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Degenerate Poisson Pencils on Curves: New Separability Theory

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Abstract

A review of a new separability theory based on degenerated Poisson pencils and the so-called separation curves is presented. This theory can be considered as an alternative to the Sklyanin theory based on Lax representations and the so-called spectral curves.

1 Introduction

The separation of variables belongs to the basic methods of mathematical physics from the previous century. Originating from the early works of D'Alambert (the $18th$ century), Fourier and Jacobi (the first half of the $19th$ century), for many decades it has been the only known method of exact solution of dynamical systems. Let us briefly recall the idea from classical mechanics. Consider a Hamiltonian mechanical system of $2n$ degrees of freedom and integrable in the sense of Liouville/Arnold theorem. This means that there exist n linearly independent functions h_i(Hamiltonians) which are in involution with respect to the canonical Poisson bracket

$$\{h_j, h_k\} = 0, \qquad j, k = 1, ..., n. \tag{1.1}$$

A system of canonical variables $(\mu_i, \lambda_i)_{i=1}^n$

$$\{\lambda_i, \lambda_j\} = \{\mu_i, \mu_j\} = 0, \quad \{\lambda_i, \mu_j\} = \delta_{ij} \tag{1.2}$$

is called separated [1] if there exist n relations of the form

$$\varphi_i(\lambda_i, \mu_i, h_1, ..., h_n) = 0, \quad i = 1, ..., n \tag{1.3}$$

joining each pair (μ_i, λ_i) of conjugate coordinates and all Hamiltonians h_k, $k = 1, ..., n$. Fixing the values of Hamiltonians $h_j = const_j = a_j$ one obtains from (1.3) an explicit factorization of the Liouville tori given by the equations

$$\varphi_i(\lambda_i, \mu_i, a_1, ..., a_n) = 0, \quad i = 1, ..., n. \tag{1.4}$$

In the Hamilton-Jacobi method, for a given Hamiltonian function $h_r(\mu,\lambda)$ we are looking for a canonical transformation $(\mu,\lambda)\to(a,b)$ in the form $b_i=\frac{\partial W}{\partial a_i}, \mu_i=\frac{\partial W}{\partial \lambda_i}$, where $W(\lambda,a)$ is a generating function given by the related Hamilton-Jacobi equation

$$h_r(\lambda,\frac{\partial W}{\partial \lambda})=a_r. \tag{1.5}$$

If (μ,λ) are separated coordinates, then

$$W(\lambda,a)=\sum_{i=1}^{n} W_i(\lambda_i,a) \tag{1.6}$$

and the partial differential equation (1.5) splits into n ordinary differential equations

$$\varphi_i(\lambda_i,\frac{\partial W}{\partial \lambda_i},a_1,...,a_n)=0, \quad i=1,...,n \tag{1.7}$$

just of the form (1.4). In (a,b) coordinates the flow is trivial

$$(a_j)_{t_r}=0, \quad (b_j)_{t_r}=\delta_{jr} \tag{1.8}$$

and the implicit form of the trajectories $\lambda_i(t_r)$ is the following

$$b_j(\lambda,a)=\frac{\partial W}{\partial a_j}=\delta_{jr}t_r+const, \quad j=1,...,n. \tag{1.9}$$

So, given a set of separated variables, it is possible to solve a related dynamical system by quadratures. In the 19*th* century and most of the present century, for a number of models of classical mechanics the separated variables were either guessed or found by some *ad hoc* methods. For example, in the second half of the 19*th* century, Neumann's investigation of a particle moving on a sphere under the action of a linear force [2] and Jacobi's study of the geodesics motion on an ellipsoid [3] exploited the separability of the Hamilton-Jacobi equation to solve the equations of motion by quadratures. In 1891 Stäckel initiated the program dealing the classification of Hamiltonian systems according to their separability or nonseparability, presenting conditions for separability of the Hamilton-Jacobi equation in orthogonal coordinates [4]. For three dimensional flat space, the 11 possible coordinate systems in which separation may take place were deduced in a paper by Eisenhart [5]. They were all obtained as degenerations of the confocal ellipsoidal coordinates [6]. For each of the coordinates Eisenhart [7] determined the form of the potential that permitted a separation of variables. These potentials, designated Stäckel or separable potentials, played a crucial role in Hamiltonian mechanics before the development of more qualitative geometric methods for differential equations. Although in all classical papers the transformation to separated coordinates was searched in the form of point transformations, nevertheless they can be produced by an arbitrary canonical transformation involving both coordinates and momenta.

A fundamental progress in the theory of separability was made in 1985, when Sklyanin adopted the method of soliton systems, i.e. the Lax representation, to systematic derivation of separated variables (see his review article [1]). It was the first constructive theory of separated coordinates for dynamical systems. In his approach, the appropriate Hamiltonians appear as coefficients of the spectral curve, i.e. the characteristic equation of Lax matrix. His method was successfully applied to separation of variables for many old and new integrable systems [8]-[12].

In this paper we present a new constructive separability theory, which has been recently intensely developed, based on a bi-Hamiltonian property of integrable systems. In last decade a considerable progress has been made in construction of new integrable finite dimensional dynamical systems showing bi-Hamiltonian property. The majority of them originate from stationary flows, restricted flows or nonlinearization of Lax equations of underlying soliton systems [13]-[25]. Quite recently a fundamental property of such systems has been discovered, i.e. their separability. It was proved [26]-[31] that most bi-Hamiltonian finite dimensional chains, which start with a Casimir of the first Poisson structure and terminate with a Casimir of the second Poisson structure, are integrable by quadratures, through the solutions of the appropriate Hamilton-Jacobi equation.

The presented review article is based on results of papers [26]-[31] systematizing and unifying them into a compact separability theory in the frame of the set of canonical coordinates. The results derived so far are sufficiently promising to consider the theory as an alternative or complement to the Sklyanin ones.

2 Preliminaries

Let us re-examine some facts about bi-Hamiltonian systems. We recall some definitions. Let M be a differentiable manifold, TM and T^*M its tangent and cotangent bundle. At any point $u \in M$, the tangent and cotangent spaces are denoted by T_uM and T_u^*M, respectively. The pairing between them is given by the map $< \cdot, \cdot >: T_u^*M \times T_uM \to R$. For each smooth function $F \in C^\infty(M)$, dF denotes the differential of F. M is said to be a Poisson manifold if it is endowed with a Poisson bracket $\{\cdot,\cdot\} : C^\infty(M)\times C^\infty(M) \to C^\infty(M)$, in general degenerate. The related Poisson tensor π is defined by $\{F, G\}\pi(u) :=< dG, \pi \circ dF > (u) =< dG(u), \pi(u)dF(u) >$. So, at each point u, $\pi(u)$ is a linear map $\pi(u) : T_u^*M \to T_uM$ which is skew-symmetric and fulfils the Jacobi identity. Any function $c \in C^\infty(M)$, such that $dc \in \ker \pi$, is called a Casimir of π. Let π_0,$\pi_1 : T^*M \to TM$ be two Poisson tensors on M. A vector field K is said to be a bi-Hamiltonian with respect to π_0 and π_1 if there exist two smooth functions H,$F \in C^\infty(M)$ such that

$$K = \pi_0 \circ dH = \pi_1 \circ dF. \tag{2.1}$$

Poisson tensors π_0 and π_1 are said to be compatible if the associated pencil $\pi\lambda = \pi_1 - \lambda\pi_0$ is itself a Poisson tensor for any $\lambda \in R$. Moreover, if π_0 is invertible, the tensor $N = \pi_1 \circ \pi_0^{-1}$, called a recursion operator, is a Nijenhuis (hereditary) tensor of such a property that when it acts on a given bi-Hamiltonian vector field K, it produces another bi-Hamiltonian vector field being a symmetry generator of K. Hence, having the invariant Nijenhuis tensor, one can construct a hierarchy of Hamiltonian symmetries and related hierarchy of constants of motion for an underlying system, so important for its integrability.

Unfortunately, for majority of bi-Hamiltonian finite dimensional systems, both Poisson structures are degenerate, so one cannot construct the recursion Nijenhuis tensor inverting one of the Poisson structures. Nevertheless, due to the nonuniqueness of Hamiltonian functions, determined up to an appropriate Casimir function, it is always possible to construct a finite bi-Hamiltonian chain starting and terminating with Casimirs of π_0 and π_1, respectively.

3 One-Casimir chains

Let us consider a Poisson manifold M of $\dim M = 2n+1$ equipped with a linear Poisson pencil

$$\pi\lambda = \pi_1 - \lambda\pi_0 \tag{3.1}$$

of maximal rank, where π_0 and π_1 are compatible Poisson structures and λ is a continuous parameter. As was first shown by Gel'fand and Zakharevich [32], a Casimir of the pencil is a polynomial in λ of an order n

$$h\lambda = h_0\lambda^n + h_1\lambda^{n-1} + ... + h_n \tag{3.2}$$

and generates a bi-Hamiltonian chain

$$\pi\lambda \circ dh\lambda = 0 \Longleftrightarrow \begin{array}{l} \pi_0 \circ dh_0 = 0 \\ \pi_0 \circ dh_1 = K_1 = \pi_1 \circ dh_0 \\ \pi_0 \circ dh_2 = K_2 = \pi_1 \circ dh_1 \\ \qquad\vdots \\ \pi_0 \circ dh_n = K_n = \pi_1 \circ dh_{n-1} \\ \qquad\qquad\quad 0 = \pi_1 \circ dh_n, \end{array} \tag{3.3}$$

where $K \equiv K_1,\ H \equiv h_1$and $F \equiv h_0$. In the paper we restrict our considerations to a class of canonical coordinates (q,p,c), where $q = (q_1, ..., q_n)^T$, $p = (p_1, ..., p_n)^T$ are generalized coordinates and c is a Casimir coordinate. Hence, π_0 always stays a canonical Poisson matrix. This restriction simplifies the theory in the sense that it makes a Marsden-Ratiu projection procedure [33], [34] trivial.

3.1 Darboux-Nijenhuis representation

To understand the theory better, let us start from the end in some sense, i.e. from a system written in separated coordinates $(\mu_i, \lambda_i)_{i=1}^n$. An interesting observation is that such a system can be represented by n different points of some curve. We start from Gel'fand and Zakharevich case. Actually, let us consider a curve in (λ, μ) plane, in the particular form

$$f(\lambda,\mu) = h\lambda, \qquad h\lambda = c\lambda^n + h_1\lambda^{n-1} + ... + h_n, \tag{3.4}$$

where $f(\lambda, \mu)$ is an arbitrary smooth function. Then, let us take n different points (μ_i, λ_i) from the curve:

$$f(\lambda_i, \mu_i) = c\lambda_i^n + h_1\lambda_i^{n-1} + ... + h_n, \quad i = 1, ..., n, \tag{3.5}$$

which will define our separated coordinates as they are of the form (1.3). The explicit dependence of h_k on $(\mu_i, \lambda_i, c)_{i=1}^n$ is given by the solution of n linear equations (3.5), while

for fixed values of $h_k = a_k$ and $\mu_i = \frac{\partial W_i}{\partial \lambda_i}$ the system (3.5) allows us to solve the appropriate Hamilton-Jacobi equations.
Here, we have to mention that the idea to relate the multi-Hamiltonian property to an m-parameter family of curves comes from P. Vanhaecke [35].

In refs. [26]-[29] the bi-Hamiltonian chain was constructed for a *separation curve* in the form (3.4). Actually, the Hamiltonian functions h_k found from the system (3.5) take the following compact form

$$\begin{aligned} h_k(\lambda, \mu, c) &= \sum_{i=1}^{n} \rho_{k-1}^{i}(\lambda) \frac{f(\lambda_i, \mu_i)}{\Delta_i(\lambda)} + c\rho_k(\lambda), \quad k = 1, ..., n \\ h_0 &= c, \end{aligned} \tag{3.6}$$

where

$$\Delta_i(\lambda) := \prod_{j \neq i} (\lambda_i - \lambda_j), \tag{3.7}$$

$$\rho_k(\lambda) := (-1)^k \sum_{\substack{j_1, .., j_k \\ j_1 < .. < j_k}} \lambda_{j_1} \cdot ... \cdot \lambda_{j_k}, \qquad k = 1, ..., n, \tag{3.8}$$

are the so-called Viete polynomials (symmetric polynomials) and

$$\rho_{k-1}^{i}(\lambda) := \rho_{k-1}(\lambda_i = 0) = -\frac{\partial \rho_k(\lambda)}{\partial \lambda_i}. \tag{3.9}$$

For example, for $n = 2$ we have

$$\rho_1 = -\lambda_1 - \lambda_2, \quad \rho_2 = \lambda_1 \lambda_2 \tag{3.10}$$

and for $n = 3$

$$\rho_1 = -\lambda_1 - \lambda_2 - \lambda_3, \quad \rho_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3, \quad \rho_3 = -\lambda_1\lambda_2\lambda_3. \tag{3.11}$$

The bi-Hamiltonian chain (3.3) is constructed with respect to the following compatible Poisson matrix

$$\pi_0 = \begin{pmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \pi_1 = \begin{pmatrix} 0 & \Lambda & h_{1,\mu} \\ -\Lambda & 0 & -h_{1,\lambda} \\ -(h_{1,\mu})^T & (h_{1,\lambda})^T & 0 \end{pmatrix}, \tag{3.12}$$

where $\Lambda = diag(\lambda_1, ..., \lambda_n)$ and $h_{1,\mu} := \left(\frac{\partial h_1}{\partial \mu_1}, ..., \frac{\partial h_1}{\partial \mu_n}\right)^T$. Notice that the last column of π_1 is just the first vector field K_1. All Hamiltonians h_k are in involution with respect to both Poisson structures π_0 and π_1.

Applying the following important relations

$$\rho_r(\lambda) = -\sum_{i=1}^{n} \rho_{r-1}^{i} \frac{\lambda_i^n}{\Delta_i}, \tag{3.13}$$

$$-\sum_{i=1}^{n} \frac{\partial \rho_r}{\partial \lambda_i} \frac{\lambda_i^m}{\Delta_i} = \sum_{i=1}^{n} \frac{\rho_{r-1}^{i}(\lambda)\, \lambda_i^m}{\Delta_i} = \left\{ \begin{array}{ll} 1, & m = n - r \\ 0, & m \neq n - r \end{array} \right\}, r = 1, .., n \tag{3.14}$$

and the decomposition (1.6), the Hamilton-Jacobi equations

$$h_r(\lambda, \frac{\partial W}{\partial \lambda}, c) = a_r, \quad r = 1, ..., n \tag{3.15}$$

turn into the form

$$\sum_{k=1}^{n} \frac{\rho_{r-1}^{k}(\lambda)[f(\lambda_k, \partial W_k/\partial \lambda_k) - c\lambda_k^n]}{\Delta_k} = a_r, \;\; r = 1, ..., n \tag{3.16}$$

with the solution

$$f(\lambda_k, \partial W_k/\partial \lambda_k, c) = c\lambda_k^n + a_1\lambda_k^{n-1} + ... + a_n, \quad k = 1, ..., n. \tag{3.17}$$

Hence, $W(\lambda, a)$ can be obtained by solving n decoupled first-order ODEs (3.17) and the family of dynamical systems (3.3) can be solved by quadratures.

Now, let us pass to the projection of the Poisson pencil $\pi\lambda$ onto a symplectic leaf S of π_0 ($\dim S = 2n$) fixing the value of c. Generally, one has to apply the Marsden-Ratiu theorem which in this case is trivial, as obviously $\theta\lambda = \theta_1 - \lambda\theta_0$, where

$$\theta_0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad \theta_1 = \begin{pmatrix} 0 & \Lambda \\ -\Lambda & 0 \end{pmatrix}, \tag{3.18}$$

is a nondegenerate Poisson pencil on S. Hence, the related Nijenhuis tensor

$$N = \theta_1 \circ \theta_0^{-1} = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix} \tag{3.19}$$

is diagonal and this is the reason why we will refer below to the separated coordinates as to the Darboux-Nijenhuis (DN) coordinates. Notice that ρ_i (3.8) are coefficients of minimal polynomial of the Nijenhuis tensor

$$(\det(N - \lambda))^{1/2} = \lambda^n + \sum_{i=1}^{n} \rho_i \lambda^{n-i} = \prod_{i=1}^{n} (\lambda - \lambda_i). \tag{3.20}$$

On S the chain (3.3) turns into the form

$$\begin{aligned} \theta_0 \circ dh_1 &= \overline{K}_1 = -\frac{1}{\rho_n}\theta_1 \circ dh_n \\ \theta_0 \circ dh_2 &= \overline{K}_2 = -\frac{\rho_1}{\rho_n}\theta_1 \circ dh_n + \theta_1 \circ dh_1 \\ &\;\;\vdots \\ \theta_0 \circ dh_n &= \overline{K}_n = -\frac{\rho_{n-1}}{\rho_n}\theta_1 \circ dh_n + \theta_1 \circ dh_{n-1}. \end{aligned} \tag{3.21}$$

The last equation terminates the sequence of vector fields $\overline{K}_r$ in the hierarchy as for the next equation from the chain we have

$$\theta_0 \circ dh_{n+1} = \overline{K}_{n+1} = -\theta_1 \circ dh_n + \theta_1 \circ dh_n = 0. \tag{3.22}$$

Obviously N is not a recursion operator for the hierarchy (3.21). Because of the form of the first equation, the vector field $\overline{K}_1$ is called a quasi-bi-Hamiltonian [36]-[39] and the chain (3.21) could be treated alternatively as a starting point of the separability theory for the case of $c = 0$.

3.2 Arbitrary canonical representation

Now, let us consider an arbitrary canonical transformation

$$(q,p) \rightarrow (\lambda,\mu) \tag{3.23}$$

independent of a Casimir coordinate c (not necessarily a point transformation!). The advantage of staying inside such a class of transformations is that the clear structure of a pencil is preserved and the Marsden-Ratiu projection of the Poisson pencil is still trivial. Of course, the most general case of multi-Hamiltonian separability theory takes place when one goes beyond the set of canonical coordinates. i.e. when one tries to find DN coordinates starting from the pencil written in a non-canonical representation. But then the simple structure of degenerated Poisson pencil is lost and the nontrivial problem of the Marsden-Ratiu projection for such pencil appears (see for example ref. [41]).

Applying the transformation (3.23) to Hamiltonian functions (3.6) and Poisson matrices (3.12) one finds that

$$h_k(q,p,c) = h_k(q,p) + cb_k(q,p), \quad k = 1,...,n \tag{3.24}$$

and

$$\begin{aligned} &\pi_0 = \begin{pmatrix} \theta_0 & 0 \\ 0 & 0 \end{pmatrix}, \; \theta_0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \\ &\pi_1 = \begin{pmatrix} \theta_1 & \overline{K}_1 \\ -\overline{K}_1^T & 0 \end{pmatrix}, \; \theta_1 = \begin{pmatrix} D(q,p) & A(q,p) \\ -A^T(q,p) & B(q,p) \end{pmatrix}, \end{aligned} \tag{3.25}$$

where A, B and D are $n \times n$ matrices. The nondegenerate Poisson pencil $\theta\lambda$ on S gives rise to the related Nijenhuis tensor N and its adjoint N^* in (q,p) coordinates in the form

$$N = \theta_1 \circ \theta_0^{-1} = \begin{pmatrix} A & -D \\ B & A^T \end{pmatrix}, \; N^* = \theta_0^{-1} \circ \theta_1 = \begin{pmatrix} A^T & -B \\ D & A \end{pmatrix}. \tag{3.26}$$

Obviously, in a real situation we start from a given bi-Hamiltonian chain (3.24)-(3.26) in canonical coordinates (q,p,c), derived by some method (see for example [13]-[25]), and trying to find the DN coordinates which diagonalize the appropriate Nijenhuis tensor and are separated coordinates for the considered system. So now we pass to a systematic derivation of the inverse of transformation (3.23).

The important intermediate step of the construction of DN coordinates are the so called Hankel-Fröbenious (HF) non-canonical coordinates (u,v) [35], [40], related to the DN ones through the following transformation

$$\begin{aligned} &u_i = \rho_i(\lambda_1,...,\lambda_n), \\ &\mu_i = \sum_{k=1}^{n} v_k \lambda_i^{n-k}, \quad i = 1,...,n. \end{aligned} \tag{3.27}$$

In (u,v) coordinates one finds

$$\theta_0 = \begin{pmatrix} 0 & U \\ -U^T & 0 \end{pmatrix}, \; U = \begin{pmatrix} 0 & 0 & \cdots & 1 \\ 0 & \cdots & 1 & u_1 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & u_1 & \cdots & u_{n-1} \end{pmatrix},$$

$$N = \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}, \quad F = \begin{pmatrix} -u_1 & 1 & \cdots & \cdots & 0 \\ -u_2 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -u_{n-1} & 0 & \cdots & \cdots & 1 \\ -u_n & 0 & \cdots & \cdots & 0 \end{pmatrix},$$

$$\theta_1 = N \circ \theta_0 = \begin{pmatrix} 0 & FU \\ -FU^T & 0 \end{pmatrix}, \quad N^* = N^T.$$

Moreover the differentials du_i, dv_i satisfy the following recursion relations [40], [41]

$$\begin{array}{lcl} N^* \circ du_1 & = & du_2 - u_1 du_1, \\ N^* \circ du_2 & = & du_3 - u_2 du_1, \\ & \vdots & \\ N^* \circ du_{n-1} & = & du_n - u_{n-1} du_1, \\ N^* \circ du_n & = & -u_n du_1, \\ \\ N^* \circ dv_1 & = & dv_2 - u_1 dv_1, \\ N^* \circ dv_2 & = & dv_3 - u_2 dv_1, \\ & \vdots & \\ N^* \circ dv_{n-1} & = & dv_n - u_{n-1} dv_1, \\ N^* \circ dv_n & = & -u_n dv_1. \end{array} \tag{3.28}$$

Note that vector fields $\theta_0 \circ du_1$ and $\theta_0 \circ dv_1$ are quasi-bi-Hamiltonian.

Now we relate the canonical coordinates (q, p, c) to the DN separated coordinates (λ, μ, c). From the minimal polynomial of N (3.26) we get

$$u_k = \zeta_k(q, p), \quad k = 1, ..., n. \tag{3.29}$$

Conjugate coordinates $v_k = v_k(q, p), k = 2, ..., n$ are found from the recursion formula (3.28) while $v_1 = v_1(q, p)$ coordinate from relations

$$\{u_j, v_1\}_{\theta_k} = \delta_{j,n-k}, \qquad j = 1, ..., n, \quad k = 0, ...n-1. \tag{3.30}$$

Hence we get

$$v_k = \vartheta_k(q, p), \quad k = 1, ..., n. \tag{3.31}$$

Eliminating (u, v) coordinates from (3.27), (3.29) and (3.31) we derive the desired relations

$$\chi_i(q, p; \lambda, \mu) = 0, \quad i = 1, ..., 2n. \tag{3.32}$$

Now let us concentrate on a special but important case of point transformation between (q, p, c) and (λ, μ, c) variables. Then

$$\begin{array}{l} \theta_1 = \begin{pmatrix} 0 & A(q) \\ -A^T(q) & B(q,p) \end{pmatrix}, \\ N = \begin{pmatrix} A(q) & 0 \\ B(q,p) & A^T(q) \end{pmatrix}, \end{array} \tag{3.33}$$

where matrix elements of B are at most linear in p coordinates, so coefficients of minimal polynomial of N are equal to coefficients of the characteristic polynomial of A. Hence, we find the first part of the canonical transformation in the form

$$\rho_i(\lambda) = \eta_i(q), \ \ i = 1, ..., n \implies q_k = \psi_k(\lambda), \ \ k = 1, ..., n \,. \tag{3.34}$$

The complementary part of the transformation we get from the generating function

$$G(p, \lambda) = \sum_{i=1}^{n} p_i \psi_i(\lambda). \tag{3.35}$$

Then,

$$\mu_i = \frac{\partial G}{\partial \lambda_i}, \ i = 1, ..., n \implies p_k = \varphi_k(\lambda, \mu), \ \ k = 1, ..., n. \tag{3.36}$$

At the end of this subsection we introduce the notion of an inverse bi-Hamiltonian separable chains. In ref. [29] it was demonstrated that for each separable bi-Hamiltonian chain (3.3), (3.24), (3.25), (3.26) in canonical coordinates (q, p, c) there exists a related inverse bi-Hamiltonian separable chain

$$\begin{array}{lll}
\pi_0 \circ dh'_{n+1} = & 0 & \\
\pi_0 \circ dh'_n & = K'_n = & \pi_{-1} \circ dh'_{n+1} \\
 & \vdots & \\
\pi_0 \circ dh'_r & = K'_r = & \pi_{-1} \circ dh'_{r+1} \\
 & \vdots & \\
\pi_0 \circ dh'_1 & = K'_1 = & \pi_{-1} \circ dh'_2 \\
 & 0 = & \pi_{-1} \circ dh'_1
\end{array} \tag{3.37}$$

where $h'_{n+1} = h_0 = c$,

$$\pi_{-1} = \begin{pmatrix} \theta_{-1} & \overline{K}'_n \\ -\overline{K}'^T_n & 0 \end{pmatrix}, \ \ \theta_{-1} = \theta_0 \circ \theta_1^{-1} \circ \theta_0 = N^{-2} \circ \theta_1, \tag{3.38}$$

$\overline{K}'_n = \theta_0 \circ dh'_n$ and

$$h'_r(q, p, c) = h_r(q, p) + c b'_r(q), \quad b'_r(q) = \frac{b_{r-1}(q)}{b_n(q)}, \quad r = 1, ..., n. \tag{3.39}$$

Notice that in both chains the respective Hamiltonians (3.24), (3.39) and related vector fields differ only by the $c-$dependent parts. In the case of a point transformation to the DN coordinates, when θ_1 takes the form (3.33), we get

$$\theta_1^{-1} = \begin{pmatrix} \left(A^{-1}\right)^T \circ D \circ A^{-1} & -\left(A^{-1}\right)^T \\ A^{-1} & 0 \end{pmatrix} \tag{3.40}$$

and

$$\theta_0 \circ \theta_1^{-1} \circ \theta_0 = \begin{pmatrix} 0 & A^{-1} \\ -\left(A^{-1}\right)^T & -\left(A^{-1}\right)^T \circ D \circ A^{-1} \end{pmatrix}. \tag{3.41}$$

Notice that both chains (3.3) and (3.37) given in natural coordinates can be transformed to the Nijenhuis bi-Hamiltonian form (3.6)-(3.12), where the canonical transformation can be derived from the relation $\rho_r(\lambda) = b_r(q)$, $r = 1, ..., n$ in the first case and from the relations $\rho_r(\lambda) = \frac{b_{n-r}(q)}{b_n(q)}$, $r = 1, ..., n$ in the second case.

Consider the set of extended functions

$$h_r(q,p;c,c') = h_r(q,p) + cb_r(q) + c'b'(q). \tag{3.42}$$

They can be simultaneously put into the bi-Hamiltonian and the inverse bi-Hamiltonian hierarchies. In the first case c is treated as the Casimir variable and c' as the parameter and in the second case c is treated as the parameter and c' as the Casimir variable.

3.3 Examples

We shall illustrate the theory presented by a few representative examples. More examples can be found in refs. [26]-[29]. In all examples from this section canonical transformations between natural and separated coordinates are point like. An example of a nonpoint transformation will be given at the end of this paper.

Example 1. *The one-Casimir extension of the Henon-Heiles system.*
Let us consider the integrable case of the Henon-Heiles system generated by the Hamiltonian $H = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + q_1^3 + \frac{1}{2}q_1q_2^2$. Its one-Casimir extension reads [22]

$$(q_1)_{tt} = -3q_1^2 - \frac{1}{2}q_2^2 + c, \quad (q_2)_{tt} = -q_1q_2 \tag{3.43}$$

and belongs to the bi-Hamiltonian chain

$$\begin{aligned} \pi_0 \circ dh_0 &= 0 \\ \pi_0 \circ dh_1 &= K_1 = \pi_1 \circ dh_0 \\ \pi_0 \circ dh_2 &= K_2 = \pi_1 \circ dh_1 \\ 0 &= \pi_1 \circ dh_2, \end{aligned} \tag{3.44}$$

where

$$\begin{aligned} h_0 &= c, \\ h_1 &= h_1(q,p) + cb_1(q) = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + q_1^3 + \frac{1}{2}q_1q_2^2 - cq_1, \\ h_2 &= h_2(q,p) + cb_2(q) = \frac{1}{2}q_2p_1p_2 - \frac{1}{2}q_1p_2^2 + \frac{1}{16}q_2^4 + \frac{1}{4}q_1^2q_2^2 - \frac{1}{4}cq_2^2, \end{aligned} \tag{3.45}$$

$$\pi_1 = \begin{pmatrix} 0 & 0 & q_1 & \frac{1}{2}q_2 & p_1 \\ 0 & 0 & \frac{1}{2}q_2 & 0 & p_2 \\ -q_1 & -\frac{1}{2}q_2 & 0 & \frac{1}{2}p_2 & -h_{1,q_1} \\ -\frac{1}{2}q_2 & 0 & -\frac{1}{2}p_2 & 0 & -h_{1,q_2} \\ -p_1 & -p_2 & h_{1,q_1} & h_{1,q_2} & 0 \end{pmatrix}. \tag{3.46}$$

The construction of the related inverse bi-Hamiltonian chain, according to the results from the previous subsection, gives the following results:

$$\begin{aligned} \pi_0 \circ dh'_3 &= 0 \\ \pi_0 \circ dh'_2 &= K'_2 = \pi_{-1} \circ dh'_3 \\ \pi_0 \circ dh'_1 &= K'_1 = \pi_{-1} \circ dh'_2 \\ 0 &= \pi_{-1} \circ dh'_1, \end{aligned} \tag{3.47}$$

where

$$\begin{aligned} h'_1 &= h_1(q,p) - \frac{4}{q_2^2}c, \\ h'_2 &= h_2(q,p) + \frac{4q_1}{q_2^2}c, \\ h'_3 &= c, \end{aligned} \tag{3.48}$$

$$\pi_{-1} = \begin{pmatrix} 0 & 0 & 0 & 2/q_2 & \frac{1}{2}q_2p_2 \\ 0 & 0 & 2/q_2 & -4q_1/q_2^2 & \frac{1}{2}q_2p_1 - q_1p_2 \\ 0 & -2/q_2 & 0 & 2p_2/q_2^2 & -h'_{2,q_1} \\ -2/q_2 & 4q_1/q_2^2 & -2p_2/q_2^2 & 0 & -h'_{2,q_2} \\ -\frac{1}{2}q_2p_2 & -\frac{1}{2}q_2p_1 + q_1p_2 & h'_{2,q_1} & h'_{2,q_2} & 0 \end{pmatrix} \tag{3.49}$$

and Newton equations related with the natural Hamiltonian h'_1 are:

$$(q_1)_{tt} = -3q_1^2 - \frac{1}{2}q_2^2, \quad (q_2)_{tt} = -q_1q_2 - \left(\frac{2}{q_2}\right)^3 c, \tag{3.50}$$

being just the second well known one-Casimir extension [16] of the Henon-Heiles system considered. Both systems (3.43) and (3.50) can be transformed to the Nijenhuis chain (3.6)–(3.12) through the respective transformations

$$\begin{aligned} & q_1 = \lambda_1 + \lambda_2, \quad p_1 = \frac{\lambda_1\mu_1}{\lambda_1 - \lambda_2} + \frac{\lambda_2\mu_2}{\lambda_2 - \lambda_1}, \\ & q_2 = 2\sqrt{-\lambda_1\lambda_2}, \quad p_2 = \sqrt{-\lambda_1\lambda_2}\left(\frac{\mu_1}{\lambda_1 - \lambda_2} + \frac{\mu_2}{\lambda_2 - \lambda_1}\right), \\ & f(\lambda_i, \mu_i) = \frac{1}{2}\lambda_i\mu_i^2 + \lambda_i^4, \end{aligned} \tag{3.51}$$

and

$$\begin{aligned} & q_1 = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}, \quad p_1 = \lambda_1\lambda_2\left(\frac{\lambda_1\mu_1}{\lambda_1 - \lambda_2} + \frac{\lambda_2\mu_2}{\lambda_2 - \lambda_1}\right), \\ & q_2 = \frac{2}{\sqrt{-\lambda_1\lambda_2}}, \quad p_2 = -\sqrt{-\lambda_1\lambda_2}\left(\frac{\lambda_1^2\mu_1}{\lambda_1 - \lambda_2} + \frac{\lambda_2^2\mu_2}{\lambda_2 - \lambda_1}\right), \\ & f(\lambda_i, \mu_i) = \frac{1}{2}\lambda_i^4\mu_i^2 + \lambda_i^{-3}. \end{aligned} \tag{3.52}$$

Example 2. *One-Casimir extension of the Kepler problem in the plane.*
Let us consider the classical problem of a particle in the plane under the influence of the Kepler potential and an additional homogeneous field force. The Hamiltonian function reads

$$h_1(q,p,c)=\frac{1}{2}p_1^2+\frac{1}{2}p_2^2-\frac{a}{\sqrt{q_1^2+q_2^2}}-cq_2,\quad a=const. \tag{3.53}$$

There is a second independent integral of the motion

$$h_2(q,p,c)=-\frac{1}{2}q_2p_1^2+\frac{1}{2}q_1p_1p_2+\frac{1}{2}\frac{aq_2}{\sqrt{q_1^2+q_2^2}}-\frac{1}{4}cq_1^2, \tag{3.54}$$

which together with $h_0=c$ allows us to construct a bi-Hamiltonian chain (3.44) with the second Poisson structure in the form

$$\pi_1=\begin{pmatrix} 0 & 0 & 0 & \frac{1}{2}q_1 & p_1 \\ 0 & 0 & \frac{1}{2}q_1 & q_2 & p_2 \\ 0 & -\frac{1}{2}q_1 & 0 & -\frac{1}{2}p_1 & -h_{1,q_1} \\ -\frac{1}{2}q_1 & -q_2 & \frac{1}{2}p_1 & 0 & -h_{1,q_2} \\ -p_1 & -p_2 & h_{1,q_1} & h_{1,q_2} & 0 \end{pmatrix}. \tag{3.55}$$

The inverse bi-Hamiltonian chain (3.47) is given for functions

$$\begin{aligned} h_1'(q,p,c)&=\frac{1}{2}p_1^2+\frac{1}{2}p_2^2-\frac{a}{\sqrt{q_1^2+q_2^2}}-\frac{4}{q_1^2}c,\\ h_2'(q,p,c)&=-\frac{1}{2}q_2p_1^2+\frac{1}{2}q_1p_1p_2+\frac{1}{2}\frac{aq_2}{\sqrt{q_1^2+q_2^2}}+\frac{4q_2}{q_1^2}c,\\ h_3'&=c, \end{aligned} \tag{3.56}$$

and the second Poisson tensor in the form

$$\pi_{-1}=\begin{pmatrix} 0 & 0 & -4q_2/q_1^2 & 2/q_1 & \frac{1}{2}q_2p_1-q_1p_2 \\ 0 & 0 & 2/q_1 & 0 & \frac{1}{2}q_2p_2 \\ 4q_2/q_1^2 & -2/q_1 & 0 & -2p_1/q_1^2 & -h'_{2,q_1} \\ -2/q_1 & 0 & 2p_1/q_1^2 & 0 & -h'_{2,q_2} \\ -\frac{1}{2}q_2p_1+q_1p_2 & -\frac{1}{2}q_2p_2 & h'_{2,q_1} & h'_{2,q_2} & 0 \end{pmatrix}. \tag{3.57}$$

The transformations to DN coordinates for both chains are the following

$$\begin{aligned} q_1&=2\sqrt{\lambda_1\lambda_2},\ \ p_1=\sqrt{\lambda_1\lambda_2}\left(\frac{\mu_1}{\lambda_1-\lambda_2}+\frac{\mu_2}{\lambda_2-\lambda_1}\right),\\ q_2&=\lambda_1+\lambda_2,\ \ p_2=\frac{\lambda_1\mu_1}{\lambda_1-\lambda_2}+\frac{\lambda_2\mu_2}{\lambda_2-\lambda_1},\\ f(\lambda_i,\mu_i)&=\frac{1}{2}\lambda_i\mu_i^2+\frac{1}{2}a, \end{aligned} \tag{3.58}$$

and

$$\begin{aligned} q_1&=\frac{2}{\sqrt{-\lambda_1\lambda_2}},\ \ p_1=\sqrt{-\lambda_1\lambda_2}\left(\frac{\lambda_1^2\mu_1}{\lambda_1-\lambda_2}+\frac{\lambda_2^2\mu_2}{\lambda_2-\lambda_1}\right),\\ q_2&=\frac{\lambda_1+\lambda_2}{\lambda_1\lambda_2},\ \ p_2=\lambda_1\lambda_2\left(\frac{\lambda_1\mu_1}{\lambda_1-\lambda_2}+\frac{\lambda_2\mu_2}{\lambda_2-\lambda_1}\right),\\ f(\lambda_i,\mu_i)&=\frac{1}{2}\lambda_i^4\mu_i^2+\frac{1}{2}a\lambda_i. \end{aligned} \tag{3.59}$$

Example 3. *One-Casimir extension of elliptic separable potentials.*
In refs.[26],[28] it was proved that every natural Hamiltonian system

$$H=\frac{1}{2}\sum_{i=1}^{n}p_i^2+V(q)+\frac{1}{2}c(q,q), \tag{3.60}$$

where $(.,.)$ means the scalar product, which admits in extended phase space $M\ni(q,p,c)$ the bi-Hamiltonian formulation

$$\begin{pmatrix} q\\ p\\ c\end{pmatrix}_t=\pi_0\circ dh_1=\pi_1\circ dh_0, \tag{3.61}$$

where π_0 is a canonical Poisson structure,

$$\pi_1=\begin{pmatrix} 0 & A-\frac{1}{2}q\otimes q & h_{1,p}\\ -A+\frac{1}{2}q\otimes q & \frac{1}{2}p\otimes q-\frac{1}{2}q\otimes p & -h_{1,q}\\ -(h_{1,p})^T & (h_{1,q})^T & 0\end{pmatrix}, \tag{3.62}$$

$A=diag(\alpha_1,...,\alpha_N)$, α_i−different positive constants, $h_0=c$, $h_1=H+c\rho_1(\alpha)$, is separable in generalized elliptic coordinates. This bi-Hamiltonian formulation generates the chain (1.2) [25] of commuting bi-Hamiltonian vector fields, where

$$\begin{aligned} &h_r(q,p,c;\alpha)=h_r(q,p;\alpha)+cb_r(q;\alpha),\\ &b_r(q;\alpha)=\rho_r(\alpha)+\frac{1}{2}\sum_{k=1}^{r-1}\rho_k(\alpha)(q,A^{r-k-1}q),\ \ r=1,...,n-1,\\ &b_n(\alpha)=\rho_n(\alpha)[1-\frac{1}{2}(q,A^{-1}q)],\\ &h_r(q,p;\alpha)=\sum_{k=0}^{r}\rho_k(\alpha)\overline{h}_{r-k},\ \ \overline{h}_s=\frac{1}{2}\sum_{i=1}^{n}\alpha_i^{s-1}R_i,\\ &R_i=\sum_{i\neq j}\frac{q_ip_j-q_jp_i}{\alpha_i-\alpha_j}+p_i^2+V_i(q),\ \ \sum_{i=1}^{n}V_i(q)=V(q),\end{aligned} \tag{3.63}$$

$\{H,R_i\}_{\pi_0}=0,\ i=1,...,n$ and $\rho_r(\alpha)$ are Viete polynomials of α.

On the other hand, according to our procedure, the inverse bi-Hamiltonian chain (3.47) for one-Casimir extension of potentials separable in elliptic coordinates reads

$$h_r'(q,p,c;\alpha)=h_r(q,p;\alpha)+c\frac{b_{r-1}(q;\alpha)}{b_n(q;\alpha)}, \tag{3.64}$$

$$\pi_{-1}=\begin{pmatrix} 0 & B^{-1} & h_{n,p}\\ -B^{-1} & -B^{-1}(\frac{1}{2}p\otimes q-\frac{1}{2}q\otimes p)B^{-1} & -h_{n,q}\\ -(h_{n,p})^T & (h_{n,p})^T & 0\end{pmatrix}, \tag{3.65}$$

where

$$
\begin{aligned}
&B = A - \frac{1}{2} q \otimes q, \\
&B^{-1} = \frac{1}{|B|}\left[\partial\alpha\,|B| + \frac{1}{2}(\partial\alpha q \otimes \partial\alpha q)\,|A|\right], \\
&\partial\alpha = diag\left(\frac{\partial}{\partial\alpha_1}, ..., \frac{\partial}{\partial\alpha_n}\right), \\
&|B| = |A| - \frac{1}{2}(q, \partial\alpha\,|A|\,q).
\end{aligned}
\tag{3.66}
$$

Notice that now, the natural Hamiltonian in the inverse hierarchy is the last one of the form

$$
H' = h_1'(q,p,c) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V(q) + \frac{c}{\rho_n(\alpha)[1-(q, A^{-1}q)]}. \tag{3.67}
$$

The basic example here is the Garnier system with the potential $V(q) = \frac{1}{4}(q,q)^2 - \frac{1}{2}(q, Aq)$. This potential is a member of an infinite family of permutationally symmetric potentials separable in generalized elliptic coordinates [17]. The point transformation to DN separated coordinates can be constructed from the relations

$$
\rho_r(\lambda) = \rho_r(\alpha) + \frac{1}{2}\sum_{k=1}^{r-1}\rho_k(\alpha)(q, A^{r-k-1}q), \quad r = 1, ..., n. \tag{3.68}
$$

But so defined DN coordinates are just the generalized elliptic coordinates $\lambda_1, ..., \lambda_n$ defined by the relation

$$
1 + \frac{1}{2}\sum_{k=1}^{n}\frac{q_k^2}{z-\alpha_k} = \frac{\prod_{j=1}^{n}(z-\lambda_j)}{\prod_{k=1}^{n}(z-\alpha_k)}. \tag{3.69}
$$

The proof is given in refs. [26] and [28].

More examples of separated systems by the method presented the reader can find in refs. [26]-[29],[37]-[39].

4 Multi-Casimir unsplit chains

In the previous section a separability theory of one-Casimir bi-Hamiltonian chains was reviewed. Here we pass to the generalization of the theory and include multi-Casimir cases. This procedure considerably extends the class of separable systems and in general covers a new class of chains, i.e. the so-called split chains. In the following section we consider the simplest generalization of unsplit multi-Casimir chains related to the extension of the separation curve (3.4) of the form

$$
f(\lambda, \mu) = h\lambda, \quad h\lambda = c_n\lambda^{2n-1} + ... + c_1\lambda^n + h_1\lambda^{n-1} + ... + h_n. \tag{4.1}
$$

Choosing other admissible forms of the separation curve with more then one Casimir, one can construct split bi(multi)-Hamiltonian chain, i.e. the chain which splits onto a few bi(multi)-Hamiltonian sub-chains, each starting and terminating with some Casimir of the appropriate Poisson structure. The work on split cases is still in progress but some results are presented in the next section.

4.1 Multi-Hamiltonian Darboux-Nijenhuis chains

In ref. [30] the multi-Hamiltonian chain was constructed for the *separation curve* in the form (4.1). Actually, the Hamiltonian functions h_k found from the system

$$f(\lambda_i,\mu_i)=c_n\lambda_i^{2n-1}+...+c_1\lambda_i^n+h_1\lambda_i^{n-1}+...+h_n,\quad i=1,...,n, \tag{4.2}$$

take the following form

$$h_r(\lambda,\mu,c)=-\sum_{i=1}^n\frac{\partial\rho_r}{\partial\lambda_i}\frac{f(\lambda_i,\mu_i)}{\Delta_i}+\sum_{j=1}^n c_j\beta_{j,r}(\lambda),\quad r=1,...,n, \tag{4.3}$$

where $\beta_{1,r}(\lambda)\equiv\rho_r(\lambda)$ and $\beta_{m,r}\,,m=2,...,n,$ are defined by the recursive formula

$$\beta_{m,r}=\beta_{m-1,r+1}-\beta_{m-1,1}\cdot\beta_{1,r}. \tag{4.4}$$

Notice that Hamiltonians (4.3) are just Hamiltonians (3.6) supplemented with extra terms, linear with respect to additional Casimirs c_i, $i=2,...,n$. On the extended phase space $M\ni(\lambda_1,...,\lambda_n,\mu_1,...,\mu_n,\ c_1,...,c_n)$ functions (4.3) and $h_{1-r}(c)=c_r,\ r=1,...,n$ form $\begin{pmatrix}n+1\\2\end{pmatrix}$ bi-Hamiltonian chains, each generated by a Poisson pencil $\pi_{\lambda^{k-1}}=\pi_k-\lambda^{k-i}\pi_i$ of order $1\le(k-i)\le n$,

$$\pi_{\lambda^{k-i}}\circ dh_\lambda\Longrightarrow\begin{array}{lcl} \pi_i\circ dh_{-i} & = 0 & \\ \pi_i\circ dh_{-i+1} & = K_1 & =\pi_k\circ dh_{-k+1}\\ \pi_i\circ dh_{-i+2} & = K_2 & =\pi_k\circ dh_{-k+2}\\ & \vdots & \\ \pi_i\circ dh_{-i+j} & = K_j & =\pi_k\circ dh_{-k+j}\\ & \vdots & \\ \pi_i\circ dh_{-i+n} & = K_n & =\pi_k\circ dh_{-k+n}\\ & 0 & =\pi_k\circ dh_{-k+n+1}\end{array}\qquad 0\le i<k\le n, \tag{4.5}$$

with respect to $(n+1)$ compatible $3n\times 3n$ Poisson structures of rank $2n$

$$\pi_0=\begin{pmatrix}\theta_0 & 0 & \cdots & 0\\ 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{pmatrix},\quad \pi_1=\begin{pmatrix}\theta_1 & \overline{K}_1 & 0 & \cdots & 0\\ -\overline{K}_1^T & & & & \\ 0 & & & & \\ \vdots & & & 0 & \\ 0 & & & & \end{pmatrix},$$

$$\pi_2=\begin{pmatrix}\theta_2 & \overline{K}_2 & \overline{K}_1 & 0 & \cdots & 0\\ -\overline{K}_2^T & & & & & \\ -\overline{K}_1^T & & & & & \\ 0 & & & 0 & & \\ \vdots & & & & & \\ 0 & & & & & \end{pmatrix},\ .....,\tag{4.6}$$

$$\pi_n = \begin{pmatrix} \theta_n & \overline{K}_n & \overline{K}_{n-1} & \cdots & \overline{K}_1 \\ -\overline{K}_n^T & & & & \\ -\overline{K}_{n-1}^T & & & & \\ \vdots & & & 0 & \\ -\overline{K}_1^T & & & & \end{pmatrix},$$

where

$$\theta_m = N^m \circ \theta_0 = \begin{pmatrix} 0 & \Lambda^m \\ -\Lambda^m & 0 \end{pmatrix}, \quad \Lambda_m = diag(\lambda_1^m, ..., \lambda_n^m) \tag{4.7}$$

$\overline{K}_m = (h_{m,\mu}, -h_{m,\lambda})^T$ and each Poisson structure π_m has n Casimir functions: $c_{m+1}, c_{m+2}, \ldots, c_n, h_n, \ldots, h_{n-m+1}$. Moreover all functions $h_r(\lambda, \mu, c)$, $r = 1, ..., n$ are in involution with respect to an arbitrary Poisson tensor π_k, $k = 1, ..., n$.

Now, we integrate equations of motion from the hierarchy (4.5) solving the Hamilton-Jacobi equation for Hamiltonians (4.3)

$$h_r(\lambda, \frac{\partial W}{\partial \lambda}, c) = \sum_{k=1}^{n} \frac{\rho_{r-1}^k(\lambda) f_k(\lambda_k, \partial W/\partial \lambda_k)}{\Delta_k} + \sum_{i=1}^{n} c_i \beta_{i,r}(\lambda) = a_r. \tag{4.8}$$

First we demonstrate the separability of this equation. Taking the generating function $W(\lambda, a)$ in the form $W(\lambda, a) = \sum_{i=1}^n W_i(\lambda_i, a)$ and the following representation of $\beta_{k,r}$

$$\beta_{k,r}(\lambda) = \sum_{i=1}^{n} \frac{\partial \rho_r}{\partial \lambda_i} \frac{\lambda_i^{n+k-1}}{\Delta_i} = -\sum_{i=1}^{n} \rho_{r-1}^i \frac{\lambda_i^{n+k-1}}{\Delta_i}, \tag{4.9}$$

eq.(4.8) turns into the form

$$\sum_{k=1}^{n} \frac{\rho_{r-1}^k(\lambda)[f(\lambda_k, \partial W_k/\partial \lambda_k) - \sum_{i=1}^n c_i \lambda_k^{n-1+i}]}{\Delta_k} = a_r. \tag{4.10}$$

Applying relation (3.14) we get the solution of eq.(4.10) in the form

$$f(\lambda_k, \partial W_k/\partial \lambda_k) = g(\lambda_k), \quad k = 1, ..., n, \tag{4.11}$$

where

$$g(\xi) = c_n \xi^{2n-1} + ... + c_1 \xi^n + a_1 \xi^{n-1} + ... + a_{n-1} \xi + a_n. \tag{4.12}$$

Hence, $W(\lambda, a)$ can be obtained by solving n decoupled first-order ODEs (4.11). For example, if

$$f(\lambda_i, \mu_i) = \varphi(\lambda_i) f(\mu_i) + \psi(\lambda_i), \tag{4.13}$$

then we obtain

$$W(\lambda, a) = \sum_{k=1}^{n} \int^{\lambda_k} f^{-1}\left(\frac{g(\xi) - \psi(\xi)}{\varphi(\xi)}\right) d\xi. \tag{4.14}$$

In new canonical variables a_i, $b_i = \frac{\partial W}{\partial a_i}$, the Hamiltonians $h_r(\lambda, \mu, c)$ become $h_r = a_r$ with

$$b_i = \frac{\partial W}{\partial a_i} = \sum_{k=1}^{n} \int^{\lambda_k} \left(f^{-1}\right)' \frac{\xi^{n-i}}{\varphi(\xi)}\, d\xi, \tag{4.15}$$

where $\left(f^{-1}\right)'$ means the derivative of f^{-1}. As in the new coordinates each h_r generates a trivial flow

$$(a_j)_{t_r} = -\frac{\partial h_r}{\partial b_j} = 0, \ \ (b_j)_{t_r} = \frac{\partial h_r}{\partial a_j} = \delta_{j,r}, \ \ (c_j)_{t_r} = 0, \tag{4.16}$$

hence

$$b_i = t_i + const. \tag{4.17}$$

Combining (4.15) with (4.17) we arrive at implicit solutions for the trajectories $\lambda_i(t_r)$, with respect to the evolution parameter t_r in the form

$$\sum_{k=1}^{n} \int^{\lambda_k} \left(f^{-1}\right)' \frac{\xi^{n-i}}{\varphi(\xi)}\, d\xi = \delta_{i,r}\, t_r + const, \quad i = 1, ..., n. \tag{4.18}$$

4.2 Multi-Hamiltonian chains in arbitrary canonical coordinates

Let us introduce arbitrary canonical coordinates (q, p, c) related to the Darboux-Nijenhuis coordinates (λ, μ, c) trough some canonical transformation

$$q_k = \zeta_k(\lambda, \mu), \qquad p_k = \eta_k(\lambda, \mu), \qquad k = 1, ..., n. \tag{4.19}$$

Applying the inverse of this transformation to Hamiltonian functions (4.3) and Poisson matrices (4.6) one finds that

$$h_r(q, p, c) = h_r(q, p) + \sum_{i=1}^{n} c_i b_{i,r}(q) \tag{4.20}$$

and the nondegenerate part θ_m of rank $2n$ of each π_m(also implectic) takes now the form

$$\theta_m = N^m \circ \theta_0 = \begin{pmatrix} D_m(q,p) & A_m(q,p) \\ -A_m^T(q,p) & B_m(q,p) \end{pmatrix}, \qquad m = 1, ..., n. \tag{4.21}$$

Conversely, if we have a multi-Hamiltonian chain in (q, p, c) coordinates and the Nijenhuis tensor

$$N(q,p) = \begin{pmatrix} D_1(q,p) & A_1(q,p) \\ -A_1^T(q,p) & B_1(q,p) \end{pmatrix} \cdot \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}^{-1} = \begin{pmatrix} A_1(q,p) & -D_1(q,p) \\ B_1(q,p) & A_1^T(q,p) \end{pmatrix} \tag{4.22}$$

is nondegenerate and has n distinct eigenvalues λ_i each of multiplicity 2, then the canonical transformation (4.19) transforms a given chain to the one considered in the previous subsection.

The admissible reductions of the number of Casimir variables are the following. For arbitrary $1 \le m < n$, let $c_i \ne 0$, $1 \le i \le m$ and $c_i = 0$ for $m < i \le n$. The first m Poisson structures π_i, $i \le m$ survive the projection $(\lambda, \mu, c_1, ..., c_n) \in M \to \overline{M} \ni (\lambda, \mu, c_1, ..., c_m)$ and we have still $\binom{m+1}{2}$ bi-Hamiltonian chains (4.5). In the limit $c_1 = ... = c_n = 0$, the systems considered lose the bi-Hamiltonian property, turning into the quasi-bi-Hamiltonian systems on a symplectic manifold $M \ni (q, p)$, being still separable and integrable by quadratures. Moreover, because of the property (4.9), each of the multi-Hamiltonian systems considered on a Poisson manifold $M \ni (q, p, c)$ has a quasi-bi-Hamiltonian representation on a symplectic leaf S of π_0 ($\dim S = 2n$) fixing the values of all c_i.

4.3 Examples

The theory extended in this section will be illustrated by several representative examples of already known as well as new multi-Hamiltonian systems.

Example 4. *Stationary t_2−flow of dispersive water waves.*
The Hamiltonian functions and Poisson structures in Ostrogradsky variables are as follows [42]:

$$
\begin{aligned}
h_1(q,p,c) &= -4p_1p_2 + 5q_2p_1^2 - \frac{5}{8}q_1q_2^3 - \frac{3}{4}q_1^2q_2 - \frac{7}{64}q_2^5 + \frac{1}{2}q_2c_1 + (\frac{1}{2}q_1 + \frac{1}{8}q_2^2)c_2, \\
h_2(q,p,c) &= q_1p_1^2 + 4q_2p_1p_2 - \frac{5}{4}q_2^2p_1^2 - 2p_2^2 + \frac{5}{64}q_1q_2^4 - \frac{3}{16}q_1^2q_2^2 - \frac{1}{4}q_1^3 + \frac{45}{6\cdot 128}q_2^6 \\
&\qquad + (\frac{1}{2}q_1 + \frac{3}{8}q_2^2)c_1 - (\frac{1}{4}q_1q_2 - \frac{3}{16}q_2^3)c_2,
\end{aligned}
$$

$$
\pi_0 = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
$$

$$
\pi_1 = \begin{pmatrix}
0 & 0 & -\frac{3}{2}q_2 & -\frac{1}{2}q_1 - \frac{15}{8}q_2^2 & h_{1,p_1} & 0 \\
0 & 0 & 1 & q_2 & h_{1,p_2} & 0 \\
\frac{3}{2}q_2 & -1 & 0 & -p_1 & -h_{1,q_1} & 0 \\
\frac{1}{2}q_1 + \frac{15}{8}q_2^2 & -q_2 & p_1 & 0 & -h_{1,q_2} & 0 \\
-h_{1,p_1} & -h_{1,p_2} & h_{1,q_1} & h_{1,q_2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
$$

$$
\pi_2 = \begin{pmatrix}
0 & 0 & \frac{3}{8}q_2^2 - \frac{1}{2}q_1 & -\frac{1}{4}q_1q_2 - \frac{15}{16}q_2^3 & h_{2,p_1} & h_{1,p_1} \\
0 & 0 & -\frac{1}{2}q_2 & -\frac{1}{2}q_1 - \frac{7}{8}q_2^2 & h_{2,p_2} & h_{1,p_2} \\
-\frac{3}{8}q_2^2 + \frac{1}{2}q_1 & \frac{1}{2}q_2 & 0 & \frac{1}{2}q_2p_1 & -h_{2,q_1} & -h_{1,q_1} \\
\frac{1}{4}q_1q_2 + \frac{15}{16}q_2^3 & \frac{1}{2}q_1 + \frac{7}{8}q_2^2 & -\frac{1}{2}q_2p_1 & 0 & -h_{2,q_2} & -h_{1,q_2} \\
-h_{2,p_1} & -h_{2,p_2} & h_{2,q_1} & h_{2,q_2} & 0 & 0 \\
-h_{1,p_1} & -h_{1,p_2} & h_{1,q_1} & h_{1,q_2} & 0 & 0
\end{pmatrix}.
$$

Hence, we have three bi-Hamiltonian chains (4.5)

$$\begin{array}{llll} \pi_0\circ dc_1= & 0 & & \\ \pi_0\circ dh_1= & K_1 & =\pi_1\circ dc_1 & \\ \pi_0\circ dh_2= & K_2 & =\pi_1\circ dh_1 & \\ & 0 & =\pi_1\circ dh_2, & \end{array}\qquad \begin{array}{lll} \pi_0\circ dc_1= & 0 & \\ \pi_0\circ dh_1= & K_1 & =\pi_2\circ dc_2 \\ \pi_0\circ dh_2= & K_2 & =\pi_2\circ dc_1 \\ & 0 & =\pi_2\circ dh_1, \end{array}$$

$$\begin{array}{lll} \pi_1\circ dc_2= & 0 & \\ \pi_1\circ dc_1= & K_1 & =\pi_2\circ dc_2 \\ \pi_1\circ dh_1= & K_2 & =\pi_2\circ dc_1 \\ & 0 & =\pi_2\circ dh_1. \end{array} \tag{4.23}$$

The canonical transformation to the Darboux-Nijenhuis coordinates reads

$$\begin{aligned} q_1 &= -(3\lambda_1^2+3\lambda_2^2+4\lambda_1\lambda_2),\\ q_2 &= -2(\lambda_1+\lambda_2),\\ p_1 &= \frac{1}{2}\frac{\mu_2-\mu_1}{\lambda_1-\lambda_2},\\ p_2 &= -\frac{1}{2}\frac{\lambda_1(3\mu_2-2\mu_1)-\lambda_2(3\mu_1-2\mu_2)}{\lambda_1-\lambda_2}, \end{aligned}$$

where now h_r, $r=1,2$ take the form (4.3) with

$$f_i(\lambda_i,\mu_i)=2\lambda_i^6-\frac{1}{2}\mu_i^2,$$

$\beta_{1,1}=\rho_1, \beta_{1,2}=\rho_2$ (3.10) and

$$\begin{aligned} \beta_{2,1} &= -\lambda_1^2-\lambda_2^2-\lambda_1\lambda_2,\\ \beta_{2,2} &= \lambda_1\lambda_2(\lambda_1+\lambda_2). \end{aligned} \tag{4.24}$$

Example 5. *Two-Casimir extension of the Henon-Heiles system.*
In Example 1 one-Casimir extension of the Henon-Heiles system was considered in the form

$$(q_1)_{tt}=-3q_1^2-\frac{1}{2}q_2^2+c, \qquad (q_2)_{tt}=-q_1q_2,$$

with two constants of motion (3.45) and the related transformation to Darboux-Nijenhuis coordinates (3.51). For a two-Casimir extension we get immediately

$$\begin{aligned} \beta_{2,1} &= -[(\lambda_1+\lambda_2)^2-\lambda_1\lambda_2] = -(q_1^2+\frac{1}{4}q_2^2),\\ \beta_{2,2} &= \lambda_1\lambda_2(\lambda_1+\lambda_2) = -\frac{1}{4}q_1q_2^2. \end{aligned} \tag{4.25}$$

Hence

$$\begin{aligned} h_1 = H &= \frac{1}{2}p_1^2+\frac{1}{2}p_2^2+q_1^3+\frac{1}{2}q_1q_2^2-c_1q_1-(q_1^2+\frac{1}{4}q_2^2)c_2,\\ h_2 &= \frac{1}{2}q_2p_1p_2-\frac{1}{2}q_1p_2^2+\frac{1}{16}q_2^4+\frac{1}{4}q_1^2q_2^2-\frac{1}{4}q_2^2c_1-\frac{1}{4}q_1qc_2, \end{aligned} \tag{4.26}$$

where the Newton's equations related to the energy H are

$$(q_1)_{tt} = -3q_1^2 - \frac{1}{2}q_2^2 + c_1 + 2q_1c_2, \qquad (q_2)_{tt} = -q_1q_2 + \frac{1}{2}q_2c_2. \tag{4.27}$$

This is tri-Hamiltonian system with the following Poisson structures

$$\pi_0 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\pi_1 = \begin{pmatrix} 0 & 0 & q_1 & \frac{1}{2}q_2 & h_{1,p_1} & 0 \\ 0 & 0 & \frac{1}{2}q_2 & 0 & h_{1,p_2} & 0 \\ -q_1 & -\frac{1}{2}q_2 & 0 & \frac{1}{2}p_2 & -h_{1,q_1} & 0 \\ -\frac{1}{2}q_2 & 0 & -\frac{1}{2}p_2 & 0 & -h_{1,q_2} & 0 \\ -h_{1,p_1} & -h_{1,p_2} & h_{1,q_1} & h_{1,q_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\pi_2 = \begin{pmatrix} 0 & 0 & q_1^2 + \frac{1}{4}q_2^2 & \frac{1}{2}q_1q_2 & h_{2,p_1} & h_{1,p_1} \\ 0 & 0 & \frac{1}{2}q_1q_2 & \frac{1}{4}q_2^2 & h_{2,p_2} & h_{1,p_2} \\ -q_1^2 - \frac{1}{4}q_2^2 & -\frac{1}{2}q_1q_2 & 0 & \frac{1}{2}q_1p_2 & -h_{2,q_1} & -h_{1,q_1} \\ -\frac{1}{2}q_1q_2 & -\frac{1}{4}q_2^2 & -\frac{1}{2}q_1p_2 & 0 & -h_{2,q_2} & -h_{1,q_2} \\ -h_{2,p_1} & -h_{2,p_2} & h_{2,q_1} & h_{2,q_2} & 0 & 0 \\ -h_{1,p_1} & -h_{1,p_2} & h_{1,q_1} & h_{1,q_2} & 0 & 0 \end{pmatrix}.$$

The first two of them come from one-Casimir extension and the last one was constructed according to formula (4.21). Notice that again we have three bi-Hamiltonian chains (4.23).

Also in Example 1 the inverse one-Casimir extension of the Henon-Heiles system was considered in the form

$$(q_1)_{tt} = -3q_1^2 - \frac{1}{2}q_2^2, \qquad (q_2)_{tt} = -q_1q_2 - \frac{8}{q_2^3}c,$$

with two constants of motion (3.48) and the respective transformation to the DN coordinates (3.52). The two-Casimir extension we get by adding new terms

$$\beta_{2,1} = -[(\lambda_1+\lambda_2)^2 - \lambda_1\lambda_2] = \frac{4}{q_2^2} - \frac{16q_1^2}{q_2^4},$$
$$\beta_{2,2} = \lambda_1\lambda_2(\lambda_1+\lambda_2) = \frac{16q_1}{q_2^4},$$

to the constants of motion (3.48)

$$h_1'(q,p,c_1,c_2) = h_1'(q,p,c=c_1) + \frac{16q_1}{q_2^4}\,c_2,$$
$$h_2'(q,p,c_1,c_2) = h_2'(q,p,c=c_1) + \left(\frac{4}{q_2^2} - \frac{16q_1^2}{q_2^4}\right)c_2,$$

where the Newton equations related to the energy h_1' are

$$(q_1)_{tt} = -3q_1^2 - \frac{1}{2}q_2^2 - \frac{16}{q_2^4}c_2, \qquad (q_2)_{tt} = -q_1 q_2 - \frac{8}{q_2^3}c_1 + \frac{64 q_1}{q_2^5}c_2.$$

Again this is inverse tri-Hamiltonian system

$$\begin{array}{llll} \pi_0 \circ dc_1 = 0 & & \pi_0 \circ dc_1 = 0 & \\ \pi_0 \circ dh_2' = K_2' = \pi_{-1} \circ dc_1 & & \pi_0 \circ dh_2' = K_2' = \pi_{-2} \circ dc_2 & \\ \pi_0 \circ dh_1' = K_1' = \pi_{-1} \circ dh_2' & & \pi_0 \circ dh_1' = K_1' = \pi_{-2} \circ dc_1 & \\ \qquad\qquad 0 \;\; = \pi_{-1} \circ dh_1', & & \qquad\qquad 0 \;\; = \pi_{-2} \circ dh_2', & \end{array} \tag{4.28}$$

$$\begin{array}{l} \pi_{-1} \circ dc_2 = 0 \\ \pi_{-1} \circ dc_1 = K_2' = \pi_{-2} \circ dc_2 \\ \pi_{-1} \circ dh_2' = K_1' = \pi_{-2} \circ dc_1 \\ \qquad\qquad 0 \;\; = \pi_{-2} \circ dh_2', \end{array}$$

where the Poisson structure $\pi_{-1}$is given by (3.49), with additional last row and column with zeros, while the new third Poisson structure π_{-2} reads

$$\pi_{-2} = \begin{pmatrix} 0 & 0 & 4/q_2^2 & -8q/q_2^3 & h'_{1,p_1} & h'_{2,p_1} \\ 0 & 0 & -8q/q_2^3 & (16q_1^2+4q_2^2)/q_2^4 & h'_{1,p_2} & h'_{2,p_2} \\ -4/q_2^2 & 8q_1/q_2^3 & 0 & -8q_1p_2/q_2^4 & -h'_{1,q_1} & -h'_{2,q_1} \\ 8q_1/q_2^3 & -(16q_1^2+4q_2^2)/q_2^4 & 8q_1p_2/q_2^4 & 0 & -h'_{1,q_2} & -h'_{2,q_2} \\ -h'_{1,p_1} & -h'_{1,p_2} & h'_{1,q_1} & h'_{1,q_2} & 0 & 0 \\ -h'_{2,p_1} & -h'_{2,p_2} & h'_{2,q_1} & h'_{2,q_2} & 0 & 0 \end{pmatrix}.$$

Till now all examples presented were bi(multi)-Hamiltonian Stäckel systems [27], i.e. the systems with all Hamiltonian functions quadratic in momenta in DN coordinates: $f(\lambda_i, \mu_i) = \varphi(\lambda_i)\mu_i^2 + \psi(\lambda_i)$. Here we present the first example of non-Stäckel system.

Example 6. *m-Casimir extension of the relativistic n-body problem.*
Consider the Hamiltonian dynamical system with the Hamiltonian given by

$$H = \sum_{i=1}^{n} \frac{\varphi_i(\lambda_i)}{\Delta_i} e^{a\mu_i}, \tag{4.29}$$

where φ_i are arbitrary smooth functions and a is an arbitrary constant. The corresponding dynamical system takes the form

$$(\lambda_i)_{tt} = 2\sum_{k\neq i} \frac{(\lambda_i)_t(\lambda_k)_t}{\lambda_i - \lambda_k}, \qquad i = 1, ..., n, \tag{4.30}$$

which depends explicitly on velocities. The derivation of formula (4.30) is given in ref. [26]. Notice that equations (4.30) do not depend on φ_i functions, hence the dynamics is not influenced by φ_i terms.

Dynamics (4.30) is a special case of the integrable relativistic n-body problems introduced by Ruijsenaars and Schneider [43]. Now, comparing (4.29) with (4.3) one immediately concludes that (λ, μ) is a Darboux-Nijenhuis chart for the Hamiltonian H, and as a consequence, system (4.30) is quasi-bi-Hamiltonian and separable, with the solution given by the implicit formulae

$$\frac{1}{a}\sum_{k=1}^{n}\int^{\lambda_k}\frac{\xi^{n-i}}{g(\xi)}\,d\xi=\delta_{i,1}\,t+const,\quad i=1,...,n, \tag{4.31}$$

where $g(\xi)=a_n+a_{n-1}\xi+...+a_1\xi^{n-1}$. This fact was noticed for the first time by Morosi and Tondo [38]. Notice, that trajectories $\lambda_i(t)$ do not depend on the $\varphi_i(\lambda_i)$ factors, as expected, so without a loss of generality one can put $\varphi_i(\lambda_i)=1$.

The system (4.30) can be naturally extended to an m-Casimir one with the Hamiltonian

$$H=\sum_{i=1}^{n}\frac{\varphi_i(\lambda_i)}{\Delta_i}e^{a\mu_i}+\sum_{j=1}^{m}c_j\beta_{j,1}(\lambda),\qquad 1\le m\le n \tag{4.32}$$

and the related Newton equations of motion

$$(\lambda_i)_{tt}=2\sum_{k\neq i}\frac{(\lambda_i)_t(\lambda_k)_t}{\lambda_i-\lambda_k}-a\sum_{j=1}^{m}c_j\frac{\partial\beta_{j,1}}{\partial\lambda_i}\,(\lambda_i)_t. \tag{4.33}$$

The dynamical system (4.33) has n constants of motion

$$h_r(\lambda,\mu,c)=-\sum_{i=1}^{n}\frac{\partial\rho_r(\lambda)}{\partial\lambda_i}\frac{e^{a\mu_i}}{\Delta_i}+\sum_{j=1}^{m}c_j\beta_{j,r}(\lambda),\quad r=1,...,n,$$

$(m+1)$ Poisson structures (4.6) and the solution given by implicit formula (4.31), where now

$$g(\xi)=a_n+a_{n-1}\xi+...+a_1\xi^{n-1}+c_1\xi^n+...+c_m\xi^{n+m-1}.$$

The one-Casimir extension, which is bi-Hamiltonian, has the following Newton equations of motion

$$(\lambda_i)_{tt}=2\sum_{k\neq i}\frac{(\lambda_i)_t(\lambda_k)_t}{\lambda_i-\lambda_k}+ac(\lambda_i)_t,\qquad i=1,...,n.$$

The two-Casimir extension, which is tri-Hamiltonian, has the Newton equations in the form

$$(\lambda_i)_{tt}=2\sum_{k\neq i}\frac{(\lambda_i)_t(\lambda_k)_t}{\lambda_i-\lambda_k}+ac_1(\lambda_i)_t+ac_2\left(\lambda_i+\sum_{k=1}^{n}\lambda_k\right)(\lambda_i)_t,\qquad i=1,...,n.$$

Example 7. $(m+1)$-*Hamiltonian formulation for elliptic separable potentials.*
In Example 3 we presented bi-Hamiltonian systems separable in generalized elliptic coordinates. In the following example we extend the result onto appropriate multi-Hamiltonian chains.

According to the theory presented in this section, let us generalize the Hamiltonian system (3.60) to the form

$$H(q,p,c)=\frac{1}{2}(p,p)+V(q)+\sum_{k=1}^{m}c_kb_{k,1}(q),\qquad m=1,...,n \tag{4.34}$$

being multi-Hamiltonian and separable. The few first $b_{k,1}(q)$ functions are as follows

$$\begin{aligned} b_{1,1}(q) &= \frac{1}{2}(q,q), \\ b_{2,1}(q) &= \frac{1}{2}(q,Aq) - \frac{1}{4}(q,q)^2, \\ b_{3,1}(q) &= \frac{1}{2}(q,q)(q,Aq) - \frac{1}{2}(q,A^2q) - \frac{1}{8}(q,q)^3, \dots . \end{aligned} \tag{4.35}$$

For example, the three Poisson structures of the system (4.34) with two Casimirs read

$$\pi_0 = \begin{pmatrix} 0 & I & 0 & 0 \\ -I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\pi_1 = \begin{pmatrix} 0 & A - \frac{1}{2}q\otimes q & h_{1,p} & 0 \\ -A + \frac{1}{2}q\otimes q & \frac{1}{2}p\otimes q - \frac{1}{2}q\otimes p & -h_{1,q} & 0 \\ -(h_{1,p})^T & (h_{1,q})^T & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\pi_2 = \begin{pmatrix} 0 & (A - \frac{1}{2}q\otimes q)^2 & h_{2,p} & h_{1,p} \\ -(A - \frac{1}{2}q\otimes q)^2 & \begin{matrix} (A - \frac{1}{2}q\otimes q)(\frac{1}{2}p\otimes q - \frac{1}{2}q\otimes p) \\ +(\frac{1}{2}p\otimes q - \frac{1}{2}q\otimes p)(A - \frac{1}{2}q\otimes q) \end{matrix} & -h_{2,q} & -h_{1,q} \\ -(h_{2,p})^T & (h_{2,q})^T & 0 & 0 \\ -(h_{1,p})^T & (h_{1,q})^T & 0 & 0 \end{pmatrix}.$$

The functions $h_r(q,p,c)$, forming three admissible bi-Hamiltonian chains, are given by formulas (3.63), where now

$$\overline{h}_r = \frac{1}{2}\sum_{i=1}^{n} \alpha_i^{r-1} K_i + c_1 b_{1,r}(q) + c_2 b_{2,r}(q).$$

5 Multi-Casimir split chains

In the following section we extend the results from previous sections onto the so-called split bi(multi)-Hamiltonian chains [31]. Potentially, the variety of such systems is much richer than the one of unsplit chains, but still less recognized. The reason is that systems from this family are generally non-physical in the sense that are either non-Stäckel, or even if are of Stäckel type, the underlying Stäckel space is never flat (at most conformally flat).

Let M be a $(2n+k)$ dimensional manifold endowed with a linear Poisson pencil $\pi\lambda = \pi_1 - \lambda\pi_0$. We suppose that it admits k polynomial Casimir functions

$$h\lambda^{(\alpha)} = \sum_{j=0}^{n\alpha} h_j^{(\alpha)} \lambda^{n\alpha - j}, \qquad \alpha = 1, \dots, k \tag{5.1}$$

with $n = n_1 + ... + n_k$. From beeing Casimir of a pencil it follows that the set $\{h_0^{(\alpha)}\}_{\alpha=1}^k$ is a set of Casimirs of π_0, while $\{h_{n\alpha}^{(\alpha)}\}_{\alpha=1}^k$ is a set of Casimirs of π_1, respectively. In canonical coordinates (q, p, c) $h_0^{(\alpha)} = c\alpha$, $\alpha = 1, ..., k$,

$$\pi_0 = \begin{pmatrix} \theta_0 & 0 & ... & 0 \\ 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{pmatrix}, \quad \pi_1 = \begin{pmatrix} \theta_1 & K_1^{(1)} & \cdots & K_1^{(k)} \\ -\left(K_1^{(1)}\right)^T & & & \\ \vdots & & 0 & \\ -\left(K_1^{(k)}\right)^T & & & \end{pmatrix}, \tag{5.2}$$

where $K_1^{(i)} = \theta_0 dh_1^{(i)}$ and θ_0, θ_1 are given by (3.25). Now, we are looking for a separation curve in the form

$$f(\lambda, \mu) = \sum \alpha \vartheta \alpha h \lambda^{(\alpha)} = h\lambda, \tag{5.3}$$

where $\vartheta\alpha$ are admissible functions of λ and μ. Further on we concentrate on particular two-Casimir cases, but it will be enough to make some insight into the theory.

Let us start from a separation curve for the unsplit two-Casimir case

$$f(\lambda, \mu) = h\lambda, \quad h\lambda = c_2\lambda^{n+1} + c_1\lambda^n + h_1\lambda^{n-1} + ... + h_n. \tag{5.4}$$

An arbitrary shift of c_1 Casimir variable along the polynomial $h\lambda$ leads to a separation curve

$$f(\lambda, \mu) = c_2\lambda^{n+1} + h_1\lambda^n + ... + h_i\lambda^{n-i+1} + c_1\lambda^{n-i} + h_{i+1}\lambda^{n-i-1} + ... + h_n \tag{5.5}$$

of some split bi-Hamiltonian chain. This is the case (5.3) with $k = 2$, $n_1 = n - i$, $n_2 = i$, $\vartheta_1 = 1$, $\vartheta_2 = \lambda^{n+1-i}$, $h_l^{(2)} = h_l, l = 1, ..., i$ and $h_j^{(1)} = h_{i+j}, j = 1, ..., n - i$. We illustrate the situation for $i = 1$. The following results were obtained. For a separation curve in the form

$$f(\lambda, \mu) = c_2\lambda^{n+1} + h_1\lambda^n + c_1\lambda^{n-1} + h_2\lambda^{n-2} + ... + h_n \equiv h\lambda \tag{5.6}$$

Hamiltonian functions $h_r, r = 1, ..., n$ read

$$h_r(\lambda, \mu, c) = h_r(\lambda, \mu) + \gamma_{1,r}(\lambda)c_1 + \gamma_{2,r}(\lambda)c_2, \tag{5.7}$$

where

$$\begin{aligned} &h_r(\lambda, \mu) = \sum_{i=1}^{n} \alpha_{r-1}^i(\lambda)\frac{f(\lambda_i, \mu_i)}{\Omega_i(\lambda)}, \\ &\Omega_i(\lambda) = \left(\sum_{k=1}^{n} \lambda_k\right) \prod_{k\neq i}(\lambda_i - \lambda_k), \\ &\gamma_{1,1}(\lambda) = \frac{1}{\rho_1}, \quad \gamma_{1,r}(\lambda) = \frac{\rho_r}{\rho_1}, \quad r = 2, ..., n, \\ &\gamma_{2,1}(\lambda) = \frac{\rho_1^2 - \rho_2}{\rho_1}, \quad \gamma_{2,r}(\lambda) = \frac{\rho_1\rho_{r+1} - \rho_2\rho_r}{\rho_1}, \quad r = 2, ..., n, \\ &\alpha_r^i(\lambda) = \alpha_r(\lambda_i = 0), \quad \alpha_r(\lambda) \equiv \beta_{2,r}(\lambda) = \rho_{r+1} - \rho_1\rho_r. \end{aligned} \tag{5.8}$$

Two compatible Poisson structures

$$\pi_0 = \begin{pmatrix} 0 & I & 0 & 0 \\ -I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \pi_1 = \begin{pmatrix} 0 & \Lambda & h_{2,\mu} & h_{1,\mu} \\ -\Lambda & 0 & -h_{2,\lambda} & -h_{1,\lambda} \\ -(h_{2,\mu})^T & (h_{2,\lambda})^T & 0 & 0 \\ -(h_{1,\mu})^T & (h_{1,\lambda})^T & 0 & 0 \end{pmatrix}, \tag{5.9}$$

give rise to a linear Poisson pencil $\pi\lambda = \pi_1 - \lambda\pi_0$, which acting on its Casimir $h\lambda$ generates a bi-Hamiltonian chain

$$\pi\lambda \circ dh\lambda \implies \begin{array}{rcl} \pi_0 \circ dc_2 = & 0 & \\ \pi_0 \circ dh_1 = & K_1 & = \pi_1 \circ dc_2 \\ & 0 & = \pi_1 \circ dh_1 \\ \pi_0 \circ dc_1 = & 0 & \\ \pi_0 \circ dh_2 = & K_2 & = \pi_1 \circ dc_1 \\ & \vdots & \\ \pi_0 \circ dh_n = & K_n & = \pi_1 \circ dh_{n-1} \\ & 0 & = \pi_1 \circ dh_n \end{array} \tag{5.10}$$

which splits onto two sub-chains, each starting and terminating with a Casimir of an appropriate Poisson structure.

Example 8. *The case of three degrees of freedom.*
We construct a system comparable with the Newton representation of the $7th$ order KdV [26], [28]. Thus let us take

$$f(\mu_i, \lambda_i) = \frac{1}{8}\mu_i^2 + 16\lambda_i^7, \quad i = 1, 2, 3 \tag{5.11}$$

and the point transformation generated by relations

$$\begin{aligned} q_1 &= \lambda_1 + \lambda_2 + \lambda_3, \\ q_2 &= -\frac{1}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) + (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3), \\ q_3 &= \frac{1}{2}(\lambda_1 - \lambda_2 - \lambda_3)(\lambda_2 - \lambda_1 - \lambda_3)(\lambda_3 - \lambda_1 - \lambda_2). \end{aligned} \tag{5.12}$$

Then, from (5.8), we arrive at the following Hamiltonians in natural coordinates

$$\begin{aligned} h_1 &= \frac{1}{q_1}\left(\frac{1}{2}p_2^2 + p_1p_3\right) + 10q_1q_3 + 8q_2^2 - 10q_1^2q_2 + 3q_1^4 - 4\frac{q_2q_3}{q_1} - \frac{1}{q_1}c_1 \\ &\quad + \left(\frac{1}{2}\frac{q_2}{q_1} - \frac{3}{4}q_1\right)c_2, \\ h_2 &= \frac{1}{8}\left(\frac{2q_2}{q_1} - 3q_1\right)p_2^2 + \frac{1}{2}q_3p_3^2 + \frac{1}{4}\left(\frac{2q_2}{q_1} - q_1\right)p_1p_3 - \frac{1}{2}p_1p_2 - \frac{1}{2}q_2p_2p_3 \end{aligned}$$

$$- 2q_1q_2q_3 - q_1^2q_2^2 + \frac{3}{2}q_1^4q_2 + 2q_2^3 + \frac{5}{2}q_1^3q_3 + q_3^2 - \frac{1}{2}q_1^6 - 2\frac{q_2^2q_3}{q_1}$$
$$- \frac{1}{2}\left(\frac{1}{2}q_1 + \frac{q_2}{q_1}\right)c_1 + \frac{1}{2}\left(q_1q_2 - \frac{1}{2}q_3 + \frac{1}{8}q_1^3 + \frac{1}{2}\frac{q_2^2}{q_1}\right)c_2,$$
$$h_3 = \frac{1}{8}p_1^2 + \frac{1}{8}\left(q_1^2 - q_2 - \frac{q_3}{q_1}\right)p_2^2 + \frac{1}{8}q_2^2p_3^2 + \frac{1}{4}q_1p_1p_2 - \frac{1}{4}\frac{q_3}{q_1}p_1p_3$$
$$- \frac{1}{4}(2q_3 + q_1q_2) + \frac{1}{2}q_1^4q_3 - q_1^2q_2q_3 - \frac{1}{2}q_1q_3^2 + \frac{1}{2}q_1^5q_2 - \frac{1}{2}q_1^3q_2^2 - q_1q_2^3$$
$$+ \frac{q_2q_3^2}{q_1} + \frac{1}{4}\left(q_2 - \frac{q_3}{q_1}\right)c_1 + \frac{1}{16}(q_1^2q_2 - q_1q_3 + 2q_2^2 - 2\frac{q_2q_3}{q_1})c_2.$$

They form bi-Hamiltonian split chain (5.10), where now

$$\pi_1 = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2}q_1 & -\frac{1}{2} & 0 & h_{2,p_1} & h_{1,p_1} \\ 0 & 0 & 0 & \frac{1}{2}q_2 & 0 & -\frac{1}{2} & h_{2,p_2} & h_{1,p_2} \\ 0 & 0 & 0 & q_3 & \frac{1}{2}q_2 & \frac{1}{2}q_1 & h_{2,p_3} & h_{1,p_3} \\ -\frac{1}{2}q_1 & -\frac{1}{2}q_2 & -q_3 & 0 & \frac{1}{2}p_2 & \frac{1}{2}p_3 & -h_{2,q_1} & -h_{1,q_1} \\ \frac{1}{2} & 0 & -\frac{1}{2}q_2 & -\frac{1}{2}p_2 & 0 & 0 & -h_{2,q_2} & -h_{1,q_2} \\ 0 & \frac{1}{2} & -\frac{1}{2}q_1 & -\frac{1}{2}p_3 & 0 & 0 & -h_{2,q_3} & -h_{1,q_3} \\ -h_{2,p_1} & -h_{2,p_2} & -h_{2,p_3} & h_{2,q_1} & h_{2,q_2} & h_{2,q_3} & 0 & 0 \\ -h_{1,p_1} & -h_{1,p_2} & -h_{1,p_3} & h_{1,q_1} & h_{1,q_2} & h_{1,q_3} & 0 & 0 \end{pmatrix},$$

and are conformally related to these of first Newton representation of $7th$ order stationary KdV, i.e. in separated coordinates the metrices of respective Stäckel spaces are conformally related.

Another admissible form of the separation curve with two Casimirs, leading to bi-Hamiltonian split chain of non-Stäckel type, reads

$$f(\lambda, \mu) = \mu\left(c_2\lambda^{n-i} + h_1\lambda^{n-i-1} + ... + h_{n-i}\right) + c_1\lambda^i + h_{n-i-1}\lambda^{i-1} + ... + h_n.$$

It has the form (5.3) with $k = 2$, $n_1 = i$, $n_2 = n - i$, $\vartheta_1 = 1$, $\vartheta_2 = \mu$, $h_l^{(2)} = h_l, l = 1, ..., n - i$, and $h_j^{(1)} = h_{n-i+j}$, $j = 1, ..., i$. Here we concentrate on the case $i = n - 1$

$$f(\lambda, \mu) = \mu(c_2\lambda + h_1) + c_1\lambda^{n-1} + h_2\lambda^{n-2} + ... + h_n. \tag{5.13}$$

The Poisson pencil and the chain are the same as in the previous split case, i.e. are of the form (5.9) and (5.10), but $h_r(\lambda, \mu, c)$, being the solution of the system

$$f(\lambda_i, \mu_i) = \mu_i(c_2\lambda_i + h_1) + c_1\lambda_i^{n-1} + h_2\lambda_i^{n-2} + ... + h_n, \quad i = 1, ..., n,$$

are of more complicated form and we omit here the explicit formulas.

We illustrate the case (5.13) and $n = 3$ with the example where the transformation to DN coordinates will be of non-point nature.

Example 9. *The 4th order Boussinesq stationary flow.*
The Hamiltonians and compatible Poisson structures for the $4th$ order Boussinesq stationary flow, written in generalized Ostrogradsky canonical coordinates, are the following [44]

$$\begin{aligned}
h_1 =& -27p_2p_3 - 27p_2^2 - 9p_3^2 + 2q_1^2p_3 + 3q_1^2p_2 + q_2p_1 - \frac{2}{9}q_1q_3^2 - \frac{1}{9}q_1q_2^2 - \frac{8}{81}q_1^4 \\
&+ \frac{1}{3}q_1c_1 + \frac{1}{3}q_3c_2, \\
h_2 =& -9p_1p_3 + q_1^2p_1 + 2q_1(q_2 - 2q_3)p_2 + \frac{4}{3}q_1q_2p_3 - \frac{4}{27}q_3^3 + \frac{8}{81}q_1^3q_3 - \frac{16}{81}q_1^3q_2 \\
&+ \frac{1}{27}q_2^3 + \frac{2}{9}q_2q_3^2 - \frac{4}{27}q_2^2q_3 + \frac{1}{3}(2q_3 - q_2)c_1 + (3p_2 - \frac{1}{9}q_1^2)c_2, \\
h_3 =& -54p_2^3 - 27p_2p_3^2 - 81p_2^2p_3 + 3q_1p_1^2 + 18q_1^2p_2^2 + 2q_1^2p_3^2 + 3q_2p_1p_2 + 3(q_2 - q_3)p_1p_3 \\
&+ 15q_1^2p_2p_3 - q_1^2(q_3 + \frac{2}{3}q_2)p_1 - (\frac{44}{27}q_1^4 + \frac{2}{3}q_1q_2q_3)p_2 + (\frac{2}{3}q_1q_3^2 - \frac{1}{9}q_1q_2^2 \\
&- \frac{4}{9}q_1q_2q_3 - \frac{16}{27}q_1^4)p_3 + \frac{32}{243}q_1^3q_2q_3 + \frac{32}{729}q_1^6 + \frac{16}{243}q_1^3q_3^3 + \frac{4}{243}q_1^3q_2^2 \\
&- \frac{4}{81}q_2^2q_3^2 + \frac{1}{81}q_2^3q_3 + \frac{2}{27}q_2q_3^3 - \frac{1}{27}q_3^4 + (\frac{4}{27}q_1^3 + \frac{1}{9}q_3^2 - \frac{1}{9}q_2q_3 - 2q_1p_2 \\
&- q_1p_3)c_1 + (q_1p_1 + q_3p_2 - \frac{5}{27}q_1^2q_3 - \frac{2}{27}q_1^2q_2)c_2,
\end{aligned}$$

$$\pi_0 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\pi_1 = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{3}q_3 & \frac{1}{3}q_1 & -\frac{2}{3}q_1 & h_{2,p_1} & h_{1,p_1} \\ 0 & 0 & -3q_1 & A & \frac{1}{3}q_2 - \frac{1}{3}q_3 & -\frac{1}{3}q_2 & h_{2,p_2} & h_{1,p_2} \\ 0 & 3q_1 & 0 & \frac{2}{9}q_1^2 & 0 & -\frac{1}{3}q_3 & h_{2,p_3} & h_{1,p_3} \\ \frac{1}{3}q_3 & -A & -\frac{2}{9}q_1^2 & 0 & B & C & -h_{2,q_1} & -h_{1,q_1} \\ -\frac{1}{3}q_1 & -\frac{1}{3}q_2 + \frac{1}{3}q_3 & 0 & -B & 0 & -\frac{8}{81}q_1^2 & -h_{2,q_2} & -h_{1,q_2} \\ \frac{2}{3}q_1 & \frac{1}{3}q_2 & \frac{1}{3}q_3 & -C & \frac{8}{81}q_1^2 & 0 & -h_{2,q_3} & -h_{1,q_3} \\ -h_{2,p_1} & -h_{2,p_2} & -h_{2,p_3} & h_{2,q_1} & h_{2,q_2} & h_{2,q_3} & 0 & 0 \\ -h_{1,p_1} & -h_{1,p_2} & -h_{1,p_3} & h_{1,q_1} & h_{1,q_2} & h_{1,q_3} & 0 & 0 \end{pmatrix}$$

where $A = 9p_2 + 3p_3 - q_1^2$, $B = \frac{2}{27}q_1q_2 + \frac{8}{81}q_1q_3 - \frac{1}{3}p_1$, $C = -\frac{8}{81}q_1q_2 - \frac{4}{27}q_1q_3 + \frac{1}{3}p_1$.

¿From the minimal polynomial (3.20) of related Nijenhuis tensor N one finds the first part of transformation to the HF coordinates (u,v)

$$
\begin{aligned}
u_1 &= q_3 - \frac{1}{3}q_2,\\
u_2 &= \frac{1}{3}q_3^2 + \frac{5}{27}q_1^3 - 3q_1p_2 - q_1p_3 - \frac{2}{9}q_2q_3,\\
u_3 &= \frac{1}{27}q_3^3 + \frac{1}{9}q_1^3q_3 - q_1q_3p_2 - \frac{1}{3}q_1q_3p_3 + \frac{2}{81}q_1^3q_2 - \frac{1}{3}q_1^2p_1 - \frac{1}{27}q_2q_3,
\end{aligned}
\tag{5.14}
$$

while the system (3.30) with the second chain (3.28) give the second part of the transformation

$$
\begin{aligned}
v_1 &= -\frac{3}{q_1},\\
v_2 &= \frac{q_2 - 2q_3}{q_1},\\
v_3 &= 3p_3 + 6p_2 - \frac{1}{3}\frac{q_3^2}{q_1} + \frac{1}{3}\frac{q_2q_3}{q_1} - \frac{4}{9}q_1^2.
\end{aligned}
\tag{5.15}
$$

On the other hand the relation between HF and DN coordinates reads

$$
\begin{aligned}
u_1 &= -\lambda_1 - \lambda_2 - \lambda_3,\\
u_2 &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3,\\
u_3 &= -\lambda_1\lambda_2\lambda_3,\\
\mu_i &= v_1\lambda_i^2 + v_2\lambda_i + v_3, \qquad i = 1,2,3.
\end{aligned}
\tag{5.16}
$$

Hence, eliminating the HF coordinates (u,v) from the system (5.14)-(5.16) we arrive at the explicit relation between DN coordinates (λ,μ) and natural coordinates (q,p). Unfortunately, the formulas are too long to quote them in the text. In DN coordinates the two Poisson structures take the form (5.9) and the related separated curve

$$
\mu^3 - \lambda^4 = \mu(c_2\lambda + h_1) + c_1\lambda^2 + h_2\lambda + h_3. \tag{5.17}
$$

The implicit solutions of the system can be obtained by solving the three decoupled first order ODEs

$$
\left(\frac{\partial W_i}{\partial \lambda_i}\right)^3 = \frac{\partial W_i}{\partial \lambda_i}(c_2\lambda_i + a_1) + (\lambda_i^4 + c_1\lambda_i^2 + a_2\lambda_i + a_3), \;\; i = 1,2,3. \tag{5.18}
$$

6 Concluding remarks

In this review paper we presented a multi-Hamiltonian separability theory of finite dimensional dynamical systems, in the frame of the set of canonical coordinates. It reveals the important fact that the multi-Hamiltonian property of considered system is closely related to its separability. Actually, we presented the constructive method of finding a separation coordinates once having a bi(multi)-Hamiltonian representation of the underlying dynamical system, written down in some natural canonical coordinates. There is still an open question: whether each bi-Hamiltonian chain with sufficient number of constants of motion guarantees a separability of underlying Hamiltonian systems? Or, in other words:

whether the arbitrary degenerate Poisson pencil, written in noncanonical coordinates, can be restricted to the nondegenerate one on a symplectic leaf of one of the Poisson tensors from the pencil? The second important problem is how to perform effectively such a Marden-Ratiu projection if it is possible? Some progress in this direction was made recently [41] but this part of separability theory still requires further investigations.

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