

Quantized W-algebra of $\mathfrak{sl}(2, 1)$ and Quantum Parafermions of $U_q(\hat{\mathfrak{sl}}(2))$

Jintai DING [†] and Boris FEIGIN [‡]

[†] Department of Mathematical Sciences, University of Cincinnati, USA

[‡] Landau Institute, Moscow, Russia

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Abstract

In this paper, we establish the connection between the quantized W-algebra of $\mathfrak{sl}(2, 1)$ and quantum parafermions of $U_q(\hat{\mathfrak{sl}}(2))$ that a shifted product of the two quantum parafermions of $U_q(\hat{\mathfrak{sl}}(2))$ generates the quantized W-algebra of $\mathfrak{sl}(2, 1)$.

1 Introduction

The Lie algebra $\hat{\mathfrak{sl}}(2)$ has a current realization that is given by three current operators $e(z)$, $h(z)$ and $f(z)$. For $\hat{\mathfrak{sl}}(2)$, when we consider the case of bosonization of the current operator, the operators $e(z)$ and $f(z)$ can be decomposed in the following way:

$$\begin{aligned} e(z) &= v^+(z)\Psi^+(z), \\ f(z) &= v^-(z)\Psi^-(z), \end{aligned}$$

where $\Psi^\pm(z)$ are operators that commute with $h(z)$ and $v^\pm(z)$ are vertex operators generated by the Heisenberg algebra of $h(z)$. These two operators $\Psi^\pm(z)$ are called parafermions.

For the quantum affine algebra $U_q(\hat{\mathfrak{sl}}(2))$, which is a q-deformation of the universal enveloping algebra of $\hat{\mathfrak{sl}}(2)$, Drinfeld presented a loop realization of affine quantum groups with current generators [1]. This, for $U_q(\hat{\mathfrak{sl}}(2))$, gives us the quantized current operators corresponding to $e(z)$, $h(z)$ and $f(z)$ of $\hat{\mathfrak{sl}}(2)$, which are $X^+(z)$, $\varphi(z)$, $\psi(z)$ and $X^-(z)$. Here $\varphi(z)$ and $\psi(z)$ correspond to the positive and negative half of $h(z)$ respectively. For $U_q(\hat{\mathfrak{sl}}(2))$, similarly when we consider the case of bosonization of the current operator, the operators $X^+(z)$ and $X^-(z)$ can be decomposed in the following way:

$$\begin{aligned} X^+(z) &= V^+(z)\Phi^+(z), \\ X^-(z) &= V^-(z)\Phi^-(z), \end{aligned}$$

where $\Phi^\pm(z)$ are operators that commute with $\varphi(z)$, $\psi(z)$ and $V^\pm(z)$ are vertex operators generated by the Heisenberg algebra of $\varphi(z)$, $\psi(z)$ [2][3][4].

For a given affine Lie (super) algebra $\hat{\mathfrak{g}}$, there exist different ways to construct the associated W-algebra, which is an extended conformal algebra. One is the well-known Sugawara construction of Casimir type. Another is the Drinfeld Sokolov reduction. There is also another type of construction coming from the description of the associated W-algebra as the algebra generated by the current operators which commutes with the screening operators associated to $\hat{\mathfrak{g}}$. This is evident for the case $g = \mathfrak{sl}(2)$, where the construction produces the Virasoro algebra. The key idea is to consider the screening operators associated to the root of \mathfrak{g} in terms of bosonization, then we try to find the simplest current operators that commute with those screen operators also in terms of bosonization. It turns out that it also gives a W-algebra. Such a construction was first quantized for the case of Virasoro, which gives us the quantized Virasoro algebra[13] For the case of $g = \mathfrak{sl}(N)$, such a construction is also utilized in the case of constructing quantized W-algebra associated to $\mathfrak{sl}(N)$ [5]. Recently, following the idea of the description of quantum W-algebra using the quantization of screening operators [5], we derived the quantized W-algebra of $\mathfrak{sl}(2, 1)$ [6]. The generator of this algebra $l_1(z)$ is bosonized as a sum of three vertex operators, while the screening operators are a set of two fermions or a set of one quantized screening operators of $\mathfrak{sl}(2)$ and a fermion. For these two different sets of screen operators, they actually produce equivalent $l_1(z)$ [6], where $l(z)$ is given as an uniquely determined operator that commutes with the two screening operators up-to a total difference. We see clearly [6], in terms of the description of one quantized screening operators of $\mathfrak{sl}(2)$ and a fermion, the degeneration of the two screen operators, when q goes to 1, are exactly the screening operators associated to the Lie super algebra $\widehat{\mathfrak{sl}}(2, 1)$. This is the reason the algebra generated by $l_1(z)$ is called the quantized W-algebra of $\mathfrak{sl}(2, 1)$. The degeneration of this operator $l_1(z)$, when q goes to 1, still commutes with the screening operators, and gives us the associated W-algebra of $\mathfrak{sl}(2, 1)$.

It is known in mathematics as a folklore that there is connection between the W-algebra corresponding to the Lie super algebra $\widehat{\mathfrak{sl}}(2, 1)$ and these two parafermions. Namely that in the operator product expansion of $\Psi^-(w)\Psi^+(z)$, one of the components gives us a realization of the W-algebra of $\mathfrak{sl}(2, 1)$, namely if we look at the bosonized $\Psi^-(w)\Psi^+(z)$'s components in terms of product expansion, the corresponding component commutes with two fermions, which is equivalent to the screening operators of $\mathfrak{sl}(2, 1)$, which are equivalent as shown in [6] for the quantized case.

In this paper, we will use the bosonization formulas in [2] to establish again the connection between the quantized W-algebra of $\mathfrak{sl}(2, 1)$ and the two quantum parafermions. We show that the operator $l(z)$ can be identified with the operator $\int_C \Phi^-(w)\Phi^+(z)dw$, where the k is the central element or say level and the contour C is around the point $w = zq^{k+2}$. These extend the classical theory respectively.

In Section 2, we will introduce the quantized W-algebra of $\mathfrak{sl}(2, 1)$ and some basic facts. In Section 3 of this paper, we will introduce the basic theory about the algebra $U_q(\widehat{\mathfrak{sl}}(2))$ and bosonization and quantum parafermions. In Section 4, we will show the connection between the quantized W-algebra of $\mathfrak{sl}(2, 1)$ and the quantum parafermions of $U_q(\widehat{\mathfrak{sl}}(2))$, which is obtained as a corollary of our previous work [6].

2 Quantized W-algebra of $\mathfrak{sl}(2, 1)$

In this section, we introduce the definition of the quantized W-algebra of $\mathfrak{sl}(2, 1)$ presented in [6]. This is mainly from the last section of [6].

Definition 2.1. The Heisenberg algebra $H_{q,p}(2)$ is an associative algebra with generators $a_i[n], n \in \mathbb{Z}$, and the commutation relations of the generators are:

$$[a_i[n], a_j[m]] = \frac{1}{n} A_{ij}(n) \delta_{n,-m},$$

which is defined on the field of the rational functions of p and q , two generic parameters with $p, q \in \mathbb{C}^*, |q| < 1$, $A_{i,i} = 1$ and $A_{ij}(n) = a(n)$ is a rational function of p and q for $i \neq j$.

Let μ be an element of a two dimensional space A_2 generated by α_i , for $i = 1, 2$. Let α_i^* be the generator of its dual space A_2^* , such that $\alpha_i^*(\alpha_i) = 2$ and $\alpha_i^*(\alpha_j) = -b/\beta_i$. Let π_μ be the Fock representation of $H_{q,p}(2)$ generated by a vector $v_\mu (\mu \in A_2^*)$, such that $a_i[n]v_\mu = 0, n > 0$, and $a_1[0]v_\mu = \mu(\alpha_1)v_\mu$. We assume here β_1 and β_2 generic.

Introduce operators $Q_i, i=1,2$, which satisfy commutation relations $[a_i[n], Q_i] = 2\beta_i \delta_{n,0}$, $[a_j[n], Q_i] = -b\delta_{n,0}$. The operators e^{Q_i} act from π_μ to $\pi_{\mu+\beta\alpha_i}$.

We define two quantized screening currents as the generating function

$$S_i^+(z) = e^{Q_i} z^{s_i^+[0]} : \exp \left(\sum_{m \neq 0} s_i^+(m) a_i[m] z^{-m} \right) :,$$

for $i = 1, 2$, where $s_i^+[m]$ are in the polynomial ring of p, q over \mathbb{C}^* for $m \neq 0$ and $s_i^+[0] = a_i[0]$ and by $::$ we mean the normal ordered product expansion.

We impose the following equalities as an assumption on these two current operators.

Assumption:

$$S_i^+(z) S_i^+(w) = (z - w) : S_i^+(z) S_i^+(w) :,$$

$$\begin{aligned} S_2^+(z) S_1^+(w) &= z^{-b} f_{2,1}(z, w) : S_2^+(z) S_1^+(w) : \\ &= z^{-b} \exp \left(\exp 1/n \sum_{m>0} A_{2,1} s_2^+(m) s_1^+(-m) w^m / z^m \right) : S_2^+(z) S_1^+(w) :, \end{aligned}$$

$$\begin{aligned} S_1^+(z) S_2^+(w) &= z^{-b} f_{1,2}(z, w) : S_1^+(z) S_2^+(w) : \\ &= z^{-b} \exp \left(\exp \sum_{m>0} A_{1,2}(m) s_2^+(m) s_1^+(-m) w^m / z^m \right) : S_1^+(z) S_2^+(w) :, \end{aligned}$$

and

$$\lim_{q \rightarrow 1} f_{(1,2)}(z, w) = \lim_{q \rightarrow 1} f_{(2,1)}(z, w) = (1 - w/z)^b.$$

As defined in Definition 2.1, q is a complex number, which allows us to take the limit above.

Clearly $S_1^+(z)$ and $S_2^+(z)$ are two fermions, which are not the inverse of each other.

Let $l_1(z)$ be a current operator in the form:

$$l_1(z) = \Lambda_1(z) + \Lambda_2(z) + \Lambda_3(z),$$

where $\Lambda_i(z)$ are the generating functions:

$$\Lambda_i(z) = g_i : \exp \left(\sum \lambda_{ij}(m) a_j[m] z^{-m} \right) :,$$

$\lambda_i[m]$ are in $\mathbb{C}[p, q]$ for $i = 1, 2, 3$, $g_1 = 1$, the integration contour is around 0; and $l_1(z)$ commutes with the action of the quantized screening operators $\int S_i^+(z) dz/z$.

Here we can actually also define $l_1(z)$ even by a stronger condition that it commutes with the operators $S_i^+(z)$ up-to a total difference.

The main goal is to find out if such an operator actually exists; and if it exists, we would like to find out if it is unique.

In this case, the simplest situation that we have is to impose the following assumptions on the correlation functions.

Assumption: The correlation functions between $S_i^+(z)$ and $\Lambda_j(w)$ are 1, for the two pairs $i = 1, j = 3$, and $i = 2, j = 1$, which also means that for either pair of the operators, they commute with each other. The correlation functions between $S_1^+(z)$ and $\Lambda_1(z)$ satisfy the condition that the two products $\Lambda_1(z)S_1^+(w)$ and $S_1^+(w)\Lambda_1(z)$ have the same correlation functions and

$$\begin{aligned} \Lambda_1(z)S_1^+(w) &= A \frac{(z-w)}{(z-wpq^{-1})} : \Lambda_1(z)S_1^+(w) :, & |z| \gg |w|, \\ S_1^+(w)\Lambda_1(z) &= A \frac{(z-w)}{(z-wpq^{-1})} : \Lambda_1(z)S_1^+(w) :, & |w| \gg |z|, \end{aligned}$$

and

$$A = pq^{-1}.$$

Proposition 2.1 ([6]). *The correlation functions of the products $\Lambda_1(z)S_1^+(w)$ and $S_1^+(w)\Lambda_1(z)$ must be equal and the correlation functions must have only one pole and one zero. For some number A' ,*

$$\begin{aligned} \Lambda_2(z)S_1^+(w) &= A' \frac{(z-wp'_1)}{(z-wp'_2)} : \Lambda_2(z)S_1^+(w) :, & |z| \gg |w|, \\ S_1^+(w)\Lambda_2(z) &= A' \frac{(z-wp'_1)}{(z-wp'_2)} : \Lambda_2(z)S_1^+(w) :, & |w| \gg |z|, \end{aligned}$$

$$A'p'_1/p'_2 = 1.$$

$$A(1 - p_1/p_2)p_2 : \Lambda_1(z)S_1^+(zp_2^{-1}) := -A'(1 - p'_1/p'_2)p'_2 : \Lambda_2(z)S_1^+(zp_2'^{-1}) : .$$

Let $p = p'_2$ and $q = p'_2/p_2$, then $p'_1 = 1$, $g_2 = pq^{-1}(pq^{-1} - 1)$.

Let

$$\begin{aligned} S_i^+(z)\Lambda_j(w) &= S\Lambda_{ij}(z, w) : S_i^+(z)\Lambda_j(w) :, \\ \Lambda_i(z)S_j^+(w) &= \Lambda S_{ij}(z, w) : S_i^+(z)\Lambda_j(w) : . \end{aligned}$$

We also impose the following **Assumption**: for some number B ,

$$\begin{aligned}\Lambda_2(z)S_2^+(w) &= B \frac{(z-wq_1)}{(z-wq_2)} : \Lambda_2(z)S_2^+(w) :, & |z| \gg |w|, \\ S_2^+(w)\Lambda_2(z) &= B \frac{(z-wq_1)}{(z-wq_2)} : \Lambda_2(z)S_2^+(w) :, & |w| \gg |z|.\end{aligned}$$

Then, we have

Proposition 2.2. *The correlation functions of the products $\Lambda_3(z)S_2^+(w)$ and $S_2^+(w)\Lambda_3(z)$ must be equal and the correlation functions must have only one pole and one zero. For some number B' ,*

$$\begin{aligned}\Lambda_3(z)S_2^+(w) &= B' \frac{(z-wq_1)}{(z-wq_2)} : \Lambda_3(z)S_2^+(w) :, & |z| \gg |w|, \\ S_2^+(w)\Lambda_3(z) &= B' \frac{(z-wq_1)}{(z-wq_2)} : \Lambda_3(z)S_2^+(w) :, & |w| \gg |z|, \\ B'(1-q_1/q_2)q_2 : \Lambda_2(z)S_2^+(zq_2^{-1}) &:= -B'(1-q_1/q_2)q_2' : \Lambda_3(z)S_2^+(zq_2'^{-1}) :, \\ B'q_1'/q_2' &= 1.\end{aligned}$$

Let $p' = q_2'/q_2$.

Theorem 2.3 ([6]). *The operator $l_1(z)$ exists and is uniquely determined if and only if*

$$\begin{aligned}q' &= q \\ q_2 &= q_1p, \\ \Lambda S_{2,1}(z, w) &= A \frac{f_{2,1}(zq_2'^{-1}, w)}{f_{2,1}(zq_2^{-1}, w)} = \frac{f_{1,2}(w, zq_2'^{-1})}{f_{1,2}(w, zq_2^{-1})}, \\ A^{-1}\Lambda S_{2,1}(wq_2', z) &= S\Lambda_{2,2}(zp^{-1}, w).\end{aligned}$$

Proposition 2.4. *If $l_1(z)$ exists, then*

$$\begin{aligned}f_{2,1}(z, w) &= \frac{(w/z|q_2'^{-1}pq, q)_\infty}{(w/z|q_2'^{-1}q, q)_\infty}, \\ f_{1,2}(w, z) &= \frac{(z/w|q_2'p^{-1}, q)_\infty}{(z/w|q_2', q)_\infty}.\end{aligned}$$

As explained in [6], we know that the number b actually does not affect the commutation relation of $l(z)$ with itself at all. So this family of operators related to the parameter b are actually just a kind of rescaling. We call the associative algebra generated by the Fourier components of the operator $l_1(z)$, **the quantized \mathbf{W} -algebra of $\mathfrak{sl}(2, 1)$** .

3 $U_q(\mathfrak{sl}(2))$, its bosonization and quantum parafermions

For the definition, we will use directly the current realization of $U_q(\widehat{\mathfrak{sl}}(2))$ given by Drinfeld[1].

Definition 3.1. The algebra $U_q(\widehat{\mathfrak{sl}}(2))$ is an associative algebra with unit 1 and the generators: $\varphi(m), \psi(-m), X^\pm(l)$, for $i = 1, \dots, n-1, l \in \mathbb{Z}$ and $m \in \mathbb{Z}_{\leq 0}$ and a central element c . Let z be a formal variable and $X^\pm(z) = \sum_{l \in \mathbb{Z}} X^\pm(l)z^{-l}$, $\varphi(z) = \sum_{m \in \mathbb{Z}_{\leq 0}} \varphi(m)z^{-m}$ and $\psi(z) = \sum_{m \in \mathbb{Z}_{\geq 0}} \psi(m)z^{-m}$. In terms of the formal variables, the defining relations are

$$\begin{aligned} \varphi(0)\psi(0) &= \psi(0)\varphi(0) = 1, \\ \varphi(z)\varphi(w) &= \varphi(w)\varphi(z), \\ \psi(z)\psi(w) &= \psi(w)\psi(z), \\ \varphi(z)\psi(w)\varphi(z)^{-1}\psi(w)^{-1} &= \frac{g(z/wq^{-c})}{g(z/wq^c)}, \\ \varphi(z)X^\pm(w)\varphi(z)^{-1} &= g(z/wq^{\mp\frac{1}{2}c})^{\pm 1}X^\pm(w), \\ \psi(z)X^\pm(w)\psi(z)^{-1} &= g(w/zq^{\mp\frac{1}{2}c})^{\mp 1}X^\pm(w), \\ [X^+(z), X^-(w)] &= \frac{1}{q - q^{-1}} \left\{ \delta(z/wq^{-c})\psi(wq^{\frac{1}{2}c}) - \delta(z/wq^c)\varphi(zq^{\frac{1}{2}c}) \right\}, \\ (z - q^{\pm a}w)X^\pm(z)X^\pm(w) &= (q^{\pm a}z - w)X^\pm(w)X^\pm(z), \end{aligned}$$

where

$$\delta(z) = \sum_{k \in \mathbb{Z}} z^k, \quad g(z) = \frac{q^a z - 1}{z - q^a} \quad (\text{expanded around } z = 0), \quad a = 2.$$

For this current realization, Drinfeld also gave the Hopf algebra structure [7].

We first introduce the bosonic representation of $U_q(\widehat{\mathfrak{sl}}(2))$ given in [2]. We will use the notation from [2].

Let $\{\alpha_n, \bar{\alpha}_n, \beta_n | n \in \mathbb{Z}\}$ be a set of operators satisfying the following commutation relations:

$$\begin{aligned} [\alpha_m, \alpha_{-m}] &= \frac{[2m][km]}{2}, \\ [\bar{\alpha}_m, \bar{\alpha}_{-m}] &= -\frac{[2m][km]}{2}, \\ [\beta_m, \beta_{-m}] &= \frac{[2m][(k+2)m]}{2}. \end{aligned}$$

The other commutators are zero. These operators form the direct sum of three Heisenberg algebras.

We define

$$\begin{aligned} N_+ &= \mathcal{C}[\alpha_m, \bar{\alpha}_m, \beta_m]_{m>0}, \\ N_- &= \mathcal{C}[\alpha_m, \bar{\alpha}_m, \beta_m]_{m<0}. \end{aligned}$$

The left Fock module F_{l, m_1, m_2} is uniquely characterized by the following properties: there

exists a vector $|l, m_1, m_2 \rangle$ in F_{l, m_1, m_2} such that

$$\begin{aligned} \beta_0 |l, m_1, m_2 \rangle &= 2l |l, m_1, m_2 \rangle, \\ \alpha_0 |l, m_1, m_2 \rangle &= 2m_1 |l, m_1, m_2 \rangle, \\ \bar{\alpha}_0 |l, m_1, m_2 \rangle &= -2m_2 |l, m_1, m_2 \rangle, \\ N_+ |l, m_1, m_2 \rangle &= 0, \text{ and} \\ N_- |l, m_1, m_2 \rangle &\text{ is a free } N_- \text{-module of rank 1.} \end{aligned}$$

For each triple of complex numbers r, s_1 and s_2 , we define the operator $e^{2r\beta+2s_1\alpha+2s_2\bar{\alpha}}$ by the mapping $|l, m_1, m_2 \rangle$ to $|l+r, m_1+s_1, m_2+s_2 \rangle$ such that it commutes with the action of N_{\pm} . The normal ordering $: * :$ is defined according to $\alpha < \alpha_0, \bar{\alpha} < \bar{\alpha}_0, \beta < \beta_0$ and $N_- < N_+$.

Consider the operators $X^{\pm}(z) : F_{l, m_1, m_2} \rightarrow F_{l, m_1 \pm 1, m_2 \pm 1}$ defined by

$$\begin{aligned} X^+(z) &= \frac{1}{(q-q^{-1})} : Y^+(z) \left\{ Z_+(q^{-\frac{k+2}{2}}z)W_+(q^{-\frac{k}{2}}z) - W_-(q^{\frac{k}{2}}z)Z_-(q^{\frac{k+2}{2}}z) \right\} : \\ &= \frac{1}{(q-q^{-1})} (X_1^+(z) - X_2^+(z)), \\ X^-(z) &= \frac{-1}{(q-q^{-1})} : Y^-(z) \left\{ Z_+(q^{\frac{k+2}{2}}z)W_+(q^{\frac{k}{2}}z)^{-1} - W_-(q^{-\frac{k}{2}}z)^{-1}Z_-(q^{-\frac{k+2}{2}}z) \right\} : \\ &= \frac{1}{(q-q^{-1})} (X_1^-(z) - X_2^-(z)), \end{aligned}$$

where

$$\begin{aligned} Y^+(z) &= \exp \left\{ \sum_{m=1}^{\infty} q^{-\frac{km}{2}} \frac{z^m}{[km]} (\alpha_{-m} + \bar{\alpha}_{-m}) \right\} \\ &\quad e^{2(\alpha+\bar{\alpha})} z^{\frac{1}{k}(\alpha_0+\bar{\alpha}_0)} \exp \left\{ - \sum_{m=1}^{\infty} q^{-\frac{km}{2}} \frac{z^{-m}}{[km]} (\alpha_m + \bar{\alpha}_m) \right\}, \\ Y^-(z) &= \exp \left\{ - \sum_{m=1}^{\infty} q^{\frac{km}{2}} \frac{z^m}{[km]} (\alpha_{-m} + \bar{\alpha}_{-m}) \right\} \\ &\quad e^{-2(\alpha+\bar{\alpha})} z^{-\frac{1}{k}(\alpha_0+\bar{\alpha}_0)} \exp \left\{ \sum_{m=1}^{\infty} q^{\frac{km}{2}} \frac{z^{-m}}{[km]} (\alpha_m + \bar{\alpha}_m) \right\}, \\ Z_+(z) &= \exp \left\{ -(q-q^{-1}) \sum_{m=1}^{\infty} z^{-m} \frac{[m]}{[2m]} \bar{\alpha}_2 \right\} q^{-\frac{1}{2}\bar{\alpha}_0}, \\ Z_-(z) &= \exp \left\{ (q-q^{-1}) \sum_{m=1}^{\infty} z^2 \frac{[m]}{[2m]} \bar{\alpha}_{-m} \right\} q^{\frac{1}{2}\bar{\alpha}_0}, \end{aligned}$$

$$W_+(z) = \exp \left\{ -(q - q^{-1}) \sum_{m=1}^{\infty} z^{-m} \frac{[m]}{[2m]} \beta_2 \right\} q^{-\frac{1}{2}\beta_0},$$

$$W_-(z) = \exp \left\{ (q - q^{-1}) \sum_{m=1}^{\infty} z^2 \frac{[m]}{[2m]} \beta_{-m} \right\} q^{\frac{1}{2}\beta_0}.$$

The action of those operators are well defined as explained in [2] which give us a bosonization of $U_q(\widehat{\mathfrak{sl}}(2))$ of level k .

First we define two new operators:

$$\bar{Y}^+(z) = \exp \left\{ \sum_{m=1}^{\infty} q^{-\frac{km}{2}} \frac{z^m}{[km]} (\bar{\alpha}_{-m}) \right\}$$

$$e^{2(\bar{\alpha})} z^{\frac{1}{k}(\bar{\alpha}_0)} \exp \left\{ - \sum_{m=1}^{\infty} q^{-\frac{km}{2}} \frac{z^{-m}}{[km]} (\bar{\alpha}_m) \right\},$$

$$\bar{Y}^-(z) = \exp \left\{ - \sum_{m=1}^{\infty} q^{\frac{km}{2}} \frac{z^m}{[km]} (\bar{\alpha}_{-m}) \right\}$$

$$e^{-2(\bar{\alpha})} z^{-\frac{1}{k}(\bar{\alpha}_0)} \exp \left\{ \sum_{m=1}^{\infty} q^{\frac{km}{2}} \frac{z^{-m}}{[km]} (\bar{\alpha}_m) \right\},$$

Let

$$V^+(z) = \exp \left\{ \sum_{m=1}^{\infty} q^{-\frac{km}{2}} \frac{z^m}{[km]} (\alpha_{-m}) \right\}$$

$$e^{2(\alpha)} z^{\frac{1}{k}(\alpha_0)} \exp \left\{ - \sum_{m=1}^{\infty} q^{-\frac{km}{2}} \frac{z^{-m}}{[km]} (\alpha_m) \right\},$$

$$V^-(z) = \exp \left\{ - \sum_{m=1}^{\infty} q^{\frac{km}{2}} \frac{z^m}{[km]} (\alpha_{-m}) \right\}$$

$$e^{-2(\alpha)} z^{-\frac{1}{k}(\alpha_0)} \exp \left\{ \sum_{m=1}^{\infty} q^{\frac{km}{2}} \frac{z^{-m}}{[km]} (\alpha_m) \right\},$$

We define two current operators $\Phi^{\pm}(z)$ as:

Definition 3.2.

$$\Phi^+(z) = \frac{1}{(q - q^{-1})} : \bar{Y}^+(z) \left\{ Z_+(q^{-\frac{k+2}{2}} z) W_+(q^{-\frac{k}{2}} z) - W_-(q^{\frac{k}{2}} z) Z_-(q^{\frac{k+2}{2}} z) \right\} :,$$

$$\Phi^-(z) = \frac{1}{(q - q^{-1})} : \bar{Y}^-(z) \left\{ Z_+(q^{\frac{k+2}{2}} z) W_+(q^{\frac{k}{2}} z)^{-1} - W_-(q^{-\frac{k}{2}} z)^{-1} Z_-(q^{-\frac{k+2}{2}} z) \right\} : .$$

We, then, have:

$$\begin{aligned} X^+(z) &= V^+(z)\Phi^+(z), \\ X^-(z) &= V^-(z)\Phi^-(z). \end{aligned}$$

Here $V^\pm(z)$ commutes with $\Phi^\pm(z)$. Then we know that the two operators $\Phi^\pm(z)$ give us bosonic realization of the quantum parafermions associated to $U_q(\widehat{\mathfrak{sl}}(2))$ [3][4].

4 Main results

We write down the correlation functions between the vertex operators.

$$\begin{aligned} \bar{Y}^+(z)\bar{Y}^-(w) &=: \bar{Y}^+(z)\bar{Y}^-(w) : \exp \left\{ \Sigma \frac{-(w/z)^m [2m]}{[km]} \right\} z^{4/k} \\ &=: \bar{Y}^+(z)\bar{Y}^-(w) : \exp \left\{ \Sigma (w/z)^m \frac{1 - q^{(k-2)m} + q^{(k+2)m}}{m(1 - q^{2km})} \right\} z^{4/k}, \\ \bar{Y}^-(z)\bar{Y}^+(w) &=: \bar{Y}^-(z)\bar{Y}^+(w) : \exp \left\{ \Sigma \frac{-(w/z)^m [2m]}{[km]} \right\} z^{4/k} \\ &=: \bar{Y}^-(z)\bar{Y}^+(w) : \exp \left\{ \Sigma (w/z)^m \frac{1 - q^{(k-2)m} + q^{(k+2)m}}{m(1 - q^{2km})} \right\} z^{4/k}. \end{aligned}$$

$$\bar{Y}^+(z)Z^+(w) =: \bar{Y}^+(z)Z^+(w) :,$$

$$\begin{aligned} Z^+(z)\bar{Y}^+(w) &=: Z^+(z)\bar{Y}^+(w) : \exp \left\{ \frac{\Sigma (w/z)^m (q - q^{-1}) q^{-\frac{km}{2}} [m]}{m} \right\} z^2 \\ &=: Z^+(z)\bar{Y}^+(w) : \exp \left\{ \frac{\Sigma (w/z)^m (q^{-\frac{(km-2m)}{2}} - q^{-\frac{(km+2m)}{2}})}{m} \right\} z^2. \end{aligned}$$

$$Z^-(z)\bar{Y}^+(w) =: Z^-(z)\bar{Y}^+(w) :,$$

$$\begin{aligned} \bar{Y}^+(z)Z^-(w) &=: Z^-(w)\bar{Y}^+(z) : \exp \left\{ \frac{\Sigma (w/z)^m (q - q^{-1}) q^{-\frac{km}{2}} [m]}{m} \right\} z^2 \\ &=: Z^-(w)\bar{Y}^+(z) : \exp \left\{ \frac{\Sigma (w/z)^m (q^{-\frac{(km-2m)}{2}} - q^{-\frac{(km+2m)}{2}})}{m} \right\} z^2. \end{aligned}$$

$$\bar{Y}^-(z)Z^+(w) =: \bar{Y}^-(z)Z^+(w) :,$$

$$\begin{aligned} Z^+(z)\bar{Y}^-(w) &=: Z^+(z)\bar{Y}^-(w) : \exp \left\{ \frac{-\Sigma (w/z)^m (q - q^{-1}) q^{\frac{km}{2}} [m]}{m} \right\} z^2 \\ &=: Z^+(z)\bar{Y}^-(w) : \exp \left\{ \frac{\Sigma (w/z)^m (q^{\frac{km-2m}{2}} - q^{\frac{km+2m}{2}})}{m} \right\} z^2. \end{aligned}$$

$$Z^-(z)\bar{Y}^-(w) =: Z^-(z)\bar{Y}^-(w) :,$$

$$\begin{aligned} \bar{Y}^-(z)Z^-(w) &=: \bar{Y}^-(z)Z^-(w) : \exp \left\{ \frac{-\Sigma(w/z)^m (q - q^{-1}) q^{\frac{km}{2}} [m]}{m} \right\} z^2 \\ &=: \bar{Y}^-(z)Z^-(w) : \exp \left\{ \frac{\Sigma(w/z)^m (q^{\frac{km-2m}{2}} - q^{\frac{km+2m}{2}})}{m} \right\} z^2. \end{aligned}$$

$$Z^-(z)Z^+(w) =: Z^-(z)Z^+(w) :,$$

$$\begin{aligned} Z^+(z)Z^-(w) &=: Z^+(z)Z^-(w) : \exp \left\{ \frac{\Sigma(w/z)^m (q - q^{-1})^2 [m]^2 [km]}{[2m]m} \right\} \\ &=: Z^+(z)Z^-(w) : \exp \left\{ \frac{\Sigma(w/z)^m (-q^{4m} - 1 + 2q^{2m})(q^{km} - q^{-km})}{(1 - q^{4m})m} \right\} \end{aligned}$$

$$W^-(z)W^+(w) =: W^-(z)W^+(w) :,$$

$$\begin{aligned} W^+(z)W^-(w) &=: W^+(z)W^-(w) : \exp \left\{ -\frac{\Sigma(w/z)^m (q - q^{-1})^2 [m]^2 [k + 2m]}{[2m]m} \right\} \\ &=: W^+(z)W^-(w) : \exp \left\{ \frac{\Sigma(w/z)^m (q^{4m} + 1 - 2q^{2m})(q^{km+2m} - q^{-km-2m})}{(1 - q^{4m})m} \right\} \end{aligned}$$

$$\begin{aligned} \Phi^-(w)\Phi^+(z) &= \frac{1}{(q - q^{-1})} \frac{1}{(q - q^{-1})} \\ &\times [: \bar{Y}^-(w)Z_+(q^{\frac{k+2}{2}}w)W_+(q^{\frac{k}{2}}w)^{-1} :: \bar{Y}^+(z)Z_+(q^{-\frac{k+2}{2}}z)W_+(q^{-\frac{k}{2}}z) : \\ &\quad - : \bar{Y}^-(w)W_-(q^{-\frac{k}{2}}w)^{-1}Z_-(q^{-\frac{k+2}{2}}w) :: \bar{Y}^+(z)Z_+(q^{-\frac{k+2}{2}}z)W_+(q^{-\frac{k}{2}}z) : \\ &\quad - : \bar{Y}^-(w)Z_+(q^{\frac{k+2}{2}}w)W_+(q^{\frac{k}{2}}w)^{-1} :: Y^+(z)W_-(q^{\frac{k}{2}}z)Z_-(q^{\frac{k+2}{2}}z) : \\ &\quad + : \bar{Y}^-(w)W_-(q^{-\frac{k}{2}}w)^{-1}Z_-(q^{-\frac{k+2}{2}}w) :: Y^+(z)W_-(q^{\frac{k}{2}}z)Z_-(q^{\frac{k+2}{2}}z) :] \\ &= \Sigma_{i=1, \dots, 4} V_i(w, z). \end{aligned}$$

Lemma 4.1. *The correlation function of $V_i(w, z)$, $i \neq 4$ has a first order pole at $w = q^{k+2}z$, but the correlation function of $V_4(w, z)$ has neither zero nor a pole at $w = q^{k+2}z$.*

This follows from the correlation function formulas above.

Let

$$L(z) = \int_C \Phi^-(w)\Phi^+(z)dw,$$

where the contour C is around the point $w = zq^{k+2}$.

Proposition 4.2. *The current operator $L(z)$ is in the form:*

$$L(z) = \lambda_1(z) + \lambda_2(z) + \lambda_3(z),$$

where $\lambda_i(z)$, for $i = 1, 2, 3$, is a vertex operator and

$$\begin{aligned}\lambda_1(z) &= \frac{1}{(q - q^{-1})^2} : \bar{Y}^-(q^{k+2}z)Z_+(q^{\frac{2k+6}{2}}z)W_+(q^{\frac{3k+4}{2}}z)^{-1}\bar{Y}^+(z)Z_+(q^{-\frac{k+2}{2}}z)W_+(q^{-\frac{k}{2}}z) : \\ \lambda_2(z) &= -\frac{1}{(q - q^{-1})^2} : \bar{Y}^-(q^{k+2}z)W_-(q^{\frac{k+4}{2}}z)^{-1}Z_-(q^{\frac{k+2}{2}}z)\bar{Y}^+(z)Z_+(q^{-\frac{k+2}{2}}z)W_+(q^{-\frac{k}{2}}z) : \\ \lambda_3(z) &= -\frac{1}{(q - q^{-1})^2} : \bar{Y}^-(q^{k+2}z)Z_+(q^{\frac{2k+6}{2}}z)W_+(q^{\frac{3k+4}{2}}z)^{-1}Y^+(z)W_-(q^{\frac{k}{2}}z)Z_-(q^{\frac{k+2}{2}}z) : .\end{aligned}$$

This follows from the lemma above.

From this, we know that $L(z)$ is a sum of three vertex operators.

Here we will present two screening operators basically introduced in [2]. We define the operator $S^\pm(z)$ as:

$$\begin{aligned}S^+(z) &= \exp \left\{ \sum_{m=1}^{\infty} z^m \frac{1}{[2m]} \left(q^{\frac{k}{2}m} \beta_{-m} + q^{\frac{k+2}{2}m} \bar{\alpha}_{-m} \right) \right\} \\ &\quad e^{(k+2)\bar{\beta} + k\bar{\alpha}} z^{\frac{1}{2}(\beta_0 + \bar{\alpha}_0)} \exp \left\{ - \sum_{m=1}^{\infty} z^{-m} \frac{1}{[2m]} \left(q^{\frac{k}{2}m} \beta_2 + q^{\frac{k+2}{2}m} \bar{\alpha}_m \right) \right\}. \\ S^-(z) &= \exp \left\{ \sum_{m=1}^{\infty} z^m \frac{1}{[2m]} \left(q^{-\frac{k}{2}m} \beta_{-m} - q^{-\frac{k+2}{2}m} \bar{\alpha}_{-m} \right) \right\} \\ &\quad e^{(k+2)\beta - k\bar{\alpha}} z^{\frac{1}{2}(\beta_0 - \bar{\alpha}_0)} \exp \left\{ - \sum_{m=1}^{\infty} z^{-m} \frac{1}{[2m]} \left(q^{-\frac{k}{2}m} \beta_2 - q^{-\frac{k+2}{2}m} \bar{\alpha}_m \right) \right\}.\end{aligned}$$

Lemma 4.3. *The correlation function of $S^\pm(z)S^\pm(z)$ are given as:*

$$S^+(z)S^+(w) =: S^+(z)S^+(w) : (z - w), \quad S^-(z)S^-(w) =: S^-(z)S^-(w) : (z - w).$$

$S^+(z)$ and $S^-(z)$ are fermions.

We can also calculate first all the correlation functions of the products of these two operators with component of $X^\pm(z)$, which are given as:

Lemma 4.4.

$$\begin{aligned}S^+(z)X_1^+(w) &=: S^+(z)X_1^+(w) : (z - wq)^{-1} \\ S^+(z)X_1^-(w) &=: S^+(z)X_1^-(w) : (z - wq^{k+1}) \\ X_2^+(w)S^+(z) &=: X_2^+(w)S^+(z) : (w - zq)^{-1} \\ X_2^-(w)S^+(z) &=: X_2^-(w)S^+(z) : (w - zq^{k+1}) \\ S^-(z)X_1^+(w) &=: S^-(z)X_1^+(w) : (z - wq^{-k-1}) \\ S^-(z)X_1^-(w) &=: S^-(z)X_1^-(w) : (z - wq^{-1})^{-1} \\ X_2^+(w)S^-(z) &=: X_2^+(w)S^-(z) : (w - zq^{-k-1}) \\ X_2^-(w)S^-(z) &=: X_2^-(w)S^-(z) : (w - zq^{-1})^{-1}\end{aligned}$$

As shown in [2], we have

Lemma 4.5.

$$\begin{aligned} X^+(z)S^+(w) &= -S^+(w)X^+(z) \sim \frac{\partial_q}{\partial_q w} \left\{ \frac{1}{z-w} Y^+(z) B_1^+(z) \right\}, \\ X^-(z)S^+(w) &= -S^+(w)X^-(z) \sim 0, \end{aligned}$$

where

$$\begin{aligned} B_1^+(z) &= \exp \left\{ \sum_{m=1}^{\infty} z^m \frac{1}{[2m]} \left(q^{\frac{k+2}{2}m} \beta_{-m} + q^{\frac{k+4}{2}m} \bar{\alpha}_{-m} \right) \right\} \\ &e^{(k+2)\beta + k\bar{\alpha}} z^{\frac{1}{2}(\beta_0 + \bar{\alpha}_0)} \exp \left\{ - \sum_{m=1}^{\infty} z^{-m} \frac{1}{[2m]} \left(q^{\frac{k+2}{2}m} \beta_m + q^{\frac{k+4}{2}m} \bar{\alpha}_m \right) \right\}, \end{aligned}$$

and

$$\frac{\partial_p}{\partial_p z} f(z) = \frac{f(p^{1/2}z) - f(p^{-1/2}z)}{(p^{1/2} - p^{-1/2})z},$$

for a function $f(z)$ on \mathbb{C}^* and a scalar p in \mathbb{C}^* .

Similarly we have

Lemma 4.6.

$$\begin{aligned} X^+(z)S^-(w) &= -S^-(w)X^+(z) \sim 0 \\ X^-(z)S^-(w) &= -S^-(w)X^-(z) \\ &\sim \frac{\partial_q}{\partial_q w} \left\{ \frac{1}{z-w} : Y^-(z) Z_+(q^{\frac{k+2}{2}}) W_+(q^{\frac{k}{2}}z) S^+(z) : \right\}. \end{aligned}$$

Proposition 4.7. $\lambda_2(z)$ commutes with $S^-(z)$,

$$\lambda_2(z)S^-(w) =: \lambda_2(z)S^-(w) :,$$

and $L(z)$ commutes with $S^-(z)$ up-to a total difference. $\lambda_3(z)$ commutes with $S^+(z)$,

$$\lambda_3(z)S^+(w) =: \lambda_3(z)S^+(w) :,$$

and $L(z)$ commutes with $S^+(z)$ up-to a total difference.

For two current operators $A(z)$ and $B(z)$, by that $A(z)$ commutes with $B(w)$ up-to a total difference we mean:

$$[A(z), b(w)] = r \left(\delta \left(\frac{z}{w} r_1 \right) A(z) - \delta \left(\frac{z}{w} r_2 \right) A(z) \right),$$

where r_1, r_2, r are non-zero constants and $A(z)$ is a current operator.

This follows from the lemmas above.

Theorem 4.8. The operator $L(z)$ gives a realization of the quantized W-algebra of $\mathfrak{sl}(2, 1)$.

This follows from the fact that the generator for the quantized W-algebra of $\mathfrak{sl}(2, 1)$ is uniquely determined by the fact that it commutes with two fermion up-to a total difference as explained in the section above. Because $S^+(z)$ and $S^-(z)$ are both fermions, which are not inverse of each other, therefore our $l(z)$ is exactly such an operator, which gives us a realization of the quantized W-algebra of $\mathfrak{sl}(2, 1)$.

In a subsequent paper, we plan to present the quantized W-algebra for $\mathfrak{sl}(m, n)$. For some of those algebras, we expect that we can build similar constructions using the operators, which comes from the quotient of the $\widehat{\mathfrak{sl}}(n+1)$ up to $\widehat{\mathfrak{gl}}(n)$. This should be related to the construction in [8].

On the other hand, the W-algebra structure constructed from the parafermions of $\widehat{\mathfrak{sl}}(2)$ was studied from a different point of view [10][11], where the connection with $\widehat{\mathfrak{sl}}(2, 1)$ was not utilized. It is a very interesting question to look at the results of [10][11] from the point view of the W-algebra of $\mathfrak{sl}(2, 1)$.

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