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Asymptotic Solitons of the Johnson Equation

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Abstract

We prove the existence of non-decaying real solutions of the Johnson equation, vanishing as $x \to +\infty$. We obtain asymptotic formulas as $t \to \infty$ for the solutions in the form of an infinite series of asymptotic solitons with curved lines of constant phase and varying amplitude and width.

1 Introduction

The Johnson equation (JE)

$$\left(v_t + \frac{1}{4}v_{xxx} + \frac{3}{2}vv_x + \frac{v}{2t}\right)_x = -\frac{12\alpha^2}{t^2}u_{yy} \tag{1.1}$$

($\alpha^2 = \pm 1$) or the cylindrical Kadomtsev-Petviashvili equation, is the analogue of the well-known cylindrical Korteweg-de Vries equation ($\alpha = 0$) in two spatial dimensions (2D). The JE was obtained firstly in [1] under the description of the surface waves on a shallow incompressible liquid. Later it was shown that it describes the propagation of waves in the stratified media [2]. It follows from the derivation of (1.1) that the correct statement of the Cauchy problem is possible only as $t = t_0 > 0$.

In [3] and [4] the equivalence of the Kadomtsev-Petviashvili equation (KP) and the JE was established. Let $u(\xi, \eta, \tau)$ be an arbitrary solution of the KP

$$\left(u_\tau + \frac{1}{4}u_{\xi\xi\xi} + \frac{3}{2}uu_\xi\right)_\xi = -\frac{3\alpha^2}{4}u_{\eta\eta}. \tag{1.2}$$

Then the function

$$v(x, y, t) = u\left(x - \frac{y^2 t}{48\alpha^2}, \frac{yt}{4}, t\right) \tag{1.3}$$

satisfies the JE. This mapping $u(\xi,\eta,\tau) \to v(x,y,t)$ is invertible. Each solution $v(x,y,t)$ of the JE generates a solution of the KP by the formula

$$u(\xi,\eta,\tau) = v\left(\xi + \frac{\eta^2}{3\alpha^2 t}, \frac{4\eta}{\tau}, \tau\right). \tag{1.4}$$

It was shown in [3] that mappings (1.3) and (1.4) preserve the class of functions rapidly decaying at infinity (as $(x^2+y^2)^{-1} \to 0$), and all the results obtained in the theory of the KP in the corresponding class of solutions can be directly applied to solve the Johnson equation. The situation is similar for the solutions involving the Airy functions (see [5] and [6]). Obviously this is not the fact for periodic initial data, which are not invariant with respect to this transformation, and investigation of the JE is an independent interesting problem in this case (for the KP see the corresponding theory, for example, in [35]-[38]).

We are interested in the construction of a class of JE-I ($\alpha = i$ in (1.1)) non-decaying solutions, which are bounded for all (x,y,t) and vanish as $x \to +\infty$ for all fixed y and t. Such a kind of KP solutions was constructed and investigated firstly for KP-II in [7], [8], and then for KP-I in [9]-[13]. It turns out that all basic stages of the construction of the solutions of the KP and the JE, and the study of their asymptotic behaviour admit mutual recounting using the described mapping.

We apply the change of variables

$$\xi = x - \frac{y^2 t}{48\alpha^2}, \quad \eta = \frac{yt}{4}, \quad \tau = t$$

to the scheme of the V.E.Zakharov and A.B.Shabat "dressing method" [14] of integration of the KP, and obtain analogous formulas for the JE. Using them we prove the existence of a class of JE-I non-decaying solutions with the prescribed properties. The simplest one is the one-soliton solution

$$v(x,y,t) = \frac{2q^2}{\cosh^2\left[q\left(x - \left(q^2 - 3p^2 - \frac{y^2}{48} - \frac{py}{2}\right)t - \frac{1}{2q}\ln\frac{c}{2q}\right)\right]}$$

($p \in \mathbb{R}$, $q \in \mathbb{R}^+$) which corresponds to the KP plane-soliton by virtue of (1.3).

Then we study the asymptotic behaviour of the constructed solution as $t \to \infty$. The investigations of long-time asymptotic behaviour of non-decaying solutions of 2D nonlinear evolution equations is closely connected with the same investigations in one spatial dimension. A.V.Gurevich and L.P.Pitaevsky studied in 1973 a non-decaying solution of the Korteweg–de Vries (KdV) equation, which describes the evolution of an initial step–function ([15],[16]). They applied Whitham method to construct an approximation of this solution by a knoidal wave with slowly varying parameters and detected the appearance of many strong oscillations like solitons on the front of the solution for a large time. This approximate solution satisfies the KdV–equation with error vanishing as $t \to +\infty$. The mathematical ground of this phenomenon was done in 1975 by E.Ya.Khruslov in [17] and [18], where the nature of these solitons was explained. Subsequently these solitons were called asymptotic solitons. An analogous phenomenon of splitting of non-decaying initial data into infinite series of solitons was proved later for other KdV–like equations (nonlinear Schrödinger equation, sine–Gordon equation, modified KdV and the Toda lattice as a discrete analogue of the KdV) ([19]–[29]).

In [7]–[13] the method proposed by E.Ya.Khruslov ([17],[18]) was extended to the investigation of the asymptotic behaviour of non-decaying solutions of KP-type equations (KP, modified KP-I and 2D-Gardner equation) as $t \to \infty$. It was proved that they are represented as infinite series of solitons with curved lines of constant phase in the neighbourhood of the front as $t \to \infty$. These asymptotic solitons were called curved asymptotic solitons. Note that recently ([30]) V.E.Zakharov also considered a curved soliton of the KP-II equation, but in another space–time domain.

Our principal goal is to prove the phenomenon of splitting of non-decaying solutions of the JE-I into infinite series of solitons as $t \to \infty$.

We prove that there exist non-decaying real solutions of the JE-I, which split in the neighbourhood of the front into a series of solitons of the form

$$v_n(x, y, t) = \frac{2q_0(y)^2}{\cosh^2\left[q_0(y)\left(x - C(y)t + \frac{1}{2q_0(y)}\left(\ln t^{n+1/2} - \ln g(y) - \phi_n(y)\right)\right)\right]}, \qquad (1.5)$$

which depend on two parameters $C(y)$ and $g(y)$. The functions $p_0(y)$, $q_0(y)$ and $\phi_n(y)$ are completely determined by them. These solitons are diverged with the velocity $\ln t^{\frac{n+1/2}{2q_0(y)}}$. They have varying amplitude and width in the general case, but we present also examples where amplitude and width are constant. In these cases curved and weakly curved asymptotic solitons are both constructed. The lines of constant phase of the weakly curved solitons are deviated from the straight line just on a value proportional to $\ln y^2$.

Asymptotic solitons (1.5) of the JE-I and the KP-I ([9], [11]) coincide taking into account transformations (1.3) and (1.4).

2 Existence of a Johnson equation solution.

After application of the change of variables

$$\xi = x - \frac{y^2 t}{48\alpha^2}, \quad \eta = \frac{yt}{4}, \quad \tau = t$$

to the scheme of the V.E.Zakharov and A.B.Shabat "dressing method" [14] of integration of the KP we obtain the following formulas for the JE. A JE solution has the form

$$v(x, y, t) = 2\frac{\partial}{\partial x}K(x, x, y, t), \qquad (2.1)$$

where the function $K(x, s, y, t)$ is a solution of the Marchenko integral equation

$$K(x, z, y, t) + F(x, z, y, t) + \int_x^\infty K(x, \xi, y, t)F(\xi, z, y, t)d\xi = 0. \qquad (2.2)$$

This equation is an equation with respect to z and x, y, t are parameters. The kernel $F(x, z, y, t)$ of (2.2) satisfies the system of linear differential equations

$$\begin{cases} F_t + \dfrac{y^2}{48\alpha^2}(F_x + F_z) + \dfrac{y}{4\alpha}(F_{xx} - F_{zz}) + F_{xxx} + F_{zzz} = 0 \\ \alpha F_y + \dfrac{yt}{24\alpha}(F_x + F_z) + \dfrac{t}{4}(F_{xx} - F_{zz}) = 0 \end{cases} \qquad (2.3)$$

($\alpha = i$ for the JE-I, and $\alpha = 1$ for the JE-II). The correspondence of (2.1)-(2.3) to the JE can be verified by a direct method, described in [32] for the KP. The scheme (2.1)-(2.3) don't allow us to solve the Cauchy problem for JE, but it is rather convenient for the construction of classes of solutions with various properties, in particular, of rapidly decaying rational solutions described in [5] and of non-decaying solutions.

A wide class of solutions of (2.3) as $\alpha = i$ can be found as follows

$$F(x, z, y, t) = \iint_{\Omega} \exp[ip(x - z) - q(x + z) + 2qf(p, q, y)t]d\mu(p, q), \tag{2.4}$$

where $\Omega \subset \mathbb{C}^+$ ($\mathbb{C}^+ = \{\lambda \mid \lambda = p + iq, q > 0\}$ is the upper half-plane of the complex plane), $f(p, q, y) = q^2 - 3p^2 + py/2 - y^2/48$, and $d\mu(p, q)$ is some measure on Ω.

To construct a JE-I solution by the scheme (2.1)-(2.4) we must define the set Ω in (2.4) and the measure $d\mu(p, q)$ on this set. For this goal we introduce two positive functions $C(s)$ and $g(s)$ which play an important role in the construction of the solution and investigation of its asymptotic behaviour, and we formulate the following conditions.

Condition A. *The function $C(s) : \mathbb{R} \to \mathbb{R}^+$ is of class C^2 and such that*

$$C(s) \geq \delta > \varepsilon^2 \quad (\delta, \varepsilon = \text{const} > 0), \quad C''(s) > -1/24. \tag{2.5}$$

Condition B. *The set Ω has the form*

$$\Omega = \left\{(p, q) \in \mathbb{R}^2 \mid -\infty < p < \infty,\ 0 < \varepsilon \leq q \leq h(p)\right\}, \tag{2.6}$$

where $q = h(p)$ is the envelope of the family of hyperbolas

$$f(p, q, s) = C(s), \tag{2.7}$$

which touch it at the point

$$(p_0(y), q_0(y)) = \left(C'(y) + y/12,\ \sqrt{C(y) + 3\left(C'(y)\right)^2}\right).$$

Remark. The special structure of Ω (2.6) implies that

$$C(s) = \max_{(p,q)\in\Omega} f(p, q, s). \tag{2.8}$$

Condition C. *The function $g(s) : \mathbb{R} \to \mathbb{R}^+$, $g(s) < A = \text{const}$ is of class C^∞ and such that the measure $d\mu$ of the form $d\mu(p, q) = \tilde{g}(p, q)dpdq$ with real positive $\tilde{g} \in C^\infty$, $\tilde{g}(p_0(Y), q_0(Y)) = g(Y)$, satisfies the inequality*

$$\forall a = \text{const} > 0 : \quad \iint_{\Omega} e^{a(q+|p|)} d\mu(p, q) < \infty. \tag{2.9}$$

Let us show that under Conditions A-C the scheme (2.1)-(2.3) determines a smooth real solution of the JE-I vanishing as $x \to +\infty$. For a function $h(y) \in L^2[x, \infty)$ define the operator $\hat{F}$ by

$$[\hat{F}h](z) = \int_x^\infty F(s, z, y, t)h(s)ds \tag{2.10}$$

with the kernel F given by (2.4), where $h(s)$ also depends on the parameters y, t.

Lemma 1. *Assume that Conditions A-C are fulfilled.*
Then $\hat{F}$ is a self-adjoint, compact and positive operator in $L^2[x,\infty)$.

Proof. Its self-adjointness follows from the form (2.4) of $F(x,z,y,t)$. Let us show that this operator is compact. We estimate the Hilbert-Schmidt norm of $\hat{F}$:

$$\begin{aligned}||\hat{F}||^2_{L^2[x,\infty)} &= \int_x^\infty \int_x^\infty |F(x,z,y,t)|^2 ds dz \\ &\le \frac{1}{4\varepsilon^2} \iint_\Omega d\mu(p,q) \iint_\Omega \exp[4q(|x| + f(p,q,y))t] d\mu(p,q) \\ &\le \frac{1}{4\varepsilon^2} \iint_\Omega d\mu(p,q) \iint_\Omega e^{aq} d\mu(p,q) < \infty.\end{aligned}$$

Thus $\hat{F}$ is a Hilbert-Schmidt operator. Hence, it is a compact operator ([33]). To prove the positivity of $\hat{F}$, consider the scalar product

$$\begin{aligned}(\hat{F}h,h) &= \int_x^\infty \int_x^\infty F(s,z,y,t) h(s) ds \overline{h}(z) dz \\ &= \iint_\Omega e^{2qf(p,q,y)t} \left| \int_x^\infty e^{(ip-q)s} h(s) ds \right|^2 d\mu(p,q) > 0\end{aligned}$$

as $h(s) \neq 0$. ■

Under the conditions of Lemma 1 the following statement holds.

Lemma 2. *The scheme* (2.1)-(2.4) *determines a smooth real solution of the JE-I vanishing as $x \to \infty$ and bounded for all fixed x, y, t $(t > 0)$.*

Proof. Let us represent (2.3) in the operator form in $L^2[x,\infty)$

$$\varphi + \hat{F}\varphi = f, \tag{2.11}$$

where $\hat{F}$ has the form (2.10), and $\varphi = K(x,z,y,t)$, $f = -F(x,z,y,t)$. Due to the positivity of $\hat{F}$ the homogeneous equation $\varphi + \hat{F}\varphi = 0$ has only the trivial solution. Since $\hat{F}$ is a compact operator, then by the Fredholm theorem ([33]) inhomogeneous equation (2.11) has a unique solution given by

$$K(x,z,y,t) = -(I + \hat{F})^{-1} F(x,z,y,t)$$

with $||(I + \hat{F})^{-1}|| \le 1$.

Due to Condition C and the fact that Ω is inside the upper half-plane at positive distance from the q-axis (Condition B), F is an infinitely differentiable function with respect to all variables. Moreover, $D_i^\alpha F \to 0$ ($D^\alpha = \frac{\partial^\alpha}{\partial x_i^\alpha}$; $\alpha = 0, 1, \ldots$; $i = 1, \ldots, 4$) as $x + z \to \infty$, and $D_i^\alpha F$ are bounded for all fixed x, z, y, t $(t > 0)$. One can show ([34]) that the function K has the same properties.

Let us prove that $K(x,x,y,t)$ is a real function. After multiplication of (2.11) by $\overline{\varphi}$ and integration with respect to z from x to $+\infty$ we obtain

$$||\varphi||^2 + (\hat{F}\varphi, \varphi) = (f, \varphi). \tag{2.12}$$

The self-adjointness of $\hat{F}$ implies that the imaginary part of the left-hand-side of (2.12) is equal to zero:

$$\operatorname{Im} \int_x^\infty F(x,s)\overline{K(x,s)}ds = 0. \tag{2.13}$$

Application of conjugation to (2.3) for $z = x$ gives

$$\overline{K(x,x)} + F(x,x) + \int_x^\infty \overline{K(x,\xi)}F(x,\xi)d\xi = 0.$$

It follows from (2.13) and the reality of $F(x,x,y,t)$ that $K(x,x,y,t)$ is real. ■

3 Theorem about long-time asymptotic behaviour of JE-I non-decaying solutions

Our goal is to investigate the long-time asymptotic behaviour of the JE-I solution defined in the previous section. To define a domain in which we shall carry out the investigations, we introduce the following definition.

Definition. *Let $M > 2$ be an arbitrary number. The domain $G_M(t) \subset \mathbb{R}^2$ given by*

$$G_M(t) = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; |\ln g(y)| < \ln t, \;\; x > C(y)t - \frac{1}{2q_0(y)} \ln t^{M+1} \right\}$$

is called the neighbourhood of the solution front.

The following theorem describes the asymptotic behaviour of the JE-I solution defined by Lemma 2 for large time.

Theorem 1. *Assume that Conditions A-C are fulfilled.*
Then the JE-I solution $v(x,y,t)$ constructed by the scheme (2.1)–(2.4) *is represented in the domain $G_M(t)$ as $t \to \infty$ in the following way*

$$v(x,y,t) = \sum_{n=1}^{[M-1]} v_n(x,y,t) + O\left(\frac{1}{t^{1/2-\varepsilon_1}}\right), \qquad (0 < \varepsilon_1 < 1/2) \tag{3.1}$$

$$v_n(x,y,t) = \frac{2q_0(Y)^2}{\cosh^2\left[q_0(Y)\left(x - C(Y)t + \frac{1}{2q_0(Y)}\left(\ln t^{n+1/2} - \ln g(Y) - \ln\phi_n(y)\right)\right)\right]},$$

where $q_0(Y) = \sqrt{C(y) + 3\left(C'(y)\right)^2}$,

$$\phi_n(y) = \frac{(C(y) + 48(C'(y))^2)^{n-1}(1 + 24C''(y))^{n-1/2}Q^{(n)}\Gamma^{(n)}}{2^{(2n+5)/2}((n-1)!)^2(C(y) + 12(C'(y))^2)^{(10n-3)/4}Q^{(n-1)}\Gamma^{(n-1)}},$$

and $\Gamma^{(n)}, Q^{(n)} > 0$ are the determinants of the n by n matrices with entries

$$\Gamma^{(n)}_{i+1,k+1} = \Gamma\left(\frac{i+k+1}{2}\right)(1 + (-1)^{i+k}), \quad Q^{(n)}_{i+1,k+1} = \Gamma(i+k+1),$$
$$i,k = 0,\dots,n-1.$$

Here the asymptotic representation (3.1) *is uniform with respect to x and y in $G_M(t)$ for any fixed $M \geq 2$.*

Let us mark the key points of the proof. The proof consists in three steps. On the first step we show that as $t \to \infty$ the kernel $F(x,z,y,t)$ of integral equation (2.2) is represented as the sum of a degenerate kernel and a kernel with small operator norm in the space $L^2[x,\infty)$. On the second step we prove that the degenerate kernel gives the main contribution in the asymptotic representation of the solution of equation (2.2). The third step consists in the analysis of representation (2.1) for the solution $v(x,y,t)$ as $t \to \infty$, where the function $K(x,z,y,t)$ is a solution of the Marchenko integral equation with the degenerate kernel.

First step. To investigate the Marchenko equation kernel $F(x,z,y,t)$ (2.4) as $t \to \infty$, we set $x = C(y)t + \xi, \quad z = C(y)t + \zeta$ and define $\tilde{F}(\xi,\zeta,y,t) = F(\xi + C(y)t, \zeta + C(y)t, y, t)$. Then the function $\tilde{F}(\xi,\zeta,y,t)$ is written as follows:

$$\tilde{F}(\xi,\zeta,y,t) = \iint_{\Omega} \exp[ip(\xi-\zeta) - q(\xi+\zeta) - 2q(C(y) - f(p,q,y))t]d\mu(p,q).$$

For sufficiently small $\varepsilon' > 0$ let us consider the curve

$$2q(f(p,q,y) - C(y)) + \varepsilon' = 0. \tag{3.2}$$

This curve separates the domain Ω into two subdomains $O_{\varepsilon'}$ and $\Omega_{\varepsilon'}$ so that $\Omega = \overline{O_{\varepsilon'}} \cup \Omega_{\varepsilon'}$. Here $O_{\varepsilon'}$ lies between the curve $q = h(p)$ and curve (3.2), moreover $(p_0(y), q_0(y)) \in O_{\varepsilon'}$. The set $\Omega_{\varepsilon'}$ is the complement of the set $\overline{O_{\varepsilon'}}$ in the domain Ω. According to this decomposition, kernel (2.4) is the sum of two kernels which we denote $F_1(x,z,y,t)$ and $F_2(x,z,y,t)$ respectively.

Let us make a change of variables, setting

$$r = 2q(C(y) - f(p,q,y)) \tag{3.3}$$

in the kernel $F_1(x,z,y,t)$ which contains integration over the set $O_{\varepsilon'}$. Let u be the projection of a radius vector directed from the point $(p_0(y), q_0(y))$ to the point $(p,q) \in O_{\varepsilon'}$ on the tangent to the curve $h(p,q) = 0$ at the point $(p_0(y), q_0(y))$ or, that is the same, on the tangent to the curve $f(p,q,y) = C(y)$ at the same point, i.e.:

$$u = \frac{12p_0(y) - y}{\sqrt{16q_0^2(y) + (12p_0(y) - y)^2}}(q - q_0(y)) + \frac{4q_0(y)}{\sqrt{16q_0^2(y) + (12p_0(y) - y)^2}}(p - p_0(y)). \tag{3.4}$$

The system of equations (3.3), (3.4) has a unique solution with respect to p and q in O_ε as $\varepsilon' \le \frac{2\delta^3}{\sqrt{3}}$. Therefore in the neighbourhood of the point (p_0, q_0) the variables p and q can be expressed via the variables r and u:

$$p(r,u) = p_0 + k_1 r + k_2 u + k_3 ur + k_4 r^2 + k_5 u^2 \ldots,$$
$$q(r,u) = q_0 + \lambda_1 r + \lambda_2 u + \lambda_3 ur + \lambda_4 r^2 + \lambda_5 u^2 \ldots,$$

where k_n, λ_n are the coefficients of the corresponding Taylor series. The first have the form

$$
\begin{aligned}
k_1(y) &= \frac{12p_0(y) - y}{q_0(y)(16q_0^2(y) + (12p_0(y) - y)^2)}, \\
k_2(y) &= \frac{4q_0(y)}{\sqrt{16q_0^2(y) + (12p_0(y) - y)^2}}, \\
\lambda_1(y) &= -\frac{4}{16q_0^2(y) + (12p_0(y) - y)^2}, \\
\lambda_2(y) &= \frac{12p_0(y) - y}{\sqrt{16q_0^2(y) + (12p_0(y) - y)^2}}.
\end{aligned}
$$

One can obtain the expansion coefficients k_n and λ_n in an explicit form after n–times differentiation of (3.3) and (3.4) with respect to r and u. In the neighbourhood of the point $(p_0(y), q_0(y))$ the equation $q = h(p)$ can be written using variables r and u. It is easy to check that $\frac{\partial^2}{\partial p^2} h(p)|_{p=p_0(y)} \neq 0$ since $C'' > -1/24$ (Condition A). Therefore the curves $f(p, q, y) = C(y)$ and $q = h(p)$ have a contact of the first order, and $q = h(p)$ takes the form $u = u(r)$:

$$
u = \pm a(y)\sqrt{r} + b(y)r + \ldots,
$$

where

$$
\begin{aligned}
a(y) &= \left[\frac{16q_0^2(y) + (12p_0(y) - y)^2}{2q_0(y)\left(48q_0^2(y) - (12p_0(y) - y)^2 - 16h_{pp}(p_0(y))q_0^3(y)\right)} \right]^{\frac{1}{2}}, \\
h_{pp}(p_0) &= \frac{\partial^2 h(p)}{\partial p^2}|_{p=p_0}.
\end{aligned}
$$

Using the new variables u and r and the notation $E_0(\xi, \zeta, y) = e^{ip_0(y)(\xi-\zeta)-q_0(y)(\xi+\zeta)}$ we write the function $\tilde{F}_1(\xi, \zeta, y, t) = F_1(\xi + C(y)t, \zeta + C(y)t, y, t)$ as follows:

$$
\begin{aligned}
\tilde{F}_1(\xi, \zeta, y, t) &= E_0(\xi, \zeta, y) \\
&\times \int_0^{\varepsilon'} dr \int_{-a\sqrt{r}+\ldots}^{a\sqrt{r}+\ldots} du\, j(r, u, y)\tilde{g}(r, u, y)e^{i(p-p_0)(\xi-\zeta)-(q-q_0)(\xi+\zeta)-rt}
\end{aligned}
\tag{3.5}
$$

where $j(r, u, y) = j(p(r, u, y), q(r, u, y))$ is the Jacobian corresponding to the change of variables $(p, q) \to (r, u)$. Let us expand integrand in (3.5) into a series with respect to the powers of r and u in the neighbourhood of the point $(p_0(y), q_0(y))$ $(u = 0, r = 0)$:

$$
\begin{aligned}
&j(r, u, y)\tilde{g}(r, u, y)\exp[i(p - p_0(y))(\xi - \zeta) - (q - q_0(y))(\xi + \zeta)] \\
&\quad = \sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{l=0}^{j}\sum_{m=0}^{n-j} \zeta^j \xi^{n-j} r^{l+m} u^{n-l-m} \varphi_{n,j,l,m}(y)(1 + \psi_n(r, u)),
\end{aligned}
$$

where

$$\varphi_{n,j,l,m}(y) = \frac{(-1)^{n-m}}{l!m!(n-j-m)!(j-l)!}(ik_1(y)+\lambda_1(y))^l(ik_2(y)+\lambda_2(y))^{j-l}$$
$$\times(ik_1(y)-\lambda_1(y))^m(\lambda_2(y)-ik_2(y))^{n-j-m}g(y)j_0(y),$$
$$j_0(y) = j(0,0,y) = \frac{1}{q_0(y)\sqrt{16q_0^2(y)+(12p_0(y)-y)^2}},$$

$g(y) = \tilde{g}(0,0,y)$, and the functions $\psi_n(r,u)$ satisfy $|\psi_n(r,u)| \le An(r+|u|)$.

After integration with respect to u and r, and performance of natural estimates, we obtain

$$\tilde{F}_1(\xi,\zeta,y,t) = E_0(\xi,\zeta,y)\sum_{n=0}^{N-1}\sum_{j=0}^{N-n-1}\zeta^n\xi^j\frac{\psi_{nj}(y)}{t^{(n+j+3)/2}}(1+\delta_n(t)) + \Delta_N(\xi,\zeta,y,t),$$

where

$$\psi_{nj}(y) = \frac{g(y)j_0(y)a^{n+j+1}(y)}{2n!j!}\Gamma\left(\frac{n+j+1}{2}\right)(1+(-1)^{n+j}), \tag{3.6}$$

and $|\delta_n(t)| \le \dfrac{B_{nj}}{\sqrt{t}}$,

$$|\Delta_N| \le A(N)g(y)\sum_{j=0}^{N}\frac{|\zeta^j\xi^{N-j}|}{t^{(N+3)/2}}e^{-q_0(y)(\xi+\zeta)}, \tag{3.7}$$

where $A(N) \le \left(\dfrac{e^{N+3}}{(N+3)^{N+3}2^{N+4}}\right)^{1/2}$. Estimate (3.7) is valid as $|\xi| < t^{1/4}$, $|\zeta| < t^{1/4}$.

Let us estimate now the second kernel $F_2(\xi,\zeta,y,t)$ containing the integration over the set $\Omega_{\varepsilon'}$. Taking into account that $C(y) - f(p,q,y) \ge \frac{\varepsilon'}{2\varepsilon}$ $((p,q)\in\Omega_{\varepsilon'},\ q \ge \varepsilon > 0)$ and Condition C, we can write

$$|F_2(\xi,\zeta,y,t)| \le \iint_{\Omega_{\varepsilon'}} e^{-q(\xi+\zeta)-2q(C(y)-f(p,q,y))t}\, d\mu(p,q)$$
$$\le e^{-\frac{\varepsilon}{\varepsilon'}t}\iint_{\Omega_{\varepsilon'}} e^{q(|\xi|+|\zeta|)}\, d\mu(p,q) = O\left(e^{-\frac{\varepsilon}{\varepsilon'}t}\right). \tag{3.8}$$

Thus assuming $|\xi| < t^{1/4}$, $|\zeta| < t^{1/4}$ we obtain the final asymptotic formula

$$F(x,z,y,t) = E_0(\xi,\zeta,y)\sum_{n=0}^{N-1}\sum_{j=0}^{N-n-1}\zeta^n\xi^j\frac{\psi_{nj}(y)}{t^{(n+j+3)/2}}(1+\delta_n(t))$$
$$+ O\left(\sum_{j=0}^{N}\frac{|\zeta^j\xi^{N-j}|}{t^{(N+3)/2}}e^{-q_0(y)(\xi+\zeta)}\right) + O\left(e^{-\frac{\varepsilon}{\varepsilon'}t}\right) \tag{3.9}$$

with $\xi = x - C(y)t$, $\zeta = x - C(y)t$ and $|\delta_n(t)| \le t^{-1/2}$. It is not difficult to see that this asymptotic expression can be differentiated with respect to x.

The following estimates hold:

$$\int_x^\infty \int_x^\infty |\Delta_N(s,z,y,t)|^2 ds dz \le \frac{A^2(N)}{t^{1/2-\varepsilon_1}}, \qquad (0<\varepsilon_1<1/2) \tag{3.10}$$

in the domain

$$\zeta > \xi > -\frac{1}{2q_0(y)} \ln \frac{t^M}{g(y)}, \qquad |\ln g(y)| < \ln t, \qquad M = \frac{2N+5}{4},$$

and

$$\int_x^\infty \int_x^\infty |F_2(s,z,y,t)|^2 ds dz = O(e^{-\varepsilon' t}), \tag{3.11}$$

in the domain $x > C(y)t - \sqrt{t}$.

Taking into account (3.9), (3.10) and (3.11) we can formulate the following lemma.

Lemma 3. *Inside the domain*

$$\zeta > \xi > -\frac{1}{2q_0(y)} \ln t^{M+1}, \quad |\ln g(y)| < \ln t, \quad M > 1, \tag{3.12}$$

as $t \to \infty$ *the kernel* $\tilde F(\xi,\zeta,y,t)$ *is represented in the form*

$$\tilde F(\xi,\zeta,y,t) = F_N(\xi,\zeta,y,t) + \tilde G(\xi,\zeta,y,t),$$

where

$$F_N(\xi,\zeta,y,t) = e^{ip_0(\xi-\zeta)-q_0(\xi+\zeta)} \sum_{n=0}^{N-1} \sum_{j=0}^{N-n-1} \zeta^n \xi^j \frac{\psi_{nj}(y)}{t^{(n+j+3)/2}},$$

$N = [(4M-5)/2]$*, and the functions* $\psi_{nj}(y)$ *are bounded and defined in* (3.6)*. The function* $\tilde G(s,z,y,t)$ *admits the uniform estimate with respect to* y *in* (3.12)*:*

$$\int_\xi^\infty \int_\xi^\infty |\tilde G(s,z,y,t)|^2 ds dz = O\left(\frac{1}{t^{1/2-\varepsilon_1}}\right) \qquad (0<\varepsilon_1<1/2). \tag{3.13}$$

Second step. We show that after replacing the kernel $F(x,z,y,t)$ by the degenerate kernel $F_N(x,z,y,t)$ one can obtain an asymptotic representation of the solution $K(x,z,y,t)$ of the Marchenko integral equation (2.2) up to $O(t^{-1/2+\varepsilon_1})$ $(0<\varepsilon_1<1/2)$ as $t\to\infty$. Set up $\xi = x - C(y)t$, $\zeta = z - C(y)t$ and consider the domain $z > x > C(y) - \frac{1}{2q_0(y)} \ln t^{M+1}$, $|\ln g(y)| < \ln t$ with an arbitrary number $M > 1$. Let us introduce the operators

$$(\hat F_N f)(z) = \int_x^\infty F_N(s,z,y,t) f(s) ds, \qquad (\hat G_N f)(z) = \int_x^\infty G_N(s,z,y,t) f(s) ds,$$

in $L^2[x,\infty)$. Here $F_N(x,z,y,t)$ is the degenerate kernel

$$F_N(\xi,\zeta,y,t) = e^{ip_0(\xi-\zeta)-q_0(\xi+\zeta)} \sum_{n=0}^{N-1} \sum_{j=0}^{N-n-1} \zeta^n \xi^j \frac{\psi_{nj}(y)}{t^{(n+j+3)/2}}, \quad N = [(4M-5)/2], \tag{3.14}$$

with $\xi = x - C(y)t$, $\zeta = z - C(y)t$, and $\psi_{nj}(y)$ defined by (3.6). $G_N(x,z,y,t)$ is the difference between $F(x,z,y,t)$ (2.4) and $F_N(x,z,y,t)$:

$$G_N = F - F_N.$$

Now, (2.2) acquires the form

$$(I + \hat{F}_N)f + \hat{G}_N f = h_N + g_N, \tag{3.15}$$

where $f = K(x,z,y,t)$, $h_N = -F_N(x,z,y,t)$, $g_N = -G_N(x,z,y,t)$. By virtue of (3.13) one can easily obtain the following estimates for the norms of the operator $\hat{G}_N$ and the vector g_N in $L^2[x,\infty)$ as $z > x > C(y) - \dfrac{1}{2q_0(y)} \ln t^{M+1}$, $|\ln g(y)| < \ln t$, $t \to \infty$:

$$||\hat{G}_N|| \le A_1 t^{-1/2+\varepsilon_1}, \quad ||\hat{g}_N|| \le A_1 t^{-1/2+\varepsilon_1} \quad (A_1 = \text{const},\ 0 < \varepsilon_1 < 1/2). \tag{3.16}$$

The operator $I + \hat{F}_N$ is the direct sum of the two operators $I_1 + \hat{F}_N$ and I_2. The first one acts in the subspace H_1 of $L^2[x,\infty)$, which is generated by the vectors

$$e^{(ip_0 - q_0)z}, ze^{(ip_0-q_0)z}, \dots, z^{N-1}e^{(ip_0-q_0)z}.$$

The second operator I_2 ($I = I_1 \oplus I_2$) acts in the orthogonal complement $H_2 = L^2[x,\infty) \ominus H_1$. Since we have $\hat{F}_N = \hat{F} - \hat{G}_N$ and the operator $I + \hat{F}$ is invertible, we deduce that the operator $I + \hat{F}_N$ is also invertible in $L^2[x,\infty)$ and

$$||(I + \hat{F}_N)^{-1}||_{L^2[x,\infty)} \le A_2. \tag{3.17}$$

We shall look for a solution of (3.15) of the form $f = \phi_N + \psi_N$, where ϕ_N is the solution of the equation $(I + \hat{F}_N)\phi_N = h_N$. It implies that ψ_N satisfies

$$(I + \hat{F}_N)\psi_N = g_N - \hat{G}_N\phi_N.$$

According to (3.17), we have

$$||\psi_N|| \le A_2(||g_N|| + ||\hat{G}_N||\,||\psi_N||).$$

It follows from considerations presented below that ϕ_N is uniformly bounded with respect to $(x,y) \in G_M(t)$ and t in the space $L^2[x,\infty) \cap C[x,\infty)$. This fact and (3.16) allow us to conclude that as $t \to \infty$

$$f(z) = \phi_N(z) + O(t^{-1/2+\varepsilon_1}), \qquad 0 < \varepsilon_1 < 1/2. \tag{3.18}$$

<u>Third step.</u> The replacement of the kernel F by the degenerate kernel F_N in equation (2.2) and the implementation of the substitutions $x = C(y)t + \xi$, $z = C(y)t + \zeta$ allow us to obtain the following integral equation for the function $K_N(\xi,\zeta,y,t) = K(\xi + C(y)t, \zeta + C(y)t, y, t)$ ($\zeta > \xi$):

$$K_N(\xi,\zeta,y,t) + F_N(\xi,\zeta,y,t) + \int_\xi^\infty K_N(\xi,s,y,t)F_N(s,\zeta,y,t)ds = 0, \tag{3.19}$$

where F_N is given by (3.14).

According to (2.1) and (3.18), the following representation of the function $v(x, y, t)$ is valid in $G_M(t)$:

$$v(x, y, t) = 2\frac{\partial}{\partial \xi} K_N(\xi, \xi, y, t)\Big|_{\xi = x - C(y)t} + O(t^{-1/2+\varepsilon_1}). \tag{3.20}$$

We shall look for a solution of equation (3.19) of the form

$$K_N(\xi, \zeta, y, t) = \sum_{n=0}^{N-1} \gamma_n(\xi, y, t)\zeta^n e^{-(ip_0+q_0)\zeta}. \tag{3.21}$$

After substitution of (3.21) into (3.19) we obtain a system of algebraic equations for the function $\gamma_n(\xi, y, t)$:

$$\gamma_n + \sum_{m=0}^{N-1} \gamma_m \sum_{j=0}^{N-n-1} \frac{\psi_{nj}(y)}{t^{(n+j+3)/2}} \int_\xi^\infty s^{j+m} e^{-2q_0 s} ds = -\sum_{j=0}^{N-n-1} \frac{\psi_{nj}(y)}{t^{(n+j+3)/2}} \xi^j e^{(ip_0-q_0)\xi}, \tag{3.22}$$

$$n = 0, \dots, N-1.$$

The solution of (3.22) has the form

$$\gamma_l = \frac{\det[I + A(\xi, y, t)]^{(l)}}{\det[I + A(\xi, y, t)]},$$

where I is the identity matrix, $A(\xi, y, t)$ is the matrix with entries

$$\begin{aligned} &[A]_{n+1,m+1} = \sum_{j=0}^{N-n-1} \frac{\psi_{nj}}{t^{(n+j+3)/2}} I_{j+m}, \\ &I_{j+m} = \int_\xi^\infty s^{j+m} e^{-2q_0 s} ds, \quad n, m = 0, \dots, N-1. \end{aligned} \tag{3.23}$$

The matrix $[I + A(\xi, y, t)]^{(l)}$ is obtained via the substitution of the column of right-hand sides of the system (3.22) instead of l-th column of the matrix $[I + A(\xi, y, t)]$. The functions ψ_{nj} are defined by (3.6).

The substitution of $\gamma_n(\xi, y, t)$ into (3.21) gives us

$$K_N(\xi, \zeta, y, t) = \sum_{n=0}^{N-1} \frac{\det(I + A)^{(n)}}{\det(I + A)} \zeta^n e^{-(ip_0+q_0)\zeta}.$$

Hence for the solution $v(x, y, t)$ the following asymptotic representation holds true

$$v(x, y, t) = 2\frac{\partial^2}{\partial x^2} \ln \det[I + A(x - C(y)t, y, t)] + O(t^{-\frac{1}{2}+\varepsilon_1}) \qquad (t \to \infty). \tag{3.24}$$

To obtain asymptotics (3.24) we need to investigate the behaviour of $\Delta = \Delta(\xi, y, t) = \det[I + A(\xi, y, t)]$ in the domain (3.12) as $t \to \infty$. An analysis of the structure of the matrix $I + A(\xi, y, t)$ shows that $\Delta(\xi, y, t)$ can be represented in the form

$$\Delta(\xi, y, t) = 1 + \sum_{n=1}^{N} \frac{P_n(\xi, y, t)}{t^{n(n+2)/2}} e^{-2nq_0(y)\xi}, \tag{3.25}$$

where $P_n(\xi, y, t)$ are polynomials with respect to ξ of degree at most N^n with coefficients bounded with respect to y and t. We have the following asymptotic relations as $n \leq [(N+1)/2]$

$$\frac{P_n(\xi, y, t)}{t^{n(n+2)/2}} e^{-2nq_0(y)\xi} = \det[A^{(n)}(\xi, y, t)](1 + O(t^{-\frac{1}{2}+\varepsilon_1})), \tag{3.26}$$

where $A^{(n)}(\xi, y, t)$ are the n by n matrices with entries

$$A^{(n)}_{i+1,k+1} = \sum_{j=0}^{n-1} \frac{\psi_{ij} I_{j+k}}{t^{(i+j+3)/2}}.$$

The matrix $A^{(n)}(\xi, y, t)$ is obviously written as the product of two matrices. Therefore we have from (3.6) and (3.23):

$$\det[A^{(n)}(\xi, y, t)] = \frac{g^n(y) j_0^n(y) a^{n^2}(y)}{2^n (q_0(y))^{n^2} \prod_{k=0}^{n-1} (k!)^2} \frac{\Omega^{(n)} \tilde{I}^{(n)}(\xi, y)}{t^{\frac{n(n+2)}{2}}}, \tag{3.27}$$

where $\Omega^{(n)}$ and $\tilde{I}^{(n)}(\xi, y)$ are the determinants of the n by n matrices with entries

$$\begin{aligned} &\Omega^{(n)}_{i+1,k+1} = \Gamma\left(\frac{i+k+1}{2}\right)\left(1 + (-1)^{i+k}\right), \\ &\tilde{I}^{(n)}_{i+1,k+1} = \tilde{I}_{i+k} = \int_{q_0(y)\xi}^{\infty} s^{i+k} e^{-2s} ds, \\ &i, k = 0, 1, \ldots, n-1. \end{aligned} \tag{3.28}$$

The elements of the determinant $\tilde{I}^{(n)}$ satisfy the relations

$$\frac{d\tilde{I}_{i+k}}{d\xi} = -2q_0(y)\tilde{I}_{i+k} + (i+k)q_0(y)\tilde{I}_{i+k-1},$$

whence we have the equality

$$\frac{d\tilde{I}^{(n)}}{d\xi} + 2nq_0(y)\tilde{I}^{(n)} = q_0(y) \sum_{k=0}^{n-1} B^{(n)}_{k+1},$$

where $B^{(n)}_{k+1}$ is the determinant obtained from $\tilde{I}^{(n)}$ by replacing the $(k+1)$-th column by the column with entries $i\tilde{I}_{i+k-1}$ $(i = 0, 1, \ldots, n-1)$. We can show that $\sum_{k=0}^{n-1} B^{(n)}_{k+1} \equiv 0$, hence $\tilde{I}^{(n)}(\xi, y) = \tilde{I}^{(n)}(0, y)e^{-2nq_0(y)\xi}$. It follows from (3.28) that $\tilde{I}^{(n)}(0, y)$ do not depend on y, therefore we write finally

$$\tilde{I}^{(n)}(\xi, y) = \tilde{I}_0^{(n)} e^{-2nq_0(y)\xi},$$

where $\tilde{I}_0^{(n)}$ are positive numbers depending on n.

The determinant $\tilde{I}_0^{(n)}$ is, up to a factor 2^{-n^2}, the Gram determinant of the system of functions $x^k e^{-x/2}$ $(k = 0, 1, \dots, n-1)$ on the semi-axis $[0, \infty)$, therefore $\tilde{I}_0^{(n)} \neq 0$. The determinants $\Omega^{(n)}$ are also the Gram determinants of the system of functions

$$u_k = \begin{cases} (-1)^{(k+1)/4} x^{(2k-1)/4} e^{x/2}, & -\infty < x < 0 \\ x^{(2k-1)/4} e^{-x/2}, & 0 < x < \infty \end{cases}$$

on the axis $(-\infty, \infty)$, and $\Omega^{(n)} \neq 0$. Since in the region (3.12) the determinant Δ does not vanish (the operator $(I + F)^{-1}$ is bounded), it is not difficult to see, using (3.25)-(3.27), that $\Omega^{(n)} \tilde{I}_0^{(n)} > 0$.

Thus, for $n \le [(N+1)/2]$, $P_n(\xi, y, t) = P_n(y, t)$ does not depend on ξ, where

$$P_n(y, t) = \frac{g^n(y) j_0^n(y) a^{n^2}(y) \Omega^{(n)} \tilde{I}_0^{(n)}}{2^n (q_0(y))^{n^2} \prod_{k=0}^{n-1} (k!)^2} (1 + O(t^{-1/2})) > 0. \tag{3.29}$$

Let us turn back to the representation (3.24) of the JE-I solution. We can write

$$v(x, y, t) \sim 2 \frac{\partial^2}{\partial \xi^2} \ln \Delta = 2 \frac{\Delta'' \Delta - (\Delta')^2}{\Delta^2}. \tag{3.30}$$

Let us cover the domain $G_M(t)$ $(M > 2)$ by the subdomains

$$\begin{aligned}
a_1 &= \left\{ -\frac{1}{2q_0} \ln \frac{t^{2+\varepsilon}}{g(y)} < \xi < \infty \right\}, \\
a_n &= \left\{ -\frac{1}{2q_0} \ln \frac{t^{n+1+\varepsilon}}{g(y)} < \xi < -\frac{1}{2q_0} \ln \frac{t^{n-\varepsilon}}{g(y)} \right\}, \qquad n = 2, 3, \dots, m-1, \\
a_m &= \left\{ -\frac{1}{2q_0} \ln \frac{t^{M}}{g(y)} < \xi < -\frac{1}{2q_0} \ln \frac{t^{m-\varepsilon}}{g(y)} \right\},
\end{aligned}$$

where $m = \left[\dfrac{N+1}{2} \right] = [M-1]$, $\xi = x - C(y)t$. With ξ being inside any specific a_n the numerator and denominator in (3.30) have the asymptotic representations

$$\Delta'' \Delta - (\Delta')^2 = \frac{4q_0^2(y) e^{-4(n-1)q_0(y)\xi}}{t^{(n-1)(n+1)/2}} \frac{g^{2n-1} \tilde{P}_{n-1} \tilde{P}_n e^{-2q_0(y)\xi}}{t^{(2n+1)/2}} (1 + O(t^{-1/2})), \tag{3.31}$$

$$\Delta^2 = \frac{g^{2(n-1)} e^{-4(n-1)q_0(y)\xi}}{t^{(n-1)(n+1)}} \left[\tilde{P}_{n-1} + \frac{g \tilde{P}_n e^{-2q_0(y)\xi}}{t^{(2n+1)/2}} \right]^2 (1 + O(t^{-1/2})) \tag{3.32}$$

with

$$\tilde{P}_n(y) = \frac{j_0^n(y) a^{n^2}(y) \Omega^{(n)} \tilde{I}_0^{(n)}}{2^n (q_0(y))^{n^2} \prod_{k=0}^{n-1} (k!)^2}.$$

Using (3.31) and (3.32), we conclude that the solution of the JE-I possesses the following asymptotic representation uniformly with respect to ξ:

$$
\begin{aligned}
v(x,y,t) &\sim \left.\frac{8q_0^2(y)g\tilde{P}_{n-1}\tilde{P}_n\frac{e^{-2q_0(y)\xi}}{t^{-(2n+1)/2}}}{\left(\tilde{P}_{n-1}+g\tilde{P}_n\frac{e^{-2q_0(y)\xi}}{t^{-(2n+1)/2}}\right)^2}\right|_{\xi=x-C(y)t} \\
&= \left.\frac{8q_0^2(y)\frac{g\tilde{P}_n}{\tilde{P}_{n-1}}\frac{e^{-2q_0(y)\xi}}{t^{(2n+1)/2}}}{\left(1+\frac{g\tilde{P}_n}{\tilde{P}_{n-1}}\frac{e^{-2q_0(y)\xi}}{t^{(2n+1)/2}}\right)^2}\right|_{\xi=x-C(y)t} \\
&= \frac{2q_0^2(y)}{\cosh^2\left[q_0(y)\left(x-C(y)t+\frac{1}{2q_0(y)}\left(\ln t^{n+1/2}-\ln g(y)-\ln\frac{\tilde{P}_n(y)}{\tilde{P}_{n-1}(y)}\right)\right)\right]}.
\end{aligned} \tag{3.33}
$$

Thus, the JE-I solution splits in $G_M(t)$ into $[M-1]$ solitons of the form

$$
v_n(x,y,t) = \frac{2q_0^2(y)}{\cosh^2[q_0(y)\phi_n(x,y,t)]} \tag{3.34}
$$

with curved lines of constant phase

$$
\phi_n(x,y,t) = x - C(y)t + \frac{1}{2q_0(y)}\left(\ln t^{n+1/2} - \ln g(y) - \ln\tilde{\varphi}_n(y)\right)
$$

where

$$
\begin{aligned}
\tilde{\varphi}_n(y) &= \frac{j_0(y)a^{2n-1}(y)}{2(q_0(y))^{2n-1}[(n-1)!]^2}\frac{\Omega^{(n)}\tilde{I}_0^{(n)}}{\Omega^{(n-1)}\tilde{I}_0^{(n-1)}} \\
&= \frac{(C(y)+48(C'(y))^2)^{n-1}(1+24C''(y))^{n-1/2}Q^{(n)}\Gamma^{(n)}}{2^{(2n+5)/2}((n-1)!)^2(C(y)+12(C'(y))^2)^{(10n-3)/4}Q^{(n-1)}\Gamma^{(n-1)}},
\end{aligned}
$$

and $\Omega^{(n)}$, $\tilde{I}_0^{(n)}$ are the determinants of the n by n matrices with entries

$$
\begin{aligned}
{}[\Omega^{(n)}]_{i+1,k+1} &= \Gamma\left(\frac{i+k+1}{2}\right)\left(1+(-1)^{i+k}\right), \\
[\tilde{I}_0^{(n)}]_{i+1,k+1} &= \Gamma(i+k+1), \quad i,k=0,1,\dots,n-1.
\end{aligned}
$$

The theorem is proved. ■

4 Examples

Example 1. Assume that

$$
C(y) = \frac{y^2}{24} + \frac{1}{16}.
$$

Then

$$
p_0(y) = \frac{y}{6}, \quad q_0(y) = \frac{\sqrt{y^2+1}}{4},
$$

and the set Ω has the form

$$\Omega = \left\{(p,q) \,\middle|\, -\infty < p < \infty,\ 0 < \varepsilon \le q \le \frac{\sqrt{36p^2+1}}{4}\right\}.$$

Determine $g(y)$ and $d\mu$ as follows

$$g(y) = e^{-y^2}, \quad d\mu = e^{-(18p^2+2q^2-1/2)} dpdq.$$

Then Conditions A-C are fulfilled, and according to Theorem 1 there exists a JE-I solution which splits as $t \to \infty$ in the domain ($M > 2$)

$$G_M(t) = \left\{(x,y) \in \mathbb{R}^2 \,\middle|\, |y| < \sqrt{\ln t},\ x > \frac{y^2 t}{24} + \frac{t}{16} - \frac{2}{\sqrt{y^2+1}} \ln t^{M+1}\right\}$$

into $[M-1]$ curved solitons of the form

$$v_n(x,y,t) = \frac{y^2+1}{8\cosh^2\left[\frac{\sqrt{y^2+1}}{4}\psi_n(x,y,t)\right]},$$

$$\psi_n(x,y,t) = x - \frac{y^2 t}{24} - \frac{t}{16} + \frac{2}{\sqrt{y^2+1}}$$
$$\times \left(\ln t^{n+1/2} + y^2 - \ln \frac{3^{n-1/2}(6y^2+1)^{n-1} Q^{(n)}\Gamma^{(n)}}{2^{5n-3/2}(2y^2+1)^{(10n-3)/4}[(n-1)!]^2 Q^{(n-1)}\Gamma^{(n-1)}}\right).$$

Example 2. Suppose now that

$$C(y) = b^2 \quad (b = \text{const} > \varepsilon > 0).$$

Then

$$p_0(y) = \frac{y}{12},\ q_0(y) = b,$$

and the set Ω has the form

$$\Omega = \{(p,q) \mid -\infty < p < \infty, \quad 0 < \varepsilon \le q \le b\}.$$

Determine $g(y)$ and $d\mu$ as follows:

$$g(y) = e^{-y^2}, \qquad d\mu = e^{-(12p)^2} dpdq.$$

Then Conditions A-C are fulfilled, and according to Theorem 1 there exists a JE-I solution which splits as $t \to \infty$ in the domain ($M > 2$)

$$G_M(t) = \left\{(x,y) \in \mathbb{R}^2 \,\middle|\, |y| < \sqrt{\ln t},\ x > b^2 t - \frac{1}{2b^2} \ln t^{M+1}\right\}$$

into $[M-1]$ curved solitons of the form

$$v_n(x,y,t) = \frac{2b^2}{\cosh^2[b\psi_n(x,y,t)]},$$

$$\psi_n(x,y,t) = x - b^2 t$$
$$+ \frac{1}{2b}\left(\ln t^{n+1/2} + y^2 - \ln \frac{Q^{(n)}\Gamma^{(n)}}{b^{3(n-1/2)}[(n-1)!]^2 Q^{(n-1)}\Gamma^{(n-1)}}\right).$$

Example 3. We consider the function $C(y)$ (and also the functions $p_0(y)$, $q_0(y)$ and the set Ω) introduced in the previous example. Such structure of Ω allows us to formulate a weaker condition on the function $g(y)$. We suppose that $g(y) = (1 + y^{2\alpha})^{-1}$ with integer $\alpha \geq 4$ and the measure $d\mu$ instead of (2.9) satisfies the following inequality:

$$\iint_\Omega \frac{d\mu(p,q)}{1 + (12p)^{2\alpha}} < \infty.$$

Then the function $v(x, y, t)$ constructed by the scheme (2.1)-(2.3) satisfies the JE-I, but it is not infinitely differentiable. We can prove that such a JE-I solution splits as $t \to \infty$ in the domain $(M > 2)$

$$G_M(t) = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; y^{2\alpha} < t,\ x > b^2 t - \frac{1}{2b^2} \ln t^{M+1} \right\}$$

into $[M - 1]$ curved solitons of the form

$$v_n(x,y,t) = \frac{2b^2}{\cosh^2\left[b\psi_n(x,y,t)\right]},$$

$$\psi_n(x,y,t) = x - b^2 t + \frac{1}{2b}\left(\ln t^{n+1/2} + \ln(1 + y^{2\alpha}) - \ln \frac{Q^{(n)}\Gamma^{(n)}}{b^{3(n-1/2)}[(n-1)!]^2 Q^{(n-1)}\Gamma^{(n-1)}} \right).$$

Their lines of constant phase are deviated from the straight line just on the value $\frac{1}{2}\ln(1 + y^{2\alpha})$. Therefore we call them weakly curved solitons.

The approach developed in the present paper can be applied to solve inverse problems for other dispersion models, like the first-order Debye model, or more generally, the multi-resonance Lorentz and Nth-order Debye models. The generalization is straightforward and is based on the construction of proper piecewise holomorphic functions near each pole of the model.

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