Discrete uninorms with smooth underlying operators

Daniel Ruiz-Aguilera[1] **Joan Torrens**[1]

[1]Mathematics and Computer Science Department, Universitat de les Illes Balears

Carretera de Valldemossa, km 7,5, 07122 Palma, Spain

Abstract

The class of discrete uninorms U such that their underlying t-norm T and t-conorm S are smooth is studied. The different cases combining when T is the minimum or the Łukasiewicz t-norm and S is the maximum or the Łukasiewicz t-conorm, are characterized and the number of discrete uninorms with these underlying operators is given. It is also studied the general case when T and/or S are smooth, but ordinal sums.

Keywords: Discrete uninorms, smoothness condition, t-norms, t-conorms

1. Introduction

Uninorms are a especial kind of binary aggregation functions that generalize both t-norms and t-conorms. They have proved to be useful in many application fields and this have lead to an extensive study of uninorms from the pure theoretical point of view. One of the most interesting topics in this direction deals with the characterization of the different classes of uninorms, mainly uninorms in $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$ [7], idempotent uninorms [1, 16, 11], representable uninorms [2, 7, 15], uninorms continuous in the open unit square [8, 5], compensatory uninorms [4], and even those uninorms with continuous underlying operators [6].

Uninorms are usually defined on the unit interval $[0, 1]$, but they can be generalized to other domains. In particular, uninorms defined on a finite chain have been studied by some authors. The interest of operators defined on a finite chain comes from their usefulness in problems where qualitative information is used. When data is qualitative, the fuzzy linguistic approach is a good tool to model the information because then, the qualitative terms used by experts are represented via linguistic variables instead of numerical values. In these cases, linguistic variables are often interpreted to take values on a totally ordered scale like

$$\mathcal{L} = \{\textit{Extremely Bad, Very Bad, Bad, Fair, Good, Very Good, Extremely Good}\}.$$

Then the representative finite chain $L_n = \{0, 1, \ldots, n\}$ is usually considered to model these linguistic hedges.

For this reason many papers dealing with operations defined on L_n, usually called discrete operations, have appeared in last years. Dealing with discrete operators, the smoothness condition is usually considered as the discrete counterpart of continuity. In fact, in the discrete framework this property is equivalent to the divisibility property as well as to the Lipschitz condition. Thus, many classes of aggregation functions with some smoothness condition have been studied and characterized. For instance, smooth discrete t-norms and t-conorms were characterized in [13, 14], uninorms in $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$ and nullnorms in [12], idempotent discrete uninorms in [3], weighted means in [10].

For the case of uninorms, it is well known that there are no smooth uninorms on L_n and so, the smoothness condition is only possible in some partial regions of L_n^2, like in the mentioned cases of discrete uninorms in $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$ or discrete idempotent uninorms. However, the general case of discrete uninorms having smooth underlying operators has not yet investigated and this is the main goal of this work. The paper is organized as follows. In Section 2 we give some preliminaries that will be used in the paper, and the main results are presented in Section 3. Since smooth t-norms (and t-conorms) are given by the minimum, the Łukasiewicz t-norm (the maximum or the Łukasiewicz t-conorm) or an ordinal sum of these two types of operators, we have divided our results in Section 3 in some subsections devoting a subsection for each possible case of the underlying t-norm and t-conorm of the corresponding uninorm. Section 4 gives general conclusions and future work.

2. Preliminaries

We suppose the reader to be familiar with some basic results on uninorms and their classes that can be found for instance in [4, 7, 8, 11, 16].

In these preliminaries we recall some known facts on uninorms defined on finite chains, that we will also refer to discrete uninorms. In these cases, the

concrete scale to be used is not determinant and the only important fact is the number of elements of the scale (see [14]). Thus, given any positive integer n, we will deal from now on with the finite chain

$$L_n = \{0, 1, 2, \ldots, n\}.$$

We will use indistinctly the interval notation $L_n = [0, n]$ and also the usual notations $[0, e]$ and $[e, n]$ when $e \in L_n$ for the corresponding subsets of L_n.

Definition 1 *A* uninorm *on L_n is a two-place function $U : L_n^2 \to L_n$ which is associative, commutative, increasing in each place and such that there exists some element $e \in L_n$, called* neutral element*, such that $U(e, x) = x$ for all $x \in L_n$.*

It is clear that the function U becomes a t-norm when $e = n$ and a t-conorm when $e = 0$. For any uninorm on L_n we have $U(n, 0) \in \{0, n\}$ and a uninorm U is called *conjunctive* when $U(n, 0) = 0$ and *disjunctive* when $U(n, 0) = n$. The structure of any discrete uninorm U on L_n with neutral element $0 < e < n$ is always as follows. It is given by a t-norm T on the interval $[0, e]$, by a t-conorm S on the interval $[e, n]$ and it takes values between the minimum and the maximum in all other cases, that is, in the region

$$A(e) = [0, e] \times [e, n] \cup [e, n] \times [0, e].$$

Definition 2 *Let F be a binary operator on a finite totally ordered set L_n. It is said that F verifies the* 1-smoothness condition *or that F is* smooth *if whenever $F(i, j) = k$ then*

$$\{F(i-1, j), F(i, j-1)\} \subseteq \{k-1, k\}.$$

The relation between the Archimedean property ($T(x, x) < x$ for all $x \in L_n \setminus \{0, n\}$) and the smoothness property for t-norms (and t-conorms) on L_n was stated in [13] (see [14] for the current version).

Proposition 1 ([14]) *The only Archimedean smooth t-norm and t-conorm on L_n are, respectively, the Łukasiewicz t-norm*

$$T_{\mathbf{L}}(x, y) = \max(0, x + y - n) \quad \textit{for all } x, y \in L_n,$$

and the Łukasiewicz t-conorm

$$S_{\mathbf{L}}(x, y) = \min(n, x + y) \quad \textit{for all } x, y \in L_n.$$

Smooth t-norms were characterized in [13] and [14], obtaining that they are the minimum, the Łukasiewicz or an ordinal sum of t-norms of these two classes.

Theorem 1 ([14]) *A t-norm T on L_n is smooth if and only if there exists a natural number r with $0 \le r \le n-1$ and a subset I of L_n, $I = \{0 = a_0 < a_1 < \cdots < a_r < a_{r+1} = n\}$ such that T is given by $T(x, y) =$*

$$\begin{cases} \max(a_i, x + y - a_{i+1}), & \textit{if } (x, y) \in [a_i, a_{i+1}]^2 \\ & \textit{and } 0 \le i \le r, \\ \min(x, y), & \textit{otherwise.} \end{cases}$$

There is a dual result for t-conorms that states that they are the maximum, the Łukasiewicz or ordinal sums of them [14].

Respect to uninorms on L_n, only the classes of uninorms in $\mathcal{U}_{\min}$ and uninorms in $\mathcal{U}_{\max}$ have been studied and characterized through some partial smoothness conditions in [12]. Specifically, these kinds of uninorms are as follows.

Definition 3 ([12]) *A binary operation $U : L_n^2 \longrightarrow L_n$ is a* uninorm in $\mathcal{U}_{\min}$ *with neutral element $0 < e < n$ if and only if there is a t-norm T on $[0, e]$ and a t-conorm S on $[e, n]$ such that U is given by*

$$U(x, y) = \begin{cases} T(x, y), & \textit{if } (x, y) \in [0, e]^2, \\ S(x, y), & \textit{if } (x, y) \in [e, n]^2, \\ \min(x, y), & \textit{elsewhere.} \end{cases}$$

We will denote a uninorm in $\mathcal{U}_{\min}$ with neutral element e as $U \equiv \langle T, e, S \rangle_{\min}$.

Definition 4 ([12]) *A binary operation $U : L_n^2 \longrightarrow L_n$ is a uninorm in $\mathcal{U}_{\max}$ with neutral element $0 < e < n$ if and only if there is a t-norm T on $[0, e]$ and a t-conorm S on $[e, n]$ such that U is given by*

$$U(x, y) = \begin{cases} T(x, y), & \textit{if } (x, y) \in [0, e]^2, \\ S(x, y), & \textit{if } (x, y) \in [e, n]^2, \\ \max(x, y), & \textit{elsewhere.} \end{cases}$$

We will denote a uninorm in $\mathcal{U}_{\max}$ with neutral element e as $U \equiv \langle T, e, S \rangle_{\max}$.

Also idempotent discrete uninorms, that is, those such that $U(x, x) = x$ for all $x \in L_n$ were characterized in [3]. For the sake of completeness of the development of the present work, we will recall them in Section 3.1, instead of in the current preliminaries. Let just recall here some necessary concepts to give such a characterization.

Definition 5 ([3]) *Given any decreasing function $g : L_n \to L_n$, we define its* completed graph *F_g as the subset of $(L_n)^2$ defined as:*

$$F_g = (\{0\} \times [g(0), n]) \cup (\{n\} \times [0, g(n)]) \\ \cup \{(x, y) \in [0, n-1] \times [0, n] \mid g(x+1) \le y \le g(x)\}.$$

Definition 6 ([3]) *A subset F of $(L_n)^2$ is said to be* Id-symmetrical *if for all $(x, y) \in (L_n)^2$ it holds that*

$$(x, y) \in F \quad \Longleftrightarrow \quad (y, x) \in F.$$

The above definition expresses that a subset F of $(L_n)^2$ is symmetrical w.r.t. the diagonal $\{(x, x) \mid x \in L_n\}$. A similar notion of symmetry is introduced for a decreasing function $g : L_n \to L_n$.

Definition 7 *[3] A decreasing function $g : L_n \to L_n$ is said to be* Id-symmetrical *if its completed graph F_g is Id-symmetrical.*

3. Discrete uninorms with smooth underlying operators

Let us begin with some general results about discrete uninorms, that will be used in next subsections. Our first result refer to the values of the uninorm in the points (r, n) where r is an idempotent element of U in the interval $]0, e[$.

Lemma 1 *Let $U \equiv \langle T, e, S\rangle$ a uninorm on L_n. If r is an idempotent element of U, with $0 < r < e$, then $U(r, n) = r$ or $U(r, n) = n$.*

Proof: Consider $U(r, n) = \alpha$, it is known that $r \leq \alpha \leq n$. Now we distinguish two cases:

- If $\alpha \geq e$, then by one side we have $U(\alpha, n) = U(U(r, n), n) = U(r, n)$ and by the other side, by increasingness of U we have $U(\alpha, n) \geq U(e, n) = n$. Therefore, $U(r, n) = n$.
- If $\alpha < e$ then, as r is idempotent of U,

$$\begin{aligned} r &= U(\alpha, r) = U(U(n, r), r) = U(n, U(r, r)) \\ &= U(n, r), \end{aligned}$$

and in this case $U(r, n) = r$. ∎

From the previous lemma, we can deduce a particular case. This situation is depicted in Figure 1.

Lemma 2 *Let $U \equiv \langle T, e, S\rangle$ a uninorm on L_n. If r is an idempotent element of U, with $0 < r < e$ and $U(r, n) = r$, then $U(k, m) = \min(k, m)$ for all $(k, m) \in \{(i, j) | i \leq r \leq e \leq j$ or $j \leq r \leq e \leq i\}$.*

Proof: Let (k, m) be in $\{(i, j) | i \leq r \leq e \leq j\}$.

As $U(r, n) = U(r, e) = r$, we have that $U(r, x) = r$ for all $e \leq x \leq n$, and in particular $U(r, m) = r$. Now, using the associativity and commutativity of U, we have

$$\begin{aligned} k &= U(r, k) = U(U(r, m), k) = U(r, U(m, k)) = \\ &= U(r, U(k, m)) = U(U(r, k), m) = U(k, m), \end{aligned}$$

that concludes the proof. ∎

Analogously to Lemmas 1 and 2, the next results refer to the case when the idempotent element r is in (e, n). This case can be viewed in figure 2.

Lemma 3 *Let $U \equiv \langle T, e, S\rangle$ a uninorm on L_n. If r is an idempotent element of U, with $e < r < n$, then $U(r, 0) = r$ or $U(r, 0) = 0$.*

Lemma 4 *Let $U \equiv \langle T, e, S\rangle$ a uninorm on L_n. If r is an idempotent element of U, with $e < r < n$ and $U(r, 0) = r$, then $U(k, m) = \max(k, m)$ for all $(k, m) \in \{(i, j) | i \leq e \leq r \leq j$ or $j \leq e \leq r \leq i\}$.*

In view of Theorem 1 and its dual for t-conorms, to classiy all the uninorms with smooth underlying operators, we need to consider different possibilities depending on how is the smooth underlying t-norm and the smooth underlying t-conorm.

n	min	min	min	$\boldsymbol{r}$		n	n	n	n
$\vdots$	min	min	min	r		$\vdots$			n
$e{+}1$	min	min	min	r		$e{+}1$			n
e	0	1	$\cdots$	r	$\cdots$	$\boldsymbol{e}$	$e{+}1$	$\cdots$	n
$\vdots$	0					$\vdots$			
r	0			$\boldsymbol{r}$		r	r	r	$\boldsymbol{r}$
$\vdots$	0					$\vdots$	min	min	min
1	0					1	min	min	min
0	0	0	0	0	0	0	min	min	min
	0	1	$\cdots$	r	$\cdots$	e	$e{+}1$	$\cdots$	n

Figure 1: Discrete uninorm with neutral element e, idempotent element $0 < r < e$ and $U(r, n) = r$.

n	max	max	max	n	n	n	n	n	n
$n{-}1$	max	max	max	$n{-}1$					n
$\vdots$	max	max	max	$\vdots$					n
r	$\boldsymbol{r}$	max	max	r		$\boldsymbol{r}$			n
$\vdots$				$\vdots$					n
e	0	$\cdots$	$e{-}1$	e	$\cdots$	r	$\cdots$	$n{-}1$	n
$e{-}1$	0			$e{-}1$		max	max	max	max
$\vdots$	0			$\vdots$		max	max	max	max
0	0	0	0	0		$\boldsymbol{r}$	max	max	max
	0	$\cdots$	$e{-}1$	e	$\cdots$	r	$\cdots$	$n{-}1$	n

Figure 2: Discrete uninorm with neutral element e, idempotent element $e < r < n$ and $U(r, 0) = r$.

3.1. The idempotent case

In the case that $T = T_{\mathbf{M}}$ and $S = S_{\mathbf{M}}$, we have that U is an idempotent uninorm. This case was studied in [3], obtaining the following results.

Theorem 2 ([3]) *A binary operation U on L_n with neutral element $0 < e < n$ is an idempotent discrete uninorm if and only if there exists a decreasing function $g : [0, e] \to [e, n]$ with $g(e) = e$ such that*

$$U(x, y) = \begin{cases} \min(x, y), & \text{if } y \leq \overline{g}(x) \text{ and } x \leq \overline{g}(0), \\ \max(x, y), & \text{elsewhere,} \end{cases}$$

where $\overline{g}$ is the unique symmetrical extension of g,

given by $\overline{g}(x) =$

$$\begin{cases} g(x), & \text{if } x \le e, \\ \max\{z \in [0,e] | g(z) \ge x\}, & \text{if } e \le x \le g(0), \\ 0, & \text{if } x > g(0). \end{cases}$$

Theorem 3 *(i) The number of discrete idempotent uninorms on L_n, $n \ge 2$, with neutral element $e \in L_n$, is given by*

$$I_{e,n} = \binom{n}{e}.$$

(ii) The total number of discrete idempotent uninorms on L_n, $n \ge 2$, is given by

$$I_n = \sum_{e=0}^{n} I_{e,n} = 2^n.$$

3.2. The Łukasiewicz case

Now we give two more general results referred to the case that the underlying t-norm or the underlying t-conorm are Łukasiewicz operators. In the case that $T = T_{\mathbf{L}}$ and the uninorm U is conjunctive, then U must be in the class of $\mathcal{U}_{\min}$.

Proposition 2 *Let $U \equiv \langle T, e, S\rangle$ be a conjunctive uninorm on L_n. If $T = T_{\mathbf{L}}$ then $U \in \mathcal{U}_{\min}$.*

Analogously, if $S = S_{\mathbf{L}}$ and U is disjunctive, uninorm U must be in $\mathcal{U}_{\max}$.

Proposition 3 *Let $U \equiv \langle T, e, S\rangle$ be a disjunctive uninorm on L_n. If $S = S_{\mathbf{L}}$ then $U \in \mathcal{U}_{\max}$.*

From previous propositions, there will be only two uninorms with underlying operators $T = T_{\mathbf{L}}$ and $S = S_{\mathbf{L}}$. The structure of both uninorms is depicted in Figure 3.

Corollary 1 *There are only two uninorms on L_n, $n \ge 2$ with neutral element e, $0 < e < n$, such that their underlying operators are $T = T_{\mathbf{L}}$ and $S = S_{\mathbf{L}}$:*

- *If U is conjunctive, $U \equiv \langle T_{\mathbf{L}}, e, S_{\mathbf{L}}\rangle_{\min}$.*
- *If U is disjunctive, $U \equiv \langle T_{\mathbf{L}}, e, S_{\mathbf{L}}\rangle_{\max}$.*

From this result we can calculate the number of uninorms with Łukasiewicz underlying operators varying the neutral element in L_n (including the cases $e = 0$, the Łukasiewicz t-conorm and $e = 1$, the Łukasiewicz t-norm).

Corollary 2 *The total number of uninorms on L_n, $n \ge 1$, such that their underlying operators are $T = T_{\mathbf{L}}$ and $S = S_{\mathbf{L}}$ is $2n$.*

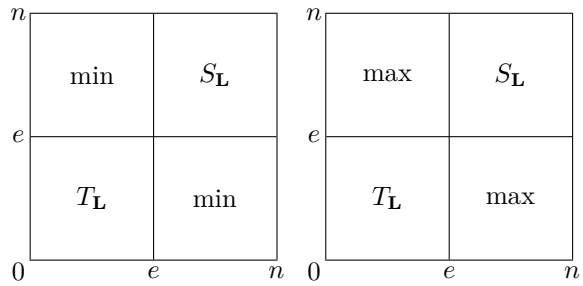

Figure 3: Discrete uninorms with underlying operators $T = T_{\mathbf{L}}$ and $S = S_{\mathbf{L}}$, conjunctive (left) and disjunctive (right).

3.3. Case $T = T_{\mathbf{M}}$ and $S = S_{\mathbf{L}}$

In this subsection we will deal with uninorms such that their underlying operators are $T = T_{\mathbf{M}}$ and $S = S_{\mathbf{L}}$.

Note that, from the previous case, if the uninorm is disjunctive and $S = S_{\mathbf{L}}$, then it must be in $\mathcal{U}_{\max}$. Considering that there will always exists $r \in L_n$ such that $0 \le r \le e$, and $U(r,n) = n$, the uninorm U restricted to $[r,n]^2$ satisfies Proposition 3. From this and Lemma 2, we can deduce the general structure of the uninorms with $T = T_{\mathbf{M}}$ and $S = S_{\mathbf{L}}$. This structure can be observed in Figure 4.

Proposition 4 *Let $U \equiv \langle T, e, S\rangle$ be a uninorm on L_n such that $T = T_{\mathbf{M}}$ and $S = S_{\mathbf{L}}$. Let $0 \le r \le e$ be the smallest element such that $U(r,n) = \max(r,n) = n$. Then U is given by*

$$U(x,y) = \begin{cases} \min(x,y), & \text{if } (x,y) \in [0,e]^2, \\ S_{\mathbf{L}}(x,y), & \text{if } (x,y) \in [e,1]^2, \\ \max(x,y), & \text{if } (x,y) \in [r,e] \times [e,n] \text{ or} \\ & (x,y) \in [e,n] \times [r,e], \\ \min(x,y), & \text{otherwise.} \end{cases}$$

Again, with this result is easy to count the total number of uninorms in this case.

Corollary 3 *(i) The number of uninorms $U \equiv \langle T, e, S\rangle$ on L_n, $n \ge 2$ with neutral element $e \in L_n$ such that their underlying operators are $T = T_{\mathbf{M}}$ and $S = S_{\mathbf{L}}$ is given by*

$$ML_{e,n} = e + 1.$$

(ii) The total number of uninorms on L_n, $n \ge 1$, such that their underlying operators are $T = T_{\mathbf{M}}$ and $S = S_{\mathbf{L}}$ is

$$ML_n = \sum_{e=0}^{n} ML_{e,n} = \frac{n(n+1)}{2} + 1.$$

3.4. Case $T = T_{\mathbf{L}}$ and $S = S_{\mathbf{M}}$

Now we study the case of uninorms such that their underlying operators are $T = T_{\mathbf{L}}$ and $S = S_{\mathbf{M}}$. It

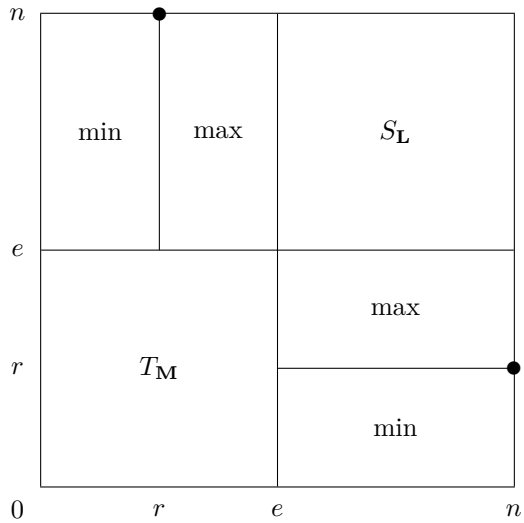

Figure 4: Structure of a discrete uninorm with underlying operators $T = T_{\mathbf{M}}$ and $S = S_{\mathbf{L}}$.

is very similar to the previous one, and it can be proved using duality. The structure of such uninorms can be viewed in Figure 5

Proposition 5 *Let $U \equiv \langle T, e, S\rangle$ such that $T = T_{\mathbf{L}}$ and $S = S_{\mathbf{M}}$. Let $e \leq r \leq n$ be the biggest element such that $U(0, r) = \min(0, r) = 0$. Then U is given by*

$$U(x,y) = \begin{cases} T_{\mathbf{L}}(x,y), & \text{if } (x,y) \in [0,e]^2, \\ \max(x,y), & \text{if } (x,y) \in [e,1]^2, \\ \min(x,y), & \text{if } (x,y) \in [0,e] \times [e,r] \text{ or} \\ & (x,y) \in [e,r] \times [0,e] \\ \max(x,y), & \text{otherwise.} \end{cases}$$

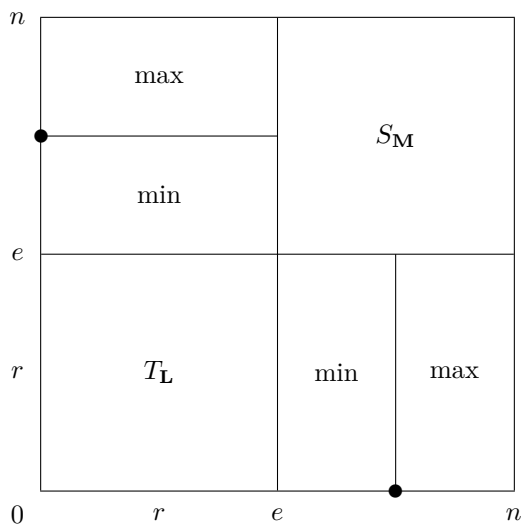

Figure 5: Structure of a discrete uninorm with underlying operators $T = T_{\mathbf{L}}$ and $S = S_{\mathbf{M}}$.

In the next corollary we give the total number of uninorms in this case.

T \ S	$S_{\mathbf{M}}$	$S_{\mathbf{L}}$
$T_{\mathbf{M}}$	$\binom{n}{e}$	$e+1$
$T_{\mathbf{L}}$	$n-e+1$	2

Table 1: Number of discrete uninorms $U \equiv \langle T, e, S\rangle$ with neutral element $e \in L_n$, $0 < e < n$, depending on which are their underlying operators.

T \ S	$S_{\mathbf{M}}$	$S_{\mathbf{L}}$
$T_{\mathbf{M}}$	2^n	$\frac{n(n+1)}{2}+1$
$T_{\mathbf{L}}$	$\frac{n(n+1)}{2}+1$	$2n$

Table 2: Total number of discrete uninorms $U \equiv \langle T, e, S\rangle$ on L_n, with T and S idempotent or Archimedean.

Corollary 4 *(i) The number of uninorms $U \equiv \langle T, e, S\rangle$ on L_n, $n \geq 2$ with neutral element $e \in L_n$ such that their underlying operators are $T = T_{\mathbf{L}}$ and $S = S_{\mathbf{M}}$ is given by*

$$LM_{e,n} = n - e + 1.$$

(ii) The total number of uninorms on L_n, $n \geq 1$, sucht that their underlying operators are $T = T_{\mathbf{L}}$ and $S = S_{\mathbf{M}}$ is

$$LM_n = \sum_{e=0}^{n} LM_{e,n} = \frac{n(n+1)}{2} + 1.$$

So, if we join all the results above, we can obtain the number of uninorms in any of the considered cases which is represented in tables 1 (with a fixed neutral element e) and 2 (in general).

Remark 1 *Note that in such a table, there are some cases counted twice. For instance, $T = T_{\mathbf{L}}$ is included in both the cases: $T = T_{\mathbf{L}}$ and $S = S_{\mathbf{L}}$, and $T = T_{\mathbf{L}}$ and $S = S_{\mathbf{M}}$.*

In view of the previous remark, we obtain the following proposition, which gives the total number of uninorms considered in this section.

Proposition 6 *The total number of discrete uninorms on L_n, $n \geq 1$, such that their underlying operators are idempotent or Archimedean is given by*

$$2^n + n(n+3) - 2.$$

3.5. General case: when T or S are ordinal sums

After studying all cases for T and S being idempotent or Archimedean operators, the remaining case

is whenever T and/or S are ordinal sums. To start with this case, we give a similar result to Proposition 2.

Proposition 7 *Let $U \equiv \langle T, e, S\rangle$ be a uninorm such that T is smooth and there exist two consecutive idempotent elements of U, $r, s \in L_n$, $0 < r < s < e$ with $s \geq r+2$. If $U(r,n) = r$, then $U(k,m) = \min(k,m)$ for all $(k,m) \in [r,s[\times[e,n] \cup [e,n] \times [r,s[$.*

By duality, we have a similar result to Proposition 3 for uninorms with idempotent elements greater than the neutral element e of U.

Proposition 8 *Let $U \equiv \langle T, e, S\rangle$ be a uninorm such that S is smooth and there exist two consecutive idempotent elements of U, $r, s \in L_n$, $e < r < s < n$ with $s \geq r+2$. If $U(s,0) = s$, then $U(k,m) = \max(k,m)$ for all $(k,m) \in]r,s] \times [0,e] \cup [0,e] \times]r,s]$.*

Now we present construction example for uninorms $U \equiv \langle T, e, S\rangle$ with T and S smooth operators, in which T is an ordinal sum and S is Łukasiewicz.

Example 1 *Consider L_7, we want to determine all possible uninorms with neutral element $e = 4$, underlying smooth operators, and idempotent elements $I = \{0, 2, 4, 7\}$, that is, $S = S_{\mathbf{L}}$ and T is an ordinal sum with one non-trivial idempotent element. This situation can be observed in figure 6. Empty squares correspond to not-known values of U. Now, we will distinguish some cases:*

- *If U is disjunctive, using Proposition 3, we have that $U \in \mathcal{U}_{\max}$.*
- *If U is conjunctive. By Proposition 7, $U(i,j) = \min(i,j)$ for all $(i,j) \in [0,2) \times [5,7] \cup [5,7] \times [0,2)$. Now, depending on the values of $U(2,7)$ we have the following two subcases:*
  - *If $U(2,7) = 2$, if we consider uninorm U on the set $[2,7]^2$, it's a conjunctive uninorm $U|_{[2,7]}$, and thus, by Theorem 2, this uninorm has to be in $\mathcal{U}_{\min}$. Then, adding all this information, $U \in \mathcal{U}_{\min}$.*
  - *If $U(2,7) = 7$, we can take the operator $U|_{[2,7]}$, that it is disjunctive, with both underlying Łukasiewicz operators and therefore $U|_{[2,7]} \in \mathcal{U}_{\max}$.*

Then, there are only three uninorms with neutral element $e = 4$, underlying smooth operators, and idempotent elements $I = \{0, 2, 4, 7\}$. This three uninorms can be viewed in Figure 7.

In view of this example, if we know the idempotent elements of the uninorm, by using Propositions 7 and 8, we could construct all the uninorms with underlying operators T and S smooth.

Note that in all the studied cases it appears that an Id-symmetrical decreasing function g on L_n with fixed point e exists, such that the uninorm in the region $A(e)$ is given by the minimum under the graph of g and by the maximum over this graph. The only restriction of g is that it must be constant in any interval where the underlying t-norm T (or the underlying t-conorm S) is given by a Łukasiewicz summand. To formalize this idea and to used it in characterizing all discrete uninorms with smooth underlying operations is part of our future work on this topic.

7					7	7	7	7
6					6	7	7	7
5					5	6	7	7
4	0	1	2	3	**4**	5	6	7
3	0	1	2	2	3			
2	0	1	2	2	2			
1	0	0	1	1	1			
0	0	0	0	0	0			
	0	1	2	3	**4**	5	6	7

Figure 6: Structure of a discrete uninorm in L_7 with neutral element 4, smooth underlying operators and idempotent elements $\{0, 2, 4, 7\}$.

4. Conclusions and future work

In this work the structure of uninorms $U \equiv \langle T, e, S\rangle$ defined on a finite chain L_n, when the underlying operators T and S are smooth has been studied, obtaining a characterization of the cases when these operations are idempotent or Archimedean. Also, the number of operations with a fixed neutral element $e \in L_n$ has been found as well as the total number of uninorms on L_n such that its underlying operators are smooth. For the case that T or S are ordinal sums, a construction example has been given, offering all possible uninorms with this configuration of underlying smooth operators.

In a future work, our goal will be to establish a general characterization of all possible uninorms with smooth underlying operations, and to compute the total number of these uninorms.

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7	7	7	7	7	7	7	7	7
6	6	6	6	6	6	7	7	7
5	5	5	5	5	5	6	7	7
4	0	1	2	3	4	5	6	7
3	0	1	2	2	3	5	6	7
2	0	1	2	2	2	5	6	7
1	0	0	1	1	1	5	6	7
0	0	0	0	0	0	5	6	7
	0	1	2	3	4	5	6	7

7	0	1	2	3	7	7	7	7
6	0	1	2	3	6	7	7	7
5	0	1	2	3	5	6	7	7
4	0	1	2	3	4	5	6	7
3	0	1	2	2	3	3	3	3
2	0	1	2	2	2	2	2	2
1	0	0	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0
	0	1	2	3	4	5	6	7

7	0	1	7	7	7	7	7	7
6	0	1	6	6	6	7	7	7
5	0	1	5	5	5	6	7	7
4	0	1	2	3	4	5	6	7
3	0	1	2	2	3	5	6	7
2	0	1	2	2	2	5	6	7
1	0	0	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0
	0	1	2	3	4	5	6	7

Figure 7: The only three uninorms in L_7 with neutral element 4, smooth underlying operators and idempotent elements $\{0, 2, 4, 7\}$. From top to bottom, the uninorm in $\mathcal{U}_{\max}$, in $\mathcal{U}_{\min}$ and a uninorm that is not in $\mathcal{U}_{\max}$ nor $\mathcal{U}_{\min}$.