

# Generalized Operator Yang-Baxter Equations, Integrable ODEs and Nonassociative Algebras

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*Received December 27, 1999; Revised March 9, 2000; Accepted March 10, 2000*

## Abstract

Reductions for systems of ODEs integrable via the standard factorization method (the Adler-Kostant-Symes scheme) or the generalized factorization method, developed by the authors earlier, are considered. Relationships between such reductions, operator Yang-Baxter equations, and some kinds of non-associative algebras are established.

## 1 Introduction

The factorization method (see [1]) is used to integrate a system of ODEs of the form

$$q_t = [q_+, q], \quad q(0) = q_0. \quad (1.1)$$

Here  $q(t)$  belongs to a Lie algebra  $\mathfrak{G}$ , which is decomposed (as a vector space) into a direct sum

$$\mathfrak{G} = \mathfrak{G}_+ \oplus \mathfrak{G}_- \quad (1.2)$$

of its subalgebras  $\mathfrak{G}_+$  and  $\mathfrak{G}_-$ . The projection of  $q$  onto  $\mathfrak{G}_+$  parallel to  $\mathfrak{G}_-$  is denoted by  $q_+$ . For simplicity, we assume that  $\mathfrak{G}$  is embedded into a matrix algebra.

The solution of (1.1) can be written in the form

$$q(t) = A(t)q_0A^{-1}(t). \quad (1.3)$$

The matrix-valued function  $A(t)$  in (1.3) is defined from the following factorization problem

$$A^{-1}B = \exp(-q_0t), \quad A \in G_+, \quad B \in G_-, \quad (1.4)$$

where  $G_+$  and  $G_-$  are the Lie groups of  $\mathfrak{G}_+$  and  $\mathfrak{G}_-$ . It is well known that for small  $t$  the functions  $A(t)$  and  $B(t)$  are uniquely determined. If the groups  $G_+$  and  $G_-$  are algebraic then the conditions

$$A \in G_+, \quad A \exp(-q_0 t) \in G_-$$

are equivalent to a system of algebraic equations for the entries of  $A(t)$ .

It follows from (1.3) that if initial data  $q_0$  for (1.1) belongs to a  $\mathfrak{G}_+$ -module  $M$ , then  $q(t) \in M$  for any  $t$ . Such a specialization of (1.1) gives rise to a system which we call  **$M$ -reduction**.

Let us denote by  $R : M \rightarrow \mathfrak{G}_+$  the projector onto  $\mathfrak{G}_+$  parallel to  $\mathfrak{G}_-$ . In terms of  $R$  the  $M$ -reduction can be written as follows

$$m_t = [R(m), m], \quad m \in M. \quad (1.5)$$

As we demonstrate in Section 2, in important cases related to  $\mathbb{Z}_2$ -graded Lie algebras, the operator  $R$  satisfies one of the generalized operator Yang-Baxter equations (see [2]-[4]). Vice versa, if  $R : \mathfrak{A} \rightarrow \mathfrak{A}$  satisfies one of several kinds of operator Yang-Baxter equation on a Lie algebra  $\mathfrak{A}$ , then equation (1.5) can be regarded as an  $\mathfrak{A}$ -reduction of (1.1) for some  $\mathfrak{G} \supseteq \mathfrak{A}$ .

The component form of (1.5) is a system of quadratic ODEs. There are a lot of works where such systems are studied by the methods based on the analysis of singularities of solutions (see [5],[6]). We believe that the concrete systems that appear in Section 2 as examples could be used for examining some of the conjectures in the literature about singularity analysis. It would be interesting also to investigate which types of singularities are possible for systems integrable by the generalized factorization method [7].

In Sections 3 and 4 we investigate properties of an algebraic operation  $*$  on  $M$ , defined by

$$m * n = [R(m), n]. \quad (1.6)$$

In terms of this operation the  $M$ -reduction (1.5) takes the form

$$m_t = m * m. \quad (1.7)$$

It turns out that the operation  $*$  can be described by algebraic identities for several examples of interest. Some generalizations of non-associative algebras such as Lie, left-symmetric and Jordan algebras naturally arise there.

In Section 5 we show that, besides (1.7), the factorization method can be applied to the following equation

$$m_{tt} = 2m * m_t + m_t * m - m * (m * m) + c_1(m_t - m * m) + c_2 m, \quad (1.8)$$

where  $c_i \in \mathbb{C}$ . In a particular case when  $*$  is a Lie operation, (1.8) has been linearized in [7].

Moreover, equations (1.7) (see [7]) and (1.8) can be reduced to linear equations with variable coefficients in the following more general case. Let (1.2) be a vector space decomposition, where  $\mathfrak{G}_+$  and  $\mathfrak{G}_-$  are not subalgebras as before, but vector spaces satisfying the following conditions

$$[\mathfrak{H}, \mathfrak{G}_+] \subset \mathfrak{G}_+, \quad [\mathfrak{H}, \mathfrak{G}_-] \subset \mathfrak{G}_-, \quad (1.9)$$

with

$$\mathfrak{H} = [\mathfrak{G}_+, \mathfrak{G}_+]_- + [\mathfrak{G}_-, \mathfrak{G}_-]_+. \quad (1.10)$$

It is easy to verify that  $\mathfrak{H}$  is a subalgebra in  $\mathfrak{G}$ . If  $\mathfrak{G}_+$  and  $\mathfrak{G}_-$  are subalgebras, then  $\mathfrak{H} = \{0\}$  and the conditions (1.9) become trivial.

As usual, the  $M$ -reduction is defined by a vector space  $M$  such that

$$[M, \mathfrak{G}_+] \subset M. \quad (1.11)$$

**Example 1.** Let us consider the case  $\mathfrak{G} = sl(n)$ ,  $\mathfrak{G}_+ = so(n)$ . Given an expansion  $n = m_1 + m_2 + \dots + m_k$ , we denote by  $\mathfrak{G}_-$  a vector space of block matrices  $a = \{a_{ij}\}$  whose entries  $a_{ij}$  are  $m_i \times m_j$ -matrices such that all  $a_{ii}$  are symmetric and  $a_{ij} = 0$  if  $i > j$ . It is easy to verify that  $\mathfrak{H}$  coincides with the Lie algebra of block-diagonal skew-symmetric matrices and (1.9) is fulfilled. One can take for  $M$  the set of all symmetric matrices from  $sl(n)$ .

In a particular case  $m_1 = m_2 = \dots = m_n = 1$  the vector space  $\mathfrak{G}_-$  coincides with the Lie algebra of all upper-triangular matrices,  $\mathfrak{H} = \{0\}$  and we deal with an ordinary decomposition of  $sl(n)$  into a direct sum of two subalgebras.

## 2 $M$ -reductions and operator Yang-Baxter equations

### 2.1 The case of $\mathbb{Z}_2$ -graded Lie algebras

Let

$$\mathfrak{G} = \mathfrak{G}_0 \oplus \mathfrak{G}_1 \quad (2.1)$$

be a  $\mathbb{Z}_2$ -graded Lie algebra. In other words, the following commutation relations

$$[\mathfrak{G}_0, \mathfrak{G}_0] \subset \mathfrak{G}_0, \quad [\mathfrak{G}_0, \mathfrak{G}_1] \subset \mathfrak{G}_1, \quad [\mathfrak{G}_1, \mathfrak{G}_1] \subset \mathfrak{G}_0$$

hold.

Let us consider an  $M$ -reduction (1.5), where  $\mathfrak{G}_+ = \mathfrak{G}_0$ ,  $M = \mathfrak{G}_1$ . It is clear that

$$\mathfrak{G}_- = \{m - R(m) | m \in M\}. \quad (2.2)$$

One can show that  $\mathfrak{G}_-$  is a subalgebra iff  $R$  satisfies the modified Yang-Baxter equation

$$R([R(X), Y] - [R(Y), X]) - [R(X), R(Y)] - [X, Y] = 0, \quad X, Y \in \mathfrak{G}_1. \quad (2.3)$$

It is important to note that in our case  $R$  is an operator defined on  $\mathfrak{G}_1$  and acting from  $\mathfrak{G}_1$  to  $\mathfrak{G}_0$ , whereas usually (see [4])  $R$  is required to be an operator on  $\mathfrak{G}$ .

It also can be shown that  $\mathfrak{G}_-$  satisfies (1.9) iff

$$[D(X, Y), R(Z)] = R([D(X, Y), Z]) \quad X, Y, Z \in M,$$

where

$$D(X, Y) = R([R(X), Y] - [R(Y), X]) - [R(X), R(Y)] - [X, Y], \quad X, Y \in \mathfrak{G}_1.$$

It is clear that for any  $\mathfrak{G}_+$ -invariant subspace  $V \in M$  the corresponding  $V$ -reduction can be solved, as well as the original  $M$ -reduction (1.5), by the factorization method. This means that if  $M$  is reducible then (1.5) admits further reductions. It is natural to expect that the most interesting reductions are related to the cases when  $M$  is irreducible with respect to  $\mathfrak{G}_+$ -action.

The structure of corresponding  $\mathbb{Z}_2$ -graded Lie algebras is described in the following statement (see [8]).

**Theorem 1.** *Let  $\mathfrak{G}$  be a  $\mathbb{Z}_2$ -graded Lie algebra. The representation of  $\mathfrak{G}_0$  on  $\mathfrak{G}_1$  is both faithful and irreducible if and only if  $\mathfrak{G}$  belongs to one of the following three types:*

- (1)  $\mathfrak{G}$  is a simple Lie algebra;
- (2)  $\mathfrak{G} = \mathfrak{F} \oplus \mathfrak{F}$ , with  $\mathfrak{F}$  being a simple Lie algebra and  $\mathfrak{G}_0 = \{(X, X)\}$ ,  $\mathfrak{G}_1 = \{(X, -X)\}$ ;
- (3)  $[\mathfrak{G}_1, \mathfrak{G}_1] = 0$  and the action of  $\mathfrak{G}_0$  on  $\mathfrak{G}_1$  is faithful and irreducible.

It follows from (2.3) that in case (3) of Theorem 1 the operator  $R$  satisfies the classical Yang-Baxter equation

$$R([R(X), Y] - [R(Y), X]) - [R(X), R(Y)] = 0 \quad (2.4)$$

on  $\mathfrak{G}_1$ . Under the additional condition that  $\mathfrak{G}_1 = \mathfrak{G}_0^*$  the last equation has been investigated in [2],[3]. The corresponding equation (1.5) on  $\mathfrak{G}_0^*$  is related to a decomposition of the double  $\mathfrak{G}_0 \oplus \mathfrak{G}_0^*$  into a sum of two subalgebras and can be solved by the factorization method.

In case (2) there exists the one-to-one correspondence between complementary subalgebras (2.2) and constant solutions of the modified Yang-Baxter equation (2.3) on  $\mathfrak{F}$ . Equation (1.5) on  $\mathfrak{F}$  was treated in [4] without a consideration of the double  $\mathfrak{F} \oplus \mathfrak{F}$ , but as we have seen above it can be solved by the standard factorization method on this double.

For all interesting examples in case 1 we have  $\mathfrak{G}_1 \neq \mathfrak{G}_0^*$ . Such non-hamiltonian  $\mathfrak{G}_1$ -reductions do not seem to be considered before.

Below we present several concrete examples of equations (1.5), related to different cases of Theorem 1.

**Example 2.** The Lie algebra  $\mathfrak{G} = sl(3)$  admits decomposition (2.1), where

$$\mathfrak{G}_0 = \left\{ \begin{pmatrix} a & c & 0 \\ d & b & 0 \\ 0 & 0 & -a-b \end{pmatrix} \right\}; \quad \mathfrak{G}_1 = \left\{ \begin{pmatrix} 0 & 0 & P \\ 0 & 0 & Q \\ R & S & 0 \end{pmatrix} \right\}.$$

Apparently, we are under conditions of Theorem 1, case (1). Let us choose the complementary subalgebra  $\mathfrak{G}_-$  as follows

$$\mathfrak{G}_- = \left\{ \begin{pmatrix} -Y + X + \alpha U & X & Y - X \\ -Z + (2 - 3\alpha)U & (1 - 2\alpha)U & Z + (3\alpha - 2)U \\ -Y + X + U & X & Y - X + (\alpha - 1)U \end{pmatrix} \right\},$$

where  $\alpha$  is a (complex) parameter.

In this case, (1.5) yields the following system of ODEs

$$\begin{cases} P_t = P^2 - RP - QS \\ Q_t = (\beta - 2)RQ + \beta PQ \\ R_t = R^2 - RP - QS \\ S_t = (3 - \beta)RS + (1 - \beta)PS \end{cases}, \quad \beta = 3\alpha, \quad (2.5)$$

with respect to entries of the matrix

$$U = \left\{ \begin{pmatrix} 0, & 0, & P \\ 0, & 0, & Q \\ R, & S, & 0 \end{pmatrix} \right\}.$$

It follows from (1.1) that  $I_1 = \text{tr } U^2 = RP + QS$  is a first integral for (2.5). Other first integrals of (1.1) which can be obtained in a standard way [1] become trivial under the  $M$ -reduction. Nevertheless it is not hard to integrate (2.5) by quadratures. Two additional first integrals are of the form

$$I_2 = \frac{P - R}{QS}, \quad I_3 = Q^{1-\beta} S^{-\beta} (R^2 - RP - QS).$$

It is interesting to note that for generic  $\beta$  the system (2.5) probably does not pass the Painlevé-Kovalevskaya test.

**Example 3.** Let us consider a  $\mathbb{Z}_2$ -graded Lie algebra  $\mathfrak{G} = \mathfrak{G}_0 \oplus \mathfrak{G}_1$ , where

$$\mathfrak{G}_0 = \left\{ \begin{pmatrix} a, & c, & 0, & 0 \\ b, & -a, & 0, & 0 \\ 0, & 0, & a, & c \\ 0, & 0, & b, & -a \end{pmatrix} \right\}; \quad \mathfrak{G}_1 = \left\{ \begin{pmatrix} a, & c, & 0, & 0 \\ b, & -a, & 0, & 0 \\ 0, & 0, & -a, & -c \\ 0, & 0, & -b, & a \end{pmatrix} \right\}.$$

It is clear that  $\mathfrak{G}$  is isomorphic to  $sl(2) \oplus sl(2)$  and, as a  $\mathbb{Z}_2$ -graded Lie algebra, belongs to the class (2) of Theorem 1. The top, corresponding (up to scaling) to

$$\mathfrak{G}_- = \left\{ \begin{pmatrix} \alpha X, & Y, & 0, & 0 \\ 0, & -\alpha X, & 0, & 0 \\ 0, & 0, & -X, & 0 \\ 0, & 0, & Z, & X \end{pmatrix} \right\}, \quad \alpha \neq -1,$$

is given by

$$\begin{cases} P_t = RQ \\ Q_t = PQ \\ R_t = \alpha RP. \end{cases} \quad (2.6)$$

The functions

$$I_1 = QR - \frac{\alpha + 1}{2} P^2, \quad I_2 = RQ^{-\alpha}$$

are first integrals for (2.6).

**Example 4.** Let us take

$$\left\{ \begin{pmatrix} *, & *, & *, & * \\ *, & *, & *, & * \\ *, & *, & *, & * \\ 0, & 0, & 0, & 0 \end{pmatrix} \right\},$$

for  $\mathfrak{G}$ . It is clear that  $\mathfrak{G} = \mathfrak{G}_0 \oplus \mathfrak{G}_1$ , where

$$\mathfrak{G}_0 = \left\{ \begin{pmatrix} *, & *, & *, & 0 \\ *, & *, & *, & 0 \\ *, & *, & *, & 0 \\ 0, & 0, & 0, & 0 \end{pmatrix} \right\}; \quad \mathfrak{G}_1 = \left\{ \begin{pmatrix} 0, & 0, & 0, & P \\ 0, & 0, & 0, & Q \\ 0, & 0, & 0, & R \\ 0, & 0, & 0, & 0 \end{pmatrix} \right\}.$$

Obviously, the algebra  $\mathfrak{G}$  belongs to the class (3) of Theorem 1. Let  $\mathfrak{G}_+ = \mathfrak{G}_0$  and

$$\mathfrak{G}_- = \left\{ \begin{pmatrix} c, & \lambda c, & a, & a \\ -\lambda c, & c, & b, & b \\ a, & b, & c, & c \\ 0, & 0, & 0, & 0 \end{pmatrix} \right\}.$$

Since  $\mathfrak{G}_-$  is not a subalgebra, we have to find the vector space  $H$  using (1.10). A simple calculation shows that

$$\mathfrak{H} = \left\{ \begin{pmatrix} 0, & -d, & 0, & 0 \\ d, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{pmatrix} \right\},$$

and the conditions (1.9) are fulfilled. The corresponding equation (1.5) is (up to scaling) of the form

$$\begin{cases} P_t = 2PR + \lambda QR \\ Q_t = 2QR - \lambda PR \\ R_t = P^2 + Q^2 + R^2. \end{cases}$$

The last equation can be linearized with the help of the generalized factorization method [7].

## 2.2 One more version of the operator Yang-Baxter equation

In [9] (see also [10]) one more version

$$R([R(X), Y] - [R(Y), X]) - [R(X), R(Y)] - R^2([X, Y]) = 0 \quad (2.7)$$

of the operator Yang-Baxter equation on a Lie algebra  $\mathfrak{F}$  has been considered. In a particular case when  $\mathfrak{F}$  is the infinite dimensional Lie algebra of vector fields, (2.7) means that the Frölicher-Nijenhuis tensor [11] of the affiner  $R$  is equal to zero.

It turns out (see [9]) that equation (1.5), where  $R : \mathfrak{F} \rightarrow \mathfrak{F}$  satisfies (2.7), can be also solved with the help of the factorization method. Namely, let  $\mathfrak{G}$  be a (finite-dimensional)

factor-algebra of the polynomial Lie algebra  $\mathfrak{F}[\lambda]$  with respect to an ideal, generated by the minimal polynomial of the operator  $R$ . For the vector space  $M$  and the subalgebras  $\mathfrak{G}_+$  and  $\mathfrak{G}_-$  we take, correspondingly, the images of  $\lambda\mathfrak{F}$  and subalgebras  $\mathfrak{F}$  and

$$\left\{ \sum_{i=1}^m \lambda^i (a_i - \lambda^{-1} R(a_i)) \mid a_i \in \mathfrak{F}, \quad m > 0 \right\}$$

under the canonical homomorphism. Then the  $M$ -reduction of (1.1) coincides with the equation under consideration.

It was shown in [9], that if a Lie algebra  $\mathfrak{G}$  is decomposed into a direct sum of subalgebras  $\mathfrak{G}_1, \dots, \mathfrak{G}_k$  such that the sum of any two subalgebras is a subalgebra, then the following operator  $R = \lambda_1 \pi_1 + \dots + \lambda_k \pi_k$ , where  $\lambda_i \in \mathbb{C}$  and  $\pi_i$  denotes the projector onto  $\mathfrak{G}_i$ , satisfies (2.7).

**Example 5.** It easy to verify that for a decomposition  $sl(2) = \mathfrak{G}_1 \oplus \mathfrak{G}_2 \oplus \mathfrak{G}_3$ , where

$$\mathfrak{G}_1 = \left\{ \begin{pmatrix} a, & 0 \\ 0, & -a \end{pmatrix} \right\}; \quad \mathfrak{G}_2 = \left\{ \begin{pmatrix} b, & 2b \\ 0, & -b \end{pmatrix} \right\}, \quad \mathfrak{G}_3 = \left\{ \begin{pmatrix} -c, & 0 \\ 2c, & c \end{pmatrix} \right\},$$

all these conditions hold. Let us define the operator  $R$  as follows  $R = \lambda_1 \pi_1 + \lambda_2 \pi_2 + \lambda_3 \pi_3$ . Then the corresponding equation (1.5) for entries of the matrix

$$m = \begin{pmatrix} P, & Q \\ R, & -P \end{pmatrix}$$

has the form

$$\begin{cases} P_t = (\nu - \mu)QR \\ Q_t = 2\mu PQ + \mu Q^2 + \nu QR \\ R_t = -2\nu PR - \nu R^2 - \mu QR, \end{cases}$$

with  $\mu = \lambda_2 - \lambda_1$ ,  $\nu = \lambda_3 - \lambda_1$ . A series of examples generalizing Example 5, has been presented in [9].

### 3 Nonassociative algebras and $M$ -reductions

We associate with each  $N$ -dimensional algebra  $\mathfrak{A}$ , defined by the structural constants  $C_{jk}^i$ , a top-like system of ODEs of the form

$$u_t^i = C_{jk}^i u^j u^k, \quad i, j, k = 1, \dots, N. \quad (3.1)$$

Here and in the sequel we assume that summation from 1 to  $N$  is carried out over repeated indices. In terms of the  $\mathfrak{A}$ -operation  $*$  the system (3.1) takes the form

$$U_t = U * U, \quad (3.2)$$

where  $U(t)$  is an  $\mathfrak{A}$ -valued function. The system (3.1) will be called the  $\mathfrak{A}$ -top.

One of our purposes is to investigate relationships between properties of  $\mathfrak{A}$  and the integrability of (3.1). Recall that in this paper the term "integrable" means integrable by the factorization or generalized factorization method.

### 3.1 Left-symmetric algebras

Let us consider the left-symmetric tops. Recall that an algebra  $\mathfrak{A}$  is said to be **left-symmetric** (see [12]-[15]) if the multiplication  $*$  in  $\mathfrak{A}$  satisfies the identity

$$As(X, Y, Z) = As(Y, X, Z), \quad (3.3)$$

where

$$As(X, Y, Z) = (X * Y) * Z - X * (Y * Z).$$

**Example 6.** Given a vector  $C$ , a vector space  $\mathfrak{V}$  with the operation

$$X * Y = (X, Y)C + (X, C)Y,$$

where  $(\cdot, \cdot)$  is the ordinary dot product, gives us an example of a left-symmetric algebra (see [16]).

**Example 7.** Let  $R : \mathfrak{G} \rightarrow \mathfrak{G}$  be a solution of the classical operator Yang-Baxter equation (2.4) on a Lie algebra  $\mathfrak{G}$ . Then  $\mathfrak{G}$  is a left-symmetric algebra with respect to the operation (1.6).

**Example 8.** If  $\mathfrak{A}$  is an associative algebra and  $R : \mathfrak{A} \rightarrow \mathfrak{A}$  satisfies the modified Yang-Baxter equation (2.3), then the operation

$$a * b = ab + ba + [R(a), b]$$

is a left-symmetric one.

**Example 9.** The previous formula can be generalized to the case of arbitrary Jordan algebra  $J$  with a multiplication  $\circ$ . Let  $Der(J)$  be the Lie algebra of all derivations of  $J$  and  $R : J \rightarrow Der(J)$ , an operator satisfying (2.3) in the structural algebra  $Lie(J)$  (see [17]). Then

$$a * b = a \circ b + R(a)(b)$$

is a left-symmetric operation on  $J$ .

Let  $\mathfrak{A}$  be a left-symmetric algebra. It follows from (3.3) that the operation  $[X, Y] = X * Y - Y * X$  is a Lie bracket. Moreover, it is easy to verify that the vector space  $\mathfrak{G} = \mathfrak{A} \oplus \mathfrak{A}$  is a Lie algebra with respect to the bracket

$$[(g_1, a_1), (g_2, a_2)] = ([g_1, g_2], g_1 * a_2 - g_2 * a_1). \quad (3.4)$$

The Lie algebra  $\mathfrak{G}$  admits a  $\mathbb{Z}_2$ -gradation:  $\mathfrak{G} = \mathfrak{G}_0 \oplus \mathfrak{G}_1$ , where  $\mathfrak{G}_0 = \{(a, 0)\}$ ,  $\mathfrak{G}_1 = \{(0, b)\}$ . Since  $[\mathfrak{G}_1, \mathfrak{G}_1] = \{0\}$ , we deal with case 3 of Theorem 1.

It follows from (3.4) that  $\mathfrak{G}_+ = \mathfrak{G}_1$  and  $\mathfrak{G}_- = \{(a, -a)\}$  are subalgebras in  $\mathfrak{G}$ . Equation (1.1), corresponding to the decomposition  $\mathfrak{G} = \mathfrak{G}_+ \oplus \mathfrak{G}_-$  has the form

$$V_t = U * V - V * U, \quad U_t = V * U + U * U, \quad (3.5)$$

where  $q = (V, U)$ . To obtain (3.2) as  $M$ -reduction of (3.5) we can choose  $M = \mathfrak{G}_1$  (i.e. put  $V = 0$ ).



### 3.2 $G$ -algebras and associated tops

In this subsection we consider algebras with the identities <sup>1</sup>

$$[X, Y, Z] + [Y, Z, X] + [Z, X, Y] = 0, \quad (3.6)$$

$$V * [X, Y, Z] = [V * X, Y, Z] + [X, V * Y, Z] + [X, Y, V * Z], \quad (3.7)$$

where

$$[X, Y, Z] = As(X, Y, Z) - As(Y, X, Z). \quad (3.8)$$

We call them  $G$ -algebras. It is clear that left-symmetric algebras belong to the class of  $G$ -algebras.

**Example 10.** Evidently, any matrix  $X$  can be decomposed into a sum of a skew-symmetric matrix  $X_+$  and an upper-triangular matrix  $X_-$ . Let us consider the operation (cf. (1.6))  $X * Y = [X_+, Y]$ . One can verify that the set of all symmetric matrices equipped with this operation is a  $G$ -algebra.

#### Theorem 2.

- (1) Let  $\mathfrak{G} = \mathfrak{G}_0 \oplus \mathfrak{G}_1$  be a  $\mathbb{Z}_2$ -graded Lie algebra. Given a splitting (1.2), where  $\mathfrak{G}_+ = \mathfrak{G}_0$  and  $\mathfrak{G}_-$  is a Lie subalgebra, we define an algebraic structure on  $\mathfrak{G}_1$  by the formula

$$X * Y = [X_+, Y], \quad (3.9)$$

where  $X_+$  denotes the projection of  $X$  onto  $\mathfrak{G}_+$  parallel to  $\mathfrak{G}_-$ . Then  $\mathfrak{G}_1$  is a  $G$ -algebra with respect to (3.9).

- (2) Any  $G$ -algebra can be obtained from a suitable  $\mathbb{Z}_2$ -graded Lie algebra by the above construction.

**Proof.** To establish the identity (3.6) it suffices to show that the operation

$$X \times Y = [X_+, Y] - [Y_+, X], \quad X, Y \in \mathfrak{G}_1$$

satisfies the Jacobi identity. In accordance with (1.2), any element  $X \in \mathfrak{G}$  can be uniquely represented as  $X = X_+ + X_-$ , where  $X_+ \in \mathfrak{G}_+$ ,  $X_- \in \mathfrak{G}_-$ . Let  $X, Y \in \mathfrak{G}_1$ . Using the formula

$$[X_-, Y_-] = [X - X_+, Y - Y_+] = [X, Y] + [X_+, Y_+] + [Y_+, X] - [X_+, Y],$$

and the fact that  $\mathfrak{G}_-$  is a Lie algebra we conclude that

$$[X_-, Y_-]_{\mathfrak{G}_1} = Y \times X, \quad (3.10)$$

where the index " $\mathfrak{G}_1$ " on the left hand side means the projection onto  $\mathfrak{G}_1$  parallel to  $\mathfrak{G}_0$ . If we apply (3.10) to project the Jacobi identity for  $X_-, Y_-, Z_- \in \mathfrak{G}_-$  onto  $\mathfrak{G}_1$ , we make certain that the Jacobi identity for the operation " $\times$ " holds.

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<sup>1</sup>The identity (3.6) means that the operation  $[X, Y] = X * Y - Y * X$  is a Lie bracket.

In order to prove (3.7) we need the following relation

$$[X, Y, Z] = [[X, Y], Z], \quad X, Y, Z \in \mathfrak{G}_1, \quad (3.11)$$

where the left hand side is defined by (3.8). It can be checked by a direct calculation. Rewriting (3.7) in terms of  $\mathfrak{G}$ -bracket with the help of (3.11) we see that (3.7) follows from the Jacobi identity for  $\mathfrak{G}$ .

It remains to prove the second part of Theorem 2. Let  $\mathfrak{G}_1$  be a  $G$ -algebra. Put

$$\mathfrak{G} = \mathfrak{G}_0 \oplus \mathfrak{G}_1, \quad (3.12)$$

where  $\mathfrak{G}_0$  is a Lie algebra generated by all operators of left multiplication in  $\mathfrak{G}_1$ . Recall that the left multiplication operator  $L_X$  is defined as follows  $L_X : \mathfrak{G}_1 \rightarrow X * \mathfrak{G}_1$ . The vector space  $\mathfrak{G}$  becomes a Lie algebra if we define

$$[(A, X), (B, Y)] = ([A, B] - [L_X, L_Y] + L_{X*Y} - L_{Y*X}, A(Y) - B(X)). \quad (3.13)$$

Obviously, the bracket (3.13) is skew-symmetric. One can easily show that the identities (3.6), (3.7) are equivalent to the Jacobi identity for (3.13). It follows from (3.13) that the decomposition (3.12) is a  $\mathbb{Z}_2$ -gradation. To define a decomposition (1.2) we take for  $\mathfrak{G}_-$  the set  $\{(-L_X, X)\}$  and  $\mathfrak{G}_0$  for  $\mathfrak{G}_+$ . As we see from (3.13),  $\mathfrak{G}_-$  is a subalgebra in  $\mathfrak{G}$ . For  $\mathfrak{G}_-$  and  $\mathfrak{G}_+$  thus defined, (3.9) is of the form  $(0, X) * (0, Y) = [(L_X, 0), (0, Y)]$ . This relation is fulfilled according to (3.13). ■

Note that in Example 10,  $\mathfrak{G}_0$ ,  $\mathfrak{G}_1$ ,  $\mathfrak{G}_-$ , and  $\mathfrak{G}$  are the sets of skew-symmetric, symmetric, upper-triangular and all matrices, respectively.

The decompositions (1.2) from Examples 2 and 3 lead to structures of  $G$ -algebras on the corresponding vector spaces  $\mathfrak{G}_1$ .

## 4 The tops associated with $SS$ -algebras

### 4.1 A relationship between $SS$ -algebras and $\mathbb{Z}_2$ -graded Lie algebras

An algebra with identity

$$[V, X, Y * Z] - [V, X, Y] * Z - Y * [V, X, Z] = 0, \quad (4.1)$$

where  $[X, Y, Z]$  is defined by (3.8), will be called an  $SS$ -algebra.

**Remark.** It follows from (4.1) that for any  $SS$ -algebra  $\mathfrak{A}$  the operator

$$K_{YZ} = [L_Y, L_Z] - L_{Y*Z} + L_{Z*Y}$$

is a derivation of  $\mathfrak{A}$  for any  $Y, Z$ . As before,  $L_X$  denotes the operator of left-multiplication by  $X$ .

Note that associative, left-symmetric, Jordan and Lie algebras are  $SS$ -algebras.

**Theorem 3.**

- (1) Let  $\mathfrak{G} = \mathfrak{G}_0 \oplus \mathfrak{G}_1$  be a  $\mathbb{Z}_2$ -graded Lie algebra, such that  $[\mathfrak{G}_1, \mathfrak{G}_1] = 0$ . Given a vector space decomposition  $\mathfrak{G} = \mathfrak{G}_+ \oplus \mathfrak{G}_-$  with  $\mathfrak{G}_+ = \mathfrak{G}_0$  and a vector space  $\mathfrak{G}_-$  satisfying (1.9), let us equip  $\mathfrak{G}_1$  with an algebraic structure with the help of the formula (3.9). Then the operation  $*$  satisfies the identity (4.1).
- (2) Any *SS*-algebra  $\mathfrak{A}$  can be obtained from a suitable  $\mathbb{Z}_2$ -graded Lie algebra by the above construction.

**Proof.** We do not give a complete proof of Theorem 3. Its first part can be proved in the same manner as the first part of Theorem 2. We explain only how to construct  $\mathfrak{G}$ ,  $\mathfrak{G}_+$ ,  $\mathfrak{G}_-$  for a given *SS*-algebra. We take for  $\mathfrak{G}_+$  the Lie algebra  $End \mathfrak{A}$  of all linear endomorphisms of  $\mathfrak{A}$ . The vector space  $\mathfrak{G} = (End \mathfrak{A}) \oplus \mathfrak{A}$  becomes a  $\mathbb{Z}_2$ -graded Lie algebra if we define

$$[(A, X), (B, Y)] = ([A, B], A(Y) - B(X)).$$

It is not difficult to show that (4.1) implies that

- a) the vector space  $\mathfrak{H}$  generated by all elements of the form

$$([L_Y, L_Z] - L_{Y*Z} + L_{Z*Y}, 0)$$

is a Lie subalgebra in  $\mathfrak{G}$ , and

- b) the vector space  $\mathfrak{G}_- = \{(-L_X, X)\}$  and defined above subalgebras  $\mathfrak{G}_+$  and  $\mathfrak{H}$  satisfy (1.9). ■

**4.2 Low-dimensional *SS*-algebras**

Straightforward computations, using the identity (4.1), prove the following

**Theorem 4.** *Up to change of basis  $e_1, e_2$ , every non-trivial two-dimensional *SS*-algebra is equivalent to one of the following algebras (for a fixed value of non-removable parameter  $\lambda$ ):*

$$\begin{aligned} \mathfrak{A}_1 : \quad & e_1 * e_1 = e_1, \quad e_1 * e_2 = (1 - \lambda)e_1 + \lambda e_2, \\ & e_2 * e_1 = \lambda e_1 + (1 - \lambda)e_2, \quad e_2 * e_2 = 0; \\ \mathfrak{A}_2 : \quad & e_1 * e_1 = e_1, \quad e_1 * e_2 = \lambda e_2, \\ & e_2 * e_1 = 0, \quad e_2 * e_2 = 0; \\ \mathfrak{A}_3 : \quad & e_1 * e_1 = e_1, \quad e_1 * e_2 = \lambda e_2, \\ & e_2 * e_1 = e_2, \quad e_2 * e_2 = 0; \\ \mathfrak{A}_4 : \quad & e_1 * e_1 = 0, \quad e_1 * e_2 = \lambda e_1, \\ & e_2 * e_1 = e_1, \quad e_2 * e_2 = e_1 + (\lambda + 1)e_2; \\ \mathfrak{A}_5 : \quad & e_1 * e_1 = 0, \quad e_1 * e_2 = e_1, \\ & e_2 * e_1 = -e_1, \quad e_2 * e_2 = 0; \\ \mathfrak{A}_6 : \quad & e_1 * e_1 = e_1, \quad e_1 * e_2 = 0, \\ & e_2 * e_1 = 0, \quad e_2 * e_2 = e_2; \\ \mathfrak{A}_7 : \quad & e_1 * e_1 = e_1, \quad e_1 * e_2 = e_1 + 2e_2, \end{aligned}$$

$$\begin{aligned} & e_2 * e_1 = 2e_1 + e_2, \quad e_2 * e_2 = e_2; \\ \mathfrak{A}_8 : \quad & e_1 * e_1 = 0, \quad e_1 * e_2 = e_2, \\ & e_2 * e_1 = 0, \quad e_2 * e_2 = 0; \end{aligned}$$

**Comments.** If  $\lambda = 0, 1$  then  $\mathfrak{A}_1$  is an associative algebra. For  $\lambda = 1/2$  it is a Jordan algebra.

The algebras  $\mathfrak{A}_2$  and  $\mathfrak{A}_3$  are associative at  $\lambda = 0, 1$ .

$\mathfrak{A}_4$  is a left-symmetric algebra for  $\lambda = 0$  and is an LT-algebra (see [17]) if  $\lambda = 1$ .

$\mathfrak{A}_5$  is a Lie algebra.

$\mathfrak{A}_6$  is a commutative associative algebra.

The algebras  $\mathfrak{A}_7$  and  $\mathfrak{A}_8$  are left-symmetric.

The most non-trivial  $\mathfrak{A}$ -top

$$\frac{dX}{dt} = X^2 + 3XY, \quad \frac{dY}{dt} = 3XY + Y^2$$

corresponds to  $\mathfrak{A}_7$ .

For dimension 3, we present two one-parameter families of  $SS$ -algebras with the multiplication table of the form

$$\begin{aligned} e_1 * e_2 &= \alpha e_3, & e_2 * e_1 &= \beta e_3, & e_1 * e_3 &= \gamma e_1, \\ e_3 * e_1 &= \delta e_1, & e_2 * e_3 &= \varepsilon e_2, & e_3 * e_2 &= \mu e_2, & e_3 * e_3 &= \nu e_3. \end{aligned}$$

The first is given by

$$\alpha = \lambda, \quad \beta = 2 - \lambda, \quad \gamma = 0, \quad \delta = 2, \quad \varepsilon = 0, \quad \mu = -2, \quad \nu = 0.$$

The second is equivalent to the  $SS$ -algebra defined by the decomposition from Example 4. This corresponds to

$$\alpha = 1, \quad \beta = 1, \quad \gamma = \lambda, \quad \delta = 2 - \lambda, \quad \varepsilon = \lambda, \quad \mu = 3\lambda - 2, \quad \nu = \lambda.$$

## 5 Integrable second order ODEs, related to $M$ -reductions

Let (1.2) be a decomposition of a Lie algebra  $\mathfrak{G}$  such that the vector subspaces  $\mathfrak{G}_+$  and  $\mathfrak{G}_-$  satisfy (1.9),(1.10) and  $M$  be a vector subspace satisfying (1.11).

It turns out that the following initial problem

$$X(0) = X_0, \quad Y(0) = Y_0$$

for equation

$$X_t = X * X + \alpha X + \beta Y, \quad Y_t = X * Y + \gamma X + \delta Y, \quad X, Y \in M \quad (5.1)$$

where  $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ ,  $\beta \neq 0$ , reduces to linear equations with variable coefficients by the generalized factorization method [7]. As usual, the operation  $*$  is defined by the formula (3.9).

Note, that expressing  $Y$  from the first equation and substituting into the second one can reduce (5.1) to the following second order equation

$$X_{tt} = 2X * X_t + X_t * X - X * (X * X) + c_1(X_t - X * X) + c_2X,$$

where  $c_1 = \alpha + \delta$ ,  $c_2 = \beta\gamma - \alpha\delta$ .

At first let us solve an auxiliary system

$$U_t = \alpha U + \beta V, \quad V_t = [U, V] + \gamma U + \delta V, \quad (5.2)$$

with initial data

$$U(0) = X_0, \quad V(0) = Y_0.$$

Here  $U, V \in \mathfrak{G}$ . It is clear that this system is equivalent to

$$U_{tt} = [U, U_t] + c_1 U_t + c_2 U. \quad (5.3)$$

The following linearization procedure for (5.3) was presented in [7]. Let  $Q(t) \in \mathfrak{G}$  be a solution of the linear equation

$$Q_{tt} = c_1 Q_t + c_2 Q,$$

where  $Q(0) = U(0)$ ,  $Q_t(0) = U_t(0)$ . Let us define  $Y(t)$  as a solution of the initial problem

$$Y_t = YQ(t), \quad Y(0) = E.$$

Then  $U(t) = YQ(t)Y^{-1}$  is a solution of (5.3).

Let us look for a solution of (5.1) in the form

$$X = A(t)U(t)A^{-1}(t), \quad Y = A(t)V(t)A^{-1}(t),$$

where  $A(t)$  satisfies the equation

$$A_t = - (AU(t)A^{-1})_- A, \quad A(0) = E \quad (5.4)$$

on the Lie group  $G$  of the Lie algebra  $\mathfrak{G}$ . Here  $U(t)$ ,  $V(t)$  is already known solution of the system (5.2), and "-" means the projection onto  $\mathfrak{G}_-$  parallel to  $\mathfrak{G}_+$ .

If  $\mathfrak{G}_+$  and  $\mathfrak{G}_-$  are subalgebras, then (5.4) can be reduced to the following factorization problem (cf. (1.4)):

$$C(t) = A^{-1}(t)B(t),$$

where  $C_t = U(t)C$ ,  $A(0) = B(0) = C(0) = E$ ,  $A(t)$  and  $B(t)$  belong to the Lie groups of the Lie algebras  $\mathfrak{G}_-$  and  $\mathfrak{G}_+$ .

In more general case, when  $\mathfrak{G}_-$  and  $\mathfrak{G}_+$  are vector spaces and conditions (1.9) are fulfilled, the equation (5.4) has been linearized in [7].

## Acknowledgements

The authors thank M.A. Semenov-Tian-Shansky and G. Marmo for fruitful discussions. The first author (I.G) was supported by the Russian Fund for Basic Research (grant 99-01-00294). The second author (V.S) was supported, in part, by RFBR grant 99-01-00294, INTAS project 99-1782 and the Research Programme of the Carlos III University, Madrid (ref. 00993). He is grateful to the Mathematics Department at this University and personally to Alberto Ibort Latre for their hospitality.

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