the north pole of the sphere.

Now let us fix a particular place for the fourth plane (that doesn't pass through the north pole). Namely, let this plane *intersect the sphere through its equator*. The resulting restrictions on the angles ϑ are the following:

$$\vartheta_1+\vartheta_2+\vartheta_3=\pi, \qquad \vartheta_4=\vartheta_5=\vartheta_6=\frac{\pi}{2}. \tag{4.2}$$

We will see that (4.2) yields degenerate operators S_{145}, S_{246} and S_{356}, but this only helps to perform our construction.

4.2 The gauge for "doubly rectangular" S-operators

The explicit form for operators entering in equation (4.1) can be found in section 5 of the work [13]. In the same work, quite a bit is said about different gauges for those operators. We will work with the following gauge for our "doubly rectangular" — i.e. having two of three angles ϑ equal to $\pi/2$ — operators: first take them as in section 5 of [13], and then change as described in the following paragraph.

Consider e.g. the operator $S_{145} = S_{145}(\vartheta_1, \vartheta_4, \vartheta_5) = S_{145}(\vartheta_1, \pi/2, \pi/2)$. Its legs corresponding to the spaces number 4 and 5 lie on the edges of the kagome lattice, in the horizontal plane (see the beginning of this Section). Let us perform a conjugation in both those spaces with the matrix $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, that is, let us change

$$S_{145} \to \mathbf{1} \otimes \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \cdot S_{145} \cdot \mathbf{1} \otimes \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1} \otimes \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}.$$

Let us perform the similar transformations for S_{246} in the spaces 4 and 6, and for S_{356} in the spaces 5 and 6. Note that such gauges are consistent with the tetrahedron equation (4.1), and that the gauge of S_{123} does not change.

Now let us forget about the old gauge of the operators S_{145}, S_{246} and S_{356}, and use these notations for the operators in the *new* gauge. The explicit form of each of these operators is like this:

$$S_{\dots} = \frac{1 - \tan(\vartheta/4)}{\cos(\vartheta/2)} \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where

$$A = \begin{pmatrix} \cos(\vartheta/2) & 0 & 0 & 0 \\ 0 & \sin(\vartheta/2) & 1 & 0 \\ 0 & 1 & \sin(\vartheta/2) & 0 \\ 0 & 0 & 0 & \cos(\vartheta/2) \end{pmatrix}, \tag{4.3}$$

$$B = \frac{\sqrt{\sin\vartheta}}{2} \begin{pmatrix} 1+i & 0 & 0 & 0 \\ 0 & 1-i & 0 & 0 \\ 0 & 0 & -1+i & 0 \\ 0 & 0 & 0 & -1-i \end{pmatrix}, \tag{4.4}$$

$$C = \frac{\sqrt{\sin\vartheta}}{2} \begin{pmatrix} 1-i & 0 & 0 & 0 \\ 0 & 1+i & 0 & 0 \\ 0 & 0 & -1-i & 0 \\ 0 & 0 & 0 & -1+i \end{pmatrix}, \tag{4.5}$$

$$D = \begin{pmatrix} \sin(\vartheta/2) & 0 & 0 & -1 \\ 0 & \cos(\vartheta/2) & 0 & 0 \\ 0 & 0 & \cos(\vartheta/2) & 0 \\ -1 & 0 & 0 & \sin(\vartheta/2) \end{pmatrix}, \tag{4.6}$$

where

$$\vartheta = \vartheta_1 \quad \text{for} \quad S_{145},$$
$$\vartheta = \vartheta_2 \quad \text{for} \quad S_{246},$$
$$\vartheta = \vartheta_3 \quad \text{for} \quad S_{356}.$$

Matrix A is the matrix of weights for all configurations of horizontal spins of an S-operator, if the spins at the vertical edges are both fixed and equal 0. Similarly, matrix B corresponds to the lower spin 1 and the upper spin 0; matrix C — to the lower spin 0 and the upper spin 1; and matrix D — to the lower spin 1 and the upper spin 1.

We see that all of these matrices except D *conserve the total spin*, or, in the other terminology, the "number of particles", within the horizontal plane.

4.3 Disappearance of particle in absence of a string

Now let us apply the auxiliary kagome transfer matrix made up of operators S_{145}, S_{246} and S_{356} to the one-particle vector from Sections 1 and 2. We will choose the following boundary conditions at the horizontal infinity: all spins at the "far enough" horizontal edges are zero.

Let us show that in this case the spins at *all* horizontal edges are zero. Recall that the one-particle state is a linear combination of tensor products of spins, of which *one* equals unity, and the others are zero. This means that matrix D — the only one that can change the total spin — cannot appear twice. In other words, non-zero spin cannot be created somewhere and then annihilated somewhere else, thus all the horizontal edges possess zero spins.

This conclusion implies that only *upper left* entries of the matrices $A, \ldots, D$ are playing rôle. For a given S-operator, four such elements form a matrix

$$\frac{1 - \tan(\vartheta/4)}{\cos(\vartheta/2)} \begin{pmatrix} \cos(\vartheta/2) & \frac{1+i}{2}\sqrt{\sin\vartheta} \\ \frac{1-i}{2}\sqrt{\sin\vartheta} & \sin(\vartheta/2) \end{pmatrix}, \tag{4.7}$$

that transforms a two-dimensional vector at the lower vertical edge to the one at the upper vertical edge, and each S-operator of the kagome lattice does this independently.

The remarkable property of the matrix (4.7) is its *degeneracy*. Thus, the whole lattice transfer matrix applied to the one-particle state yields an *infinite sum of vectors proportional to a fixed vector*. This sum must be inevitably equal to zero, due to the infiniteness of lattice in all directions, the fact that a one-particle vector just acquires some scalar factors under lattice translations, and a logical assumption

$$\sum_{n=-\infty}^{\infty} a^n = 0$$

for the sum of a geometrical progression infinite in both directions.

4.4 The marriage: an example of a string

The matrix (4.7) is not the only degenerate one. Formulae (4.3)–(4.6) show really that if we fix *any* values (0 or 1) for the four horizontal spins of an S-operator, then the 2×2-matrix corresponding to the two remaining vertical edges is degenerate. Imagine now that we have fixed the spins at all horizontal edges of *all operators*. Then edges with the spin 1 form "strings", and a remarkable conclusion is that a configuration of those strings determines the "outgoing" vector at the upper vertical edges up to a scalar factor.

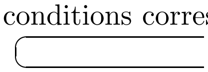
Remark. However, the "incoming" vector at the lower edges can cause this scalar factor to equal 0 for some string configurations. For example, for a one-particle incoming vector, there can be no closed strings, because they require at least two matrices D to be involved, as explained in Subsection 4.3.

We can propose the following example of a string-like state resulting from the "marriage" of a one-particle state and a string within the horizontal plane. The one-particle state is applied to the lower edges, as already explained. It is supposed that the string is "born" in the vertex where the particle is applied, due to the matrix D that can change by 2 the total "horizontal" spin. The form of the string at infinity can be fixed by two given half-lines, each going along the edges of the lattice in one of the east, north, or north-east directions.

5 Two cases of string superposition

Let us recall that we have introduced in Section 1 some "one-particle" eigenstates for the model based upon solutions of the tetrahedron equation. In the same Section, we have also constructed some "two-particle" states. However, some special condition arised in this construction, and the superposition of two *arbitrary* one-particle states was not achieved. Even the "creation operators" of Section 2 did not give a clear answer concerning multi-particle states.

On the other hand, in Section 4 we have brought in correspondence to a one-particle state some new state that could be called "one open string". It was done using some special "kagome transfer matrix". Here we will show that the superposition of such one-string states is easier to construct, because of degeneracy of kagome transfer matrix: it turns into zero the "obstacles" that hampered constructing of multi-particle states.

The scheme of string — particle "marriage" in Section 4 was as follows: take a one-particle state from Sections 1 and 2, and apply to it a kagome transfer matrix with boundary conditions corresponding to the presence of two string tails in the infinity, e.g. like this:  .

In this Section, we are going to complicate this scheme in the following way: the boundary conditions will correspond to the presence of an even number of string tails at the infinity, and instead of a one-particle state, we will use some special multi-particle vector Ψ. Its peculiarity will be in the fact that Ψ is *no longer an eigenstate* of the hedgehog transfer matrix T defined in Section 1. Instead, it will obey the condition

$$T\Psi = \lambda\Psi + \Psi', \tag{5.1}$$

where $\lambda = \text{const}$, and Ψ' is annulated by the kagome transfer matrix of Section 4, which we will here denote K.

Recall that we have defined T in such a way that its degrees could be described geometrically as "oblique slices" of the cubic lattice. The transfer matrix T can be passed through the transfer matrix K:

$$TK = KT, \tag{5.2}$$

the boundary conditions (such as the number and form of tails at the infinity) for K being intact. Define vector Φ as

$$\Phi = K\Psi.$$

This together with (5.1) and (5.2) gives

$$T\Phi = \lambda\Phi, \tag{5.3}$$

exactly as needed for an eigenvector.

5.1 Eigenvectors of the "several open strings" type for infinite lattice

Let there be n one-particle amplitudes $\varphi^{(1)}_{\dots}, \dots, \varphi^{(n)}_{\dots}$ of the same type as those described in Section 1. Let us compose an "n-particle vector" Ψ, i.e. put in correspondence to each unordered n-tuple of vertices $A^{(1)}, \dots, A^{(n)}$ of the kagome lattice some amplitude, in the following symmetrized way:

$$\psi_{A^{(1)},\dots,A^{(n)}} = \sum_s \varphi^{(1)}_{A^{s(1)}} \cdots \varphi^{(n)}_{A^{s(n)}}, \tag{5.4}$$

where s runs through the group of all permutations of the set $\{1, \dots, n\}$.

As for the boundary conditions for the transfer matrix K, let us assume that there are exactly $2n$ string tails, and they all go in positive directions, that is lie asymptotically in the first quadrant. Thus, in each of the points $A^{(1)}, \dots, A^{(n)}$ a string is created, and they are not annihilated.

The vector (5.4) is not an eigenvector of transfer matrix T due to problems arising when two or more points $A^{(k)}$ get close to one another. Nevertheless, the vector $\Phi = K\Psi$ *is* an eigenvector, because for it those problems disappear due to the simple fact: *creation of two or more strings within one triangle of the kagome lattice is geometrically forbidden.*

5.2 Eigenvectors of the "closed string" type for infinite lattice

In this Subsection, we will put in correspondence to each unordered pair of vertices of the infinite kagome lattice an "amplitude" Ψ_{AB} according to the following rules. If one of the vertices, say A, *precedes* the other one, say B, in the sense that they can be linked by a path — a broken line — going along lattice edges in positive directions, namely northward, eastward, or to the north-east, then let us put

$$\Psi_{AB} = \varphi_A \psi_B - \psi_A \varphi_B, \tag{5.5}$$

where $\varphi_{\dots}$ and $\psi_{\dots}$ are two one-particle amplitudes. In the case if vertices A and B cannot be joined by a path of such kind, let us put

$$\Psi_{AB} = 0.$$

The values Ψ_{AB} are components of the vector Ψ that belong to the two-particle subspace of tensor product of two-dimensional spaces situated in all kagome lattice vertices. What prevents Ψ_{AB} from being an eigenvector of the hedgehog transfer matrix is discrepancies arising near those pairs A, B that lie at the "border" between such pairs where one of the vertices precedes the other (so to speak, "the interval AB is timelike"), and such pairs where it does not ("the interval AB is spacelike").

Those discrepancies, however, disappear for the vector $\Phi = K\Psi$, where K is the kagome transfer matrix with boundary conditions reading *no string tails at the infinity*. This is because if a string cannot, geometrically, be created at the point A (or B) and then annihilated at the point B (or A), then the amplitude Ψ_{AB} doesn't influence at all the vector Φ. The only thing that remains to be checked for (5.3) to hold is a situation where A and B are in the same kagome lattice triangle that will be turned inside out by one of the hedgehogs of transfer matrix T, as in Fig. 11. Acting in the same manner as in Subsection 1.2, write

$$\begin{pmatrix} \varphi_{A'} \\ \varphi_{B'} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \varphi_A \\ \varphi_B \end{pmatrix}, \qquad \begin{pmatrix} \psi_{A'} \\ \psi_{B'} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}, \tag{5.6}$$

where

$$\alpha = -\delta, \qquad \alpha\delta - \beta\gamma = -1. \tag{5.7}$$

It follows from the formulas (5.6) and (5.7) that

$$\varphi_A \psi_B - \varphi_B \psi_A = \varphi_{B'} \psi_{A'} - \varphi_{A'} \psi_{B'},$$

i.e.

$$\Psi_{AB} = \Psi_{B'A'},$$

exactly what was needed to comply with the fact that an S-operator-hedgehog acts as a unity operator in the two-particle subspace.

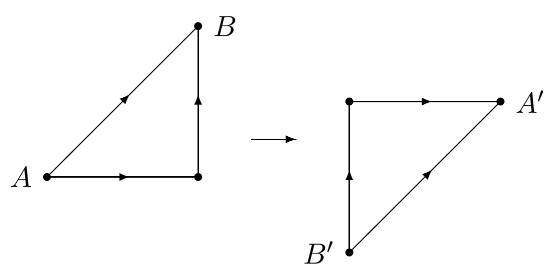


Figure 11:

6 Discussion

In this paper, we were not trying to discuss the "physical" consequences of the eigenvectors constructed. Its only modest aim was to show that some ideas of the classical Bethe ansatz could be relevant for $2+1$-dimensional models and that, on the other hand, there arise new structures, inherent for multidimensions, namely "string-like" eigenvectors.

The S-operator on which our model is based was discovered by the author in 1989 [3]. Later on, a similar but different model was discovered by J. Hietarinta [12], and then it was shown by S.M. Sergeev, V.V. Mangazeev and Yu.G. Stroganov [13] that both those models are particular cases of one model, parallel in some sense to the Zamolodchikov model. This allows one to hope that the rich mathematical structures already discovered in connection with our model will be extended some time onto the Zamolodchikov model as well.

As we were only able to construct one- and two-particle Bethe-like vectors Section 1, it was quite natural to try to do more with the help of "algebraization" of their construction in Section 2. However, here again too big complications arise when trying to actually construct more eigenvectors. In particular, even the superposition of two *arbitrary* one-particle states from Section 1 has not been obtained yet.

It is clear that the superposition of two arbitrary one-particle eigenstates from Section 1 cannot lie just in the 2-particle space as it is defined in Section 1. So, probably, some new terminology should be introduced to distinguish between the 2-*particle* space and 2-*excitation* states.

The Sections 1, 2 and 3 together show that

- even the model corresponding to the simplest solutions of tetrahedron equation possesses a large variety of eigenstates which are probably not easy to classify,
- eigenvalues seem sometimes rather trivial — maybe, it is due to relation $S^2 = \mathbf{1}$, see [6] — but probably they will be more interesting for the models from [12, 13],
- some states can be introduced that are countable sums of formal tensor products throughout the infinite lattice, but it is unclear what to do for a finite lattice,
- there probably does not exist — at least, it was not discovered in papers [3, 6, 12, 13] — a complete analog of the 6-vertex model in $1+1$ dimensions with its "conservation of particle number", but something can be built even upon the fact that that number is conserved "sometimes",
- there exists a huge amount of symmetries multiplying eigenvalues by constants (roots of unity for a finite lattice) and unknown for the $1+1$-dimensional models, and
- eigenstates can be constructed with making no use of *invertibility* of tetrahedron equation solutions — so probably it makes sense to search for non-invertible ones.

In Sections 4 and 5 we provide what seems to be the most promising way of constructing new eigenstates. Those are, in essence, strings from Section 3 fertilized by nontrivial momentum particles of Sections 1 and 2. Such strings are not just invariant under the shifts along themselves, but acquire some nontrivial multipliers.

It is clear that really a quite immense zoo of such states can be constructed.

The main point is, however, that the string — particle "marriage" makes possible a simple and clear construction of at least some multi-string states. At least some obstacles at which we have run into when trying to construct multi-*particle* states miraculously disappear when we add a string to each particle.

The mentioned states have been constructed for the infinite kagome lattice. We have to recognize that constructing such states on a finite lattice remains an open problem.

It is also unclear how to combine the results of Subsections 5.1 and 5.2, i.e. construct such states with string "creation" and "annihilation" where the total number of "creating" and "annihilating" particles would be more than two. Note that in Subsection 5.1 we use the symmetrized product of one-particle amplitudes, while in Subsection 5.2 — the antisymmetrized one.

Concerning the dispersion relations of Subsection 1.3, let us remark that perhaps there are too many of them. It is probably caused by the fact that, for now, we managed to construct not all one-particle and/or one-string states.

On the other hand, it is clear that the dispersion relations of type (1.18)–(1.19) survive also for a string "created by a particle". As for the multi-string states, all of the eigenvalues are obtained for them as products of corresponding eigenvalues for each string.

Finally, it is certainly interesting to find eigenstates for the model based on other simple solutions to the tetrahedron equation [12], and perhaps for the general model described in [13].

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