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On Symmetries of Chern-Simons and BF Topological Theories

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Abstract

We describe constructing solutions of the field equations of Chern-Simons and topological BF theories in terms of deformation theory of locally constant (flat) bundles. Maps of flat connections into one another (dressing transformations) are considered. A method of calculating (nonlocal) dressing symmetries in Chern-Simons and topological BF theories is formulated.

1 Introduction

Let X be an oriented smooth manifold of dimension n, G a semisimple Lie group, $\mathfrak{g}$ its Lie algebra, P a principal G-bundle over X, A a connection 1-form on P and $F_A = dA + A \wedge A$ its curvature. We assume that X is a compact manifold without boundary.

A connection 1-form A on P is called *flat* if its curvature F_A vanishes,

$$dA + A \wedge A = 0. \tag{1.1a}$$

Locally eqs.(1.1a) are solved trivially, and on any sufficiently small open set $U \subset X$ we have $A = -(d\psi)\psi^{-1}$, where $\psi(x) \in G$. So, locally A is a pure gauge. But globally eqs.(1.1a) are nontrivial, and finding their solutions is not an easy problem.

It is well known that in the case $n = dim_{\mathbb{R}} X = 3$ eqs.(1.1a) are the field equations of Chern-Simons theories which describe flat connections on G-bundles over 3-manifolds X. Quantum observables of these theories are topological invariants of X [1, 2].

For a generalization of Chern-Simons theories to arbitrary dimensions, in addition to a connection A on a G-bundle $P \to X$, one considers a $(n-2)$-form B with values in the adjoint bundle $\mathrm{ad}P = P \times_G \mathfrak{g}$. Using the fields B and F_A, one introduces so-called BF theories [3, 4], for which the variation of the action w.r.t. B gives eqs.(1.1a), and the variation of the action w.r.t. A gives the equations

$$d_A B := dB + A \wedge B - B \wedge A = 0, \tag{1.1b}$$

where d_A is the covariant differential on P. For more details and references see [5].

The purpose of our paper is to describe the symmetry group of eqs.(1.1) arising as equations of motion in Chern-Simons and topological BF theories. Under the symmetry group of a system of differential equations we understand the group that maps solutions of this system into one another. Only gauge (and related to them) symmetries of eqs.(1.1) forming a small subgroup in the symmetry group have been considered in the literature.

In this paper we show how *all* symmetries of eqs.(1.1) can be calculated with the help of *deformation theory* methods. In fact, finding solutions and symmetries of eqs.(1.1) will be reduced to solving functional equations on some matrices. To illustrate the method, we write down some nontrivial explicit solutions of these functional equations and describe symmetries corresponding to them.

2 Definitions and notation

In this section we recall some definitions to be used in the following and fix the notation. In Sect.2.1 we introduce bundles Int$P$ of Lie groups, bundles ad$P$ of Lie algebras and sheaves of their sections [6, 7]. In considering sheaves of non-Abelian groups we follow [6] and the papers [8, 9, 10]. In Sect.2.2 we recall definitions [6, 8, 9, 10] of cohomology sets of manifolds with values in the sheaves of Lie groups.

2.1 Some bundles and sheaves

Let $\{P_i\}$ be a set of representatives of topological equivalence classes of G-bundles over X. We fix $P \in \{P_i\}$ and choose a *good* covering $\mathfrak{U} = \{U_\alpha, \alpha \in I\}$ of the manifold X, which is always possible. For such a covering $\mathfrak{U}$ each nonempty finite intersection $U_{\alpha_1} \cap ... \cap U_{\alpha_q}$ is diffeomorphic to an open ball in $\mathbb{R}^n$. Let $f = \{f_{\alpha\beta}\}$, $f_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ be transition functions determining the bundle P in a fixed trivialization.

Recall that a map of a subset $U \subset X$ into some set is called *locally constant* if it is constant on each connected component of the set U. The fibre bundle P is called *locally constant* if its transition functions $\{f_{\alpha\beta}\}$ are locally constant. All bundles associated with such a bundle are also called locally constant. In the following, we shall consider deformations of the locally constant bundle P. They are described in terms of cohomologies of the manifold X with values in sheaves of sections of some bundles associated with P. Below we introduce these sheaves.

Let us consider the bundle of groups Int$P = P \times_G G$ (G acts on itself by internal automorphisms: $h \mapsto ghg^{-1}$, $h, g \in G$) associated with P. Consider the sheaf $\mathfrak{G}_P$ of *smooth* sections of the bundle Int$P$ and its subsheaf $\mathbb{G}_P$ of *locally constant* sections. A section ψ of the sheaf $\mathbb{G}_P$ ($\mathfrak{G}_P$) over an open set $U \subset X$ is described by a collection $\{\psi_\alpha\}$ of locally constant (smooth) G-valued functions ψ_α on $U \cap U_\alpha \neq \varnothing$ such that $\psi_\alpha = f_{\alpha\beta}\psi_\beta f_{\alpha\beta}^{-1}$ on $U \cap U_\alpha \cap U_\beta \neq \varnothing$.

We also consider the adjoint bundle ad$P = P \times_G \mathfrak{g}$ of Lie algebras and denote by $\mathcal{A}_P^q$ the sheaf of smooth q-forms on X with values in the bundle ad$P$ ($q = 1, 2, ...$). The space of sections of the sheaf $\mathcal{A}_P^q$ over an open set U is the space $\mathcal{A}_P^q(U) \equiv \Gamma(U, \mathcal{A}_P^q)$ of smooth $\mathfrak{g}$-valued q-forms $\mathcal{A}^{(\alpha)}$ on $U \cap U_\alpha \neq \varnothing$ such that $\mathcal{A}^{(\alpha)} = f_{\alpha\beta}\mathcal{A}^{(\beta)}f_{\alpha\beta}^{-1}$ on $U \cap U_\alpha \cap U_\beta \neq \varnothing$.

In particular, for global sections of the sheaves $\mathcal{A}_P^q$ over X one often uses the notation

$$\Omega^q(X, \mathrm{ad}P) := \mathcal{A}_P^q(X) \equiv \Gamma(X, \mathcal{A}_P^q). \tag{2.1}$$

The sheaf $\mathfrak{G}_P$ (and $\mathbb{G}_P$) acts on the sheaves $\mathcal{A}_P^q, q = 1, 2, ...$, with the help of the adjoint representation. In particular, for any open set $U \subset X$ we have

$$A \mapsto \mathrm{Ad}_\psi A = \psi A \psi^{-1} + \psi d\psi^{-1}, \tag{2.2a}$$

$$F \mapsto \mathrm{Ad}_\psi F = \psi F \psi^{-1}, \tag{2.2b}$$

$$B \mapsto \mathrm{Ad}_\psi B = \psi B \psi^{-1}, \tag{2.2c}$$

where $\psi \in \mathfrak{G}_P(U)$, $A \in \mathcal{A}_P^1(U)$, $F \in \mathcal{A}_P^2(U)$, $B \in \mathcal{A}_P^{n-2}(U)$. Of course, $\psi d\psi^{-1} = 0$ if $\psi \in \mathbb{G}_P(U)$.

Denote by $i : \mathbb{G}_P \to \mathfrak{G}_P$ an embedding of $\mathbb{G}_P$ into $\mathfrak{G}_P$. We define a map $\delta^0 : \mathfrak{G}_P \to \mathcal{A}_P^1$ given for any open set U of the space X by the formula

$$\delta^0 \psi := -(d\psi)\psi^{-1}, \tag{2.3}$$

where $\psi \in \mathfrak{G}_P(U)$, $\delta^0\psi \in \mathcal{A}_P^1(U)$. Let us also introduce an operator $\delta^1 : \mathcal{A}_P^1 \to \mathcal{A}_P^2$, defined for an open set $U \subset X$ by the formula

$$\delta^1 A := dA + A \wedge A, \tag{2.4}$$

where $A \in \mathcal{A}_P^1(U)$, $\delta^1 A \in \mathcal{A}_P^2(U)$. In other words, the maps of sheaves $\delta^0 : \mathfrak{G}_P \to \mathcal{A}_P^1$ and $\delta^1 : \mathcal{A}_P^1 \to \mathcal{A}_P^2$ are defined by means of localization. Finally, we denote by $\mathcal{A}_P$ the subsheaf in $\mathcal{A}_P^1$, consisting of locally defined $\mathfrak{g}$-valued 1-forms A such that $\delta^1 A = 0$, i.e. sections $A \in \mathcal{A}_P(U)$ of the sheaf $\mathcal{A}_P = \mathrm{Ker}\,\delta^1$ over $U \subset X$ satisfy eqs.(1.1a).

2.2 Čech cohomology for sheaves of non-Abelian groups

Let S be a sheaf coinciding with either the sheaf $\mathbb{G}_P$ or the sheaf $\mathfrak{G}_P$ introduced in Sect.2.1. We consider a good open covering $\mathfrak{U} = \{U_\alpha\}$ of X and sections of the sheaf S over $U_{\alpha_0} \cap ... \cap U_{\alpha_q} \neq \varnothing$. In the trivialization over U_{α_k}, we can represent $h \in \Gamma(U_{\alpha_0} \cap ... \cap U_{\alpha_m} \cap ... \cap U_{\alpha_k} \cap ... \cap U_{\alpha_q}, S)$ by matrices $h^{(\alpha_k)}_{\alpha_0 ... \alpha_m ... \alpha_k ... \alpha_q}$, and in the trivialization over U_{α_m}, h is represented by

$$h^{(\alpha_m)}_{\alpha_0 ... \alpha_m ... \alpha_k ... \alpha_q} = f_{\alpha_m \alpha_k} h^{(\alpha_k)}_{\alpha_0 ... \alpha_m ... \alpha_k ... \alpha_q} f^{-1}_{\alpha_m \alpha_k}, \tag{2.5a}$$

where $\{f_{\alpha_m \alpha_k}\}$ are the transition functions of the bundle P. If we do not write a superscript of sections of bundles, we mean that

$$h_{\alpha_0 ... \alpha_q} := h^{(\alpha_0)}_{\alpha_0 ... \alpha_q}. \tag{2.5b}$$

A *q-cochain* of the covering $\mathfrak{U}$ with values in S is a collection $h = \{h_{\alpha_0 ... \alpha_q}\}$ of sections of the sheaf S over nonempty intersections $U_{\alpha_0} \cap ... \cap U_{\alpha_q}$. A set of q-cochains is denoted by $C^q(\mathfrak{U}, S)$; it is a *group* under the pointwise multiplication. In particular, for 0-cochains $h = \{h_\alpha\} \in C^0(\mathfrak{U}, S)$ with the coefficients in S we have $h_\alpha \in \Gamma(U_\alpha, S)$ and for 1-cochains $h = \{h_{\alpha\beta}\} \in C^1(\mathfrak{U}, S)$ we have $h_{\alpha\beta} \in \Gamma(U_\alpha \cap U_\beta, S)$.

Subsets of *cocycles* $Z^q(\mathfrak{U}, S) \subset C^q(\mathfrak{U}, S)$ for $q = 0, 1$ are defined as follows:

$$Z^0(\mathfrak{U}, S) := \{h \in C^0(\mathfrak{U}, S) : h_\alpha f_{\alpha\beta} h_\beta^{-1} = f_{\alpha\beta} \text{ on } U_\alpha \cap U_\beta \neq \varnothing\}, \tag{2.6}$$

$$Z^1(\mathfrak{U}, S) := \{h \in C^1(\mathfrak{U}, S) : h_{\alpha\beta} f_{\alpha\beta} h_{\beta\gamma} f_{\beta\gamma} h_{\gamma\alpha} f_{\gamma\alpha} = 1 \text{ on } U_\alpha \cap U_\beta \cap U_\gamma \neq \varnothing\}. \tag{2.7a}$$

It follows from (2.7a) that

$$h^{(\alpha)}_{\alpha\beta} h^{(\alpha)}_{\beta\alpha} = 1 \quad \text{on} \quad U_\alpha \cap U_\beta \neq \varnothing, \tag{2.7b}$$

$$h_{\alpha\alpha} = 1 \quad \text{on} \quad U_\alpha, \tag{2.7c}$$

where $h^{(\alpha)}_{\beta\alpha} = f_{\alpha\beta} h_{\beta\alpha} f^{-1}_{\alpha\beta}$ (see (2.5)). We see from (2.6) that $Z^0(\mathfrak{U}, S)$ coincides with the group

$$H^0(X, S) := \Gamma(X, S) \tag{2.8}$$

of global sections of the sheaf S. The set $Z^1(\mathfrak{U}, S)$ is not a group if the structure group G of the bundle P is a non-Abelian group.

For $h \in C^0(\mathfrak{U}, S)$, $\phi \in Z^1(\mathfrak{U}, S)$ we define the left action T_0 of the group $C^0(\mathfrak{U}, S)$ on the set $Z^1(\mathfrak{U}, S)$ by the formula

$$T_0(h, \phi)_{\alpha\beta} = h_\alpha \phi_{\alpha\beta} (h^{(\alpha)}_\beta)^{-1} \equiv h_\alpha \phi_{\alpha\beta} f_{\alpha\beta} h_\beta^{-1} f^{-1}_{\alpha\beta}. \tag{2.9a}$$

A set of orbits of the group $C^0(\mathfrak{U}, S)$ in $Z^1(\mathfrak{U}, S)$ is called a *1-cohomology set* and denoted by $H^1(\mathfrak{U}, S)$. In other words, two cocycles $\phi, \tilde{\phi} \in Z^1(\mathfrak{U}, S)$ are called *equivalent* (cohomologous) if

$$\tilde{\phi}_{\alpha\beta} = h_\alpha \phi_{\alpha\beta} (h^{(\alpha)}_\beta)^{-1} \tag{2.9b}$$

for some $h \in C^0(\mathfrak{U}, S)$, and the 1-cohomology set

$$H^1(\mathfrak{U}, S) := C^0(\mathfrak{U}, S) \backslash Z^1(\mathfrak{U}, S) \tag{2.10}$$

is a set of equivalence classes of 1-cocycles. Since the chosen covering $\mathfrak{U} = \{U_\alpha\}$ is acyclic for the sheaf S, we have $H^1(\mathfrak{U}, S) = H^1(X, S)$. For more details see [6, 7, 8, 9, 10].

3 Moduli spaces

Chern-Simons and topological BF theories give a field-theoretic description of locally constant (flat) bundles which is discussed in this section. In Sect.3.1 we introduce the de Rham 1-cohomology set describing the moduli space of flat connections and give the de Rham description of moduli of covariantly constant fields B. Using homological algebra methods [8, 9, 10], in Sect.3.2 we present the equivalent Čech description of the moduli spaces of flat connections and covariantly constant fields B.

3.1 Flat connections, ad$P$-valued forms and de Rham cohomology

We denote by $\mathcal{G}_P$ the infinite-dimensional group of gauge transformations, i.e. smooth automorphisms of the bundle $P \to X$ which induce the identity transformation of X. Put another way, $\mathcal{G}_P$ is the group of global smooth sections of the bundle Int$P$ [11], and therefore we have

$$\mathcal{G}_P = H^0(X, \mathfrak{G}_P) \equiv \Gamma(X, \mathfrak{G}_P) = Z^0(\mathfrak{U}, \mathfrak{G}_P) \tag{3.1}$$

with the notation of Sect.2.

The space of all connections on P is an infinite-dimensional affine space modelled on the vector space $H^0(X, \mathcal{A}^1_P) = \Omega^1(X, \mathrm{ad}P)$ of global sections of the sheaf $\mathcal{A}^1_P$ and therefore the space of flat connections on P can be identified with the space $H^0(X, \mathcal{A}_P)$ of global sections of the sheaf $\mathcal{A}_P = \mathrm{Ker}\,\delta^1$ (see (2.4)). The group (3.1) acts on the space $H^0(X, \mathcal{A}_P)$ on the right by the formula

$$A \mapsto \mathrm{Ad}_{g^{-1}} A = g^{-1}Ag + g^{-1}dg,$$

where $g \in \mathcal{G}_P$, and we introduce the de Rham 1-cohomology set $H^1_{d_A;P}(X)$ as a set of orbits of the non-Abelian group $H^0(X, \mathfrak{G}_P)$ in the set $H^0(X, \mathcal{A}_P)$,

$$H^1_{d_A;P}(X) := H^0(X, \mathcal{A}_P)/H^0(X, \mathfrak{G}_P) = \{A \in \Omega^1(X, \mathrm{ad}P) : dA + A \wedge A = 0\}/\mathcal{G}_P. \tag{3.2}$$

This definition is a generalization to ad$P$-valued 1-forms A and the covariant exterior derivative d_A of the standard definition of the 1st de Rham cohomology group

$$H^1_d(X) = \frac{\{\omega \in \Omega^1(X) : d\omega = 0\}}{\{\omega = d\varphi, \varphi \in \Omega^0(X)\}}, \tag{3.3}$$

where d is the exterior derivative.

Remark. For an Abelian group G, the group (3.1) of gauge transformations and the set $H^0(X, \mathcal{A}_P)$ are Abelian groups. In this case, (3.2) is reduced in fact to the definition (3.3) of the standard de Rham cohomology.

The definition (3.2) is nothing but the definition of the moduli space $\mathcal{M}_P$ of flat connections on P,

$$\mathcal{M}_P = H^1_{d_A;P}(X) = H^0(X, \mathcal{A}_P)/H^0(X, \mathfrak{G}_P). \tag{3.4}$$

So, the moduli space $\mathcal{M}_P$ of flat connections on P is the space of gauge inequivalent solutions to eqs.(1.1a) as it should be.

Equations (1.1b) are linear in B. For any fixed flat connection A, the moduli space of solutions to eqs.(1.1b) is a vector space

$$\mathcal{B}_A = H^{n-2}_{d_A;P}(X) = \frac{\{B \in \Omega^{n-2}(X, \mathrm{ad}P) : d_A B = 0\}}{\{B = d_A\Phi, \Phi \in \Omega^{n-3}(X, \mathrm{ad}P)\}}, \tag{3.5}$$

since solutions of the form $B = d_A\Phi$ are considered as trivial (topological "symmetry" $B \mapsto B + d_A\Phi$) [5].

Nontrivial solutions of eqs.(1.1b) form the vector space $\mathcal{B}_A$ depending on a solution A of eqs.(1.1a). The space of solutions to eqs.(1.1) forms a vector bundle $\mathcal{T}_P \to H^0(X, \mathcal{A}_P)$, the base space of which is the space $H^0(X, \mathcal{A}_P)$ of solutions to eqs.(1.1a), and fibres of the bundle at points $A \in H^0(X, \mathcal{A}_P)$ are the vector spaces $\mathcal{B}_A$ of nontrivial solutions to eqs.(1.1b).

Notice that the gauge group $\mathcal{G}_P = H^0(X, \mathfrak{G}_P)$ acts on solutions B of eqs.(1.1b) by the formula

$$B \mapsto \mathrm{Ad}_{g^{-1}} B = g^{-1} B g, \tag{3.6}$$

where $g \in \mathcal{G}_P$. Therefore, identifying points $(A, B) \in \mathcal{T}_P$ and $(g^{-1}Ag + g^{-1}dg, g^{-1}Bg) \in \mathcal{T}_P$ for any $g \in \mathcal{G}_P$, we obtain the moduli space

$$\mathcal{N}_P = \mathcal{T}_P / \mathcal{G}_P \tag{3.7}$$

of solutions to eqs.(1.1). The space $\mathcal{N}_P$ is the vector bundle over the moduli space $\mathcal{M}_P$ of flat connections with fibres at points $[A] \in \mathcal{M}_P$ isomorphic to the vector space $\mathcal{B}_A$. We denote by $[A]$ the gauge equivalence class of a flat connection A.

3.2 Flat connections, ad$P$-valued forms and Čech cohomology

By using definitions of the sheaves $\mathbb{G}_P, \mathfrak{G}_P$ and $\mathcal{A}_P$, it is not difficult to verify that the sequence of sheaves

$$e \longrightarrow \mathbb{G}_P \xrightarrow{i} \mathfrak{G}_P \xrightarrow{\delta^0} \mathcal{A}_P \xrightarrow{\delta^1} e \tag{3.8}$$

is exact. Here e is a marked element of the considered sets (the identity in the sheaf $\mathbb{G}_P \subset \mathfrak{G}_P$ and zero in the sheaf $\mathcal{A}_P$). From (3.8) we obtain the exact sequence of cohomology sets [8, 9, 10]

$$e \longrightarrow H^0(X, \mathbb{G}_P) \xrightarrow{i_*} H^0(X, \mathfrak{G}_P) \xrightarrow{\delta^0_*} H^0(X, \mathcal{A}_P) \xrightarrow{\delta^1_*} H^1(X, \mathbb{G}_P) \xrightarrow{\rho} H^1(X, \mathfrak{G}_P), \tag{3.9}$$

where the map ρ coincides with the canonical embedding induced by the embedding of sheaves $i : \mathbb{G}_P \to \mathfrak{G}_P$.

By definition the set of equivalence classes of locally constant G-bundles over X is parametrized by the Čech 1-cohomology set $H^1(X, \mathbb{G}_{P_0}) \simeq H^1(X, \mathbb{G}_P)$ [6, 8, 9]. The kernel $\mathrm{Ker}\,\rho = \rho^{-1}(e)$ of the map ρ coincides with a subset of those elements $[\hat{f} f^{-1}]$ from $H^1(X, \mathbb{G}_P)$ which are mapped into the class $e \in H^1(X, \mathfrak{G}_P)$ of bundles $\hat{P}$ diffeomorphic to the bundle P defined by the 1-cocycle f. In terms of transition functions $\{f_{\alpha\beta}\}$ in P and $\{\hat{f}_{\alpha\beta}\}$ in $\hat{P}$ this means that there exists a smooth 0-cochain $\psi = \{\psi_\alpha\} \in C^0(\mathfrak{U}, \mathfrak{G}_P)$ such that

$$\hat{f}_{\alpha\beta} = \psi_\alpha^{-1} f_{\alpha\beta} \psi_\beta, \tag{3.10}$$

i.e. cocycles f and $\hat{f}$ are smoothly cohomologous.

By virtue of the exactness of the sequence (3.9), the space $\mathrm{Ker}\,\rho$ is bijective to the quotient space (3.2). So we have

$$\mathcal{M}_P = H^1_{d_A;P}(X) \simeq \mathrm{Ker}\,\rho \subset H^1(X, \mathbb{G}_P), \tag{3.11}$$

i.e. there is a one-to-one correspondence between the moduli space $\mathcal{M}_P$ of flat connections on P and the moduli space $\operatorname{Ker}\rho \subset H^1(X, \mathbb{G}_P)$ of those locally constant bundles $\hat{P}$ which are diffeomorphic to the bundle P. Notice that to $\hat{f} = \{\psi_\alpha^{-1} f_{\alpha\beta}\psi_\beta\}$, representing $[\hat{f}f^{-1}] \in \operatorname{Ker}\rho$ with fixed f, there corresponds a flat connection $A = \{A^{(\alpha)}\} = \{\psi_\alpha d\psi_\alpha^{-1}\} \in H^0(X, \mathcal{A}_P)$ representing $[A] \in H^1_{d_A;P}(X)$.

Remark. It is well known that the bundle $P \to X$ is always diffeomorphic to the trivial bundle $X \times G$ if X is a paracompact and simply connected manifold [6]. It is also known that if the structure group G is connected and simply connected, and $dim_{\mathbb{R}} X \leq 3$, then all G-bundles over X are topologically trivial, i.e. $H^1(X, \mathfrak{G}_P) = e$. In both the cases we have

$$H^1_{d_A;P}(X) \simeq H^1(X, \mathbb{G}_P), \tag{3.12}$$

since now $\operatorname{Ker}\rho = H^1(X, \mathbb{G}_P)$. The bijections (3.11) and (3.12) are non-Abelian variants of the isomorphism between Čech and de Rham cohomologies.

The moduli space (3.5) of solutions to eqs.(1.1b) can also be described in terms of Čech cohomology. Namely, notice that for a fixed flat connection $A = \{\psi_\alpha d\psi_\alpha^{-1}\}$ a general solution of eqs.(1.1b) has the form

$$B = \psi B_0 \psi^{-1} = \{\psi_\alpha B_0^{(\alpha)} \psi_\alpha^{-1}\}, \tag{3.13a}$$

where $\psi = \{\psi_\alpha\} \in C^0(\mathfrak{U}, \mathfrak{G}_P)$, and B_0 is an $\mathrm{ad}\hat{P}$-valued $(n-2)$-form satisfying the equation

$$dB_0 = 0. \tag{3.13b}$$

Therefore the space $H^{n-2}_{d_A;P}(X)$ of nontrivial solutions of eqs.(1.1b) is isomorphic to the standard $(n-2)$th de Rham cohomology group

$$H^{n-2}_{d;\hat{P}}(X) = \frac{\{B_0 \in \Omega^{n-2}(X, \mathrm{ad}\hat{P}) : dB_0 = 0\}}{\{B_0 = d\Phi_0,\ \Phi_0 \in \Omega^{n-3}(X, \mathrm{ad}\hat{P})\}} \tag{3.14}$$

for forms with values in the bundle $\mathrm{ad}\hat{P}$.

Formulae (3.13) mean that the space $H^{n-2}_{d_A;P}(X)$ is the "dressed" space $H^{n-2}_{d;\hat{P}}(X)$, and we have

$$H^{n-2}_{d_A;P}(X) \simeq H^{n-2}_{d;\hat{P}}(X). \tag{3.15a}$$

On the other hand, it is well known that

$$H^{n-2}_{d;\hat{P}}(X) \simeq H^{n-2}(X, \mathbf{g}_{\hat{P}}), \tag{3.15b}$$

where $H^{n-2}(X, \mathbf{g}_{\hat{P}})$ is the $(n-2)$th Čech cohomology group of the manifold X with the coefficients in the sheaf $\mathbf{g}_{\hat{P}}$ of locally constant sections of the bundle $\mathrm{ad}\hat{P}$.

4 Construction of solutions

In this section we discuss constructing solutions of eqs.(1.1) in some more detail. Namely, in Sect.4.1 we consider a collection of group-valued smooth functions defining maps between smooth and locally constant trivializations of flat bundles. These functions parametrize flat connections and satisfy first order differential equations. The solution space of these equations is discussed in Sect.4.1. In the Čech approach flat bundles are described by locally constant transition functions. In Sect.4.2 we discuss functional matrix equations on transition functions and the space of their solutions. In Sect.4.3 we describe the moduli space of flat connections as a double coset space and discuss the correspondence between the Čech and de Rham descriptions of flat connections.

4.1 Differential compatibility equations

As before, we fix a good covering $\mathfrak{U} = \{U_\alpha\}$ of X, trivializations of the bundle P over U_α's and locally constant transition functions $\{f_{\alpha\beta}\}$ on $U_\alpha \cap U_\beta \neq \varnothing$, $df_{\alpha\beta} = 0$, $\alpha, \beta \in I$. Any solution of equations (1.1a) on an open set U_α (a *local* solution) has the form

$$A^{(\alpha)} = -(d\psi_\alpha)\psi_\alpha^{-1}, \tag{4.1a}$$

where $\psi_\alpha(x)$ is a smooth G-valued function on U_α. If we find solutions (4.1a) for all $\alpha \in I$, we obtain a collection $\psi = \{\psi_\alpha\} \in C^0(\mathfrak{U}, \mathfrak{G}_P)$. If $U_\alpha \cap U_\beta \neq \varnothing$, then the solutions (4.1a) on U_α and U_β are not independent and must satisfy the compatibility conditions

$$A^{(\alpha)} = f_{\alpha\beta} A^{(\beta)} f_{\alpha\beta}^{-1} \tag{4.1b}$$

on $U_\alpha \cap U_\beta$. From eqs.(4.1) the *differential* compatibility equations

$$(d\psi_\alpha)\psi_\alpha^{-1} = f_{\alpha\beta}(d\psi_\beta)\psi_\beta^{-1} f_{\alpha\beta}^{-1} \tag{4.2}$$

follow. Since we are looking for *global* solutions $A = \{A^{(\alpha)}\}$ of eqs.(1.1a), eqs.(4.2) must be satisfied for any $\alpha, \beta \in I$ such that $U_\alpha \cap U_\beta \neq \varnothing$.

Any global solution of eqs.(1.1b) has the form

$$B = \psi B_0 \psi^{-1} = \{\psi_\alpha B_0^{(\alpha)} \psi_\alpha^{-1}\} = \{B^{(\alpha)}\}, \tag{4.3a}$$

where $\psi = \{\psi_\alpha\}$ satisfy eqs.(4.2), and B_0 is an arbitrary (global) solution of eqs.(3.13b). The compatibility conditions for $B^{(\alpha)}$ and $B^{(\beta)}$ on $U_\alpha \cap U_\beta \neq \varnothing$ have the form

$$B^{(\alpha)} = f_{\alpha\beta} B^{(\beta)} f_{\alpha\beta}^{-1}. \tag{4.3b}$$

After substituting (4.3a) into (4.3b), we obtain that

$$B_0^{(\alpha)} = \hat{f}_{\alpha\beta} B_0^{(\beta)} \hat{f}_{\alpha\beta}^{-1}, \tag{4.3c}$$

where $\hat{f}_{\alpha\beta} = \psi_\alpha^{-1} f_{\alpha\beta} \psi_\beta$, $d\hat{f}_{\alpha\beta} = 0$. Thus, one can easily construct solutions of eqs.(1.1b) if one knows solutions of eqs.(1.1a) or (4.2). That is why in the following we shall concentrate on describing solutions of eqs.(1.1a) and (4.2).

Differential equations (4.2) are equations on 0-cochains $\psi = \{\psi_\alpha\} \in C^0(\mathfrak{U}, \mathfrak{G}_P)$. We denote by $C^0_f(\mathfrak{U}, \mathfrak{G}_P)$ the space of solutions $\psi = \{\psi_\alpha\}$ of eqs.(4.2), $C^0_f(\mathfrak{U}, \mathfrak{G}_P) \subset C^0(\mathfrak{U}, \mathfrak{G}_P)$. Recall that according to the definitions of Sections 2 and 3, the solution space of eqs.(1.1a) is $Z^0(\mathfrak{U}, \mathcal{A}_P) \equiv H^0(X, \mathcal{A}_P)$. Restricting the map δ^0 to $C^0_f(\mathfrak{U}, \mathfrak{G}_P)$, we obtain the map

$$\delta^0 : \quad C^0_f(\mathfrak{U}, \mathfrak{G}_P) \longrightarrow Z^0(\mathfrak{U}, \mathcal{A}_P). \tag{4.4}$$

It is obvious from eqs.(4.1) that $\{\psi_\alpha^{-1}\} \in C^0_f(\mathfrak{U}, \mathfrak{G}_P)$ and $\{h_\alpha \psi_\alpha^{-1}\} \in C^0_f(\mathfrak{U}, \mathfrak{G}_P)$ with $\{h_\alpha\} \in C^0(\mathfrak{U}, \mathbb{G}_P)$ define the same solution $A = \{A^{(\alpha)}\} \in Z^0(\mathfrak{U}, \mathcal{A}_P)$ of eqs.(1.1a), which leads to the bijection

$$Z^0(\mathfrak{U}, \mathcal{A}_P) \simeq C^0(\mathfrak{U}, \mathbb{G}_P) \backslash C^0_f(\mathfrak{U}, \mathfrak{G}_P). \tag{4.5}$$

The gauge group $\mathcal{G}_P$ acts on $\psi^{-1} = \{\psi_\alpha^{-1}\}$ by the right multiplication: $\psi^{-1} \mapsto \psi^{-1} g = \{\psi_\alpha^{-1} g_\alpha\}$, $g = \{g_\alpha\} \in Z^0(\mathfrak{U}, \mathfrak{G}_P) \equiv H^0(X, \mathfrak{G}_P)$, and the definition (3.4) of the moduli space of flat connections can be reformulated in terms of the set $C^0_f(\mathfrak{U}, \mathfrak{G}_P)$ by using the bijection (4.5),

$$\mathcal{M}_P \simeq C^0(\mathfrak{U}, \mathbb{G}_P) \backslash C^0_f(\mathfrak{U}, \mathfrak{G}_P) / Z^0(\mathfrak{U}, \mathfrak{G}_P).$$

4.2 Functional matrix equations

Having a solution $\psi = \{\psi_\alpha\}$ of eqs.(4.2), one can introduce matrices $\hat{f}_{\alpha\beta} = \psi_\alpha^{-1} f_{\alpha\beta} \psi_\beta$ defined on $U_\alpha \cap U_\beta \neq \varnothing$. Using eqs.(4.2), it is not difficult to show that $d\hat{f}_{\alpha\beta} = 0$, and therefore $\hat{f} f^{-1} = \{\hat{f}_{\alpha\beta} f_{\alpha\beta}^{-1}\} \in C^1(\mathfrak{U}, \mathbb{G}_P)$. Moreover, it is easy to see that $\hat{f}_{\alpha\beta}$'s satisfy equations $\hat{f}_{\alpha\beta} \hat{f}_{\beta\gamma} \hat{f}_{\gamma\alpha} = 1$ and therefore $\{\hat{f}_{\alpha\beta}\}$ can be considered as transition functions of a locally constant bundle $\hat{P} \to X$ which is topologically equivalent to the bundle P with the transition functions $\{f_{\alpha\beta}\}$. Thus, we obtain a map of the data (f, A) into the data $(f, \hat{f})$, i.e. the connection A is encoded into the transition functions $\hat{f}$ of the locally constant bundle $\hat{P}$.

Let us now consider "free" 1-cochains $\hat{f} = \{\hat{f}_{\alpha\beta}\}$ such that $\hat{f} f^{-1} \in C^1(\mathfrak{U}, \mathbb{G}_P)$ and *functional* equations

$$\hat{f}_{\alpha\beta} \hat{f}_{\beta\gamma} \hat{f}_{\gamma\alpha} = 1 \tag{4.6a}$$

on $U_\alpha \cap U_\beta \cap U_\gamma \neq \varnothing$. The space of solutions to eqs.(4.6a) is isomorphic to the space $Z^1(\mathfrak{U}, \mathbb{G}_P)$ of 1-cocycles. Denote by $Z^1_f(\mathfrak{U}, \mathbb{G}_P)$ the subset of those solutions $\hat{f}$ to eqs.(4.6a) for which there exists a splitting

$$\hat{f}_{\alpha\beta} = \psi_\alpha^{-1} f_{\alpha\beta} \psi_\beta \tag{4.6b}$$

with some $\psi = \{\psi_\alpha\} \in C^0(\mathfrak{U}, \mathfrak{G}_P)$. Using equations $d\hat{f}_{\alpha\beta} = 0$, one can easily show that these $\{\psi_\alpha\}$ will satisfy eqs.(4.2), i.e. $\psi = \{\psi_\alpha\} \in C^0_f(\mathfrak{U}, \mathfrak{G}_P)$. Then, by introducing $A^{(\alpha)} = \delta^0 \psi_\alpha = -(d\psi_\alpha)\psi_\alpha^{-1}$, we obtain a flat connection $A = \{A^{(\alpha)}\}$ on P. Thus, for constructing flat connections on P, one can solve either *differential* equations (4.2) on $\{\psi_\alpha\} \in C^0(\mathfrak{U}, \mathfrak{G}_P)$ or *functional* equations (4.6) on $\{\hat{f}_{\alpha\beta}\}$ such that $\hat{f} f^{-1} \in C^1(\mathfrak{U}, \mathbb{G}_P)$.

4.3 Solution spaces $\longrightarrow$ moduli spaces

In Sections 4.1, 4.2 we showed that for the fixed transition functions $f = \{f_{\alpha\beta}\}$ of the locally constant bundle P there is a correspondence

$$Z^0(\mathfrak{U}, \mathcal{A}_P) \ni (f, A) \quad \leftrightarrow \quad (f, \hat{f}) \in Z^1_f(\mathfrak{U}, \mathbb{G}_P), \tag{4.7}$$

where $Z^0(\mathfrak{U}, \mathcal{A}_P)$ is the solution space of differential equations (1.1a), and $Z^1_f(\mathfrak{U}, \mathbb{G}_P)$ is the solution space of functional equations (4.6). These solution spaces are not bijective. Namely, to gauge equivalent flat connections A and $\mathrm{Ad}_{g^{-1}} A = g^{-1}Ag + g^{-1}dg$ with $g = \{g_\alpha\} \in Z^0(\mathfrak{U}, \mathfrak{S}_P)$ there corresponds the same cocycle $\hat{f} = \{\hat{f}_{\alpha\beta}\} \in Z^1_f(\mathfrak{U}, \mathbb{G}_P)$, since $\hat{f}^g_{\alpha\beta} := (\psi^g_\alpha)^{-1} f_{\alpha\beta} \psi^g_\beta = \psi^{-1}_\alpha g_\alpha f_{\alpha\beta} g^{-1}_\beta \psi_\beta = \psi^{-1}_\alpha f_{\alpha\beta} \psi_\beta = \hat{f}_{\alpha\beta}$. At the same time, the cocycles $\hat{f}$ and $\hat{f}^h = \{h_\alpha \hat{f}_{\alpha\beta} h^{-1}_\beta\}$ with $h = \{h_\alpha\} \in C^0(\mathfrak{U}, \mathbb{G}_P)$ correspond to the same flat connection A.

The correspondence (4.7) will become the bijection (3.11) if we consider the space of orbits of the group $Z^0(\mathfrak{U}, \mathfrak{S}_P) = \mathcal{G}_P$ (the gauge group of the model (1.1a)) in the space $Z^0(\mathfrak{U}, \mathcal{A}_P)$ and consider the space of orbits of the group $C^0(\mathfrak{U}, \mathbb{G}_P)$ (the gauge group of the model (4.6)) in the space $Z^1_f(\mathfrak{U}, \mathbb{G}_P)$. In other words, we should consider the pairs $(f, [A])$ and $(f, [\hat{f}])$, where $[A]$ is the gauge equivalence class of A, and $[\hat{f}]$ is the equivalence class of the 1-cocycle $\hat{f}$. In Sect.3 this correspondence was described in more abstract terms.

We obtain the following correspondence between de Rham and Čech description of flat connections:

$$\begin{array}{ccc} & C^0_f(\mathfrak{U}, \mathfrak{S}_P) & \\ & \delta^0 \swarrow \quad \searrow r & \\ & Z^0(\mathfrak{U}, \mathcal{A}_P) \qquad Z^1_f(\mathfrak{U}, \mathbb{G}_P) & \\ & \pi \downarrow \qquad\qquad \downarrow p & \\ \mathcal{M}_P = \frac{Z^0(\mathfrak{U}, \mathcal{A}_P)}{Z^0(\mathfrak{U}, \mathfrak{S}_P)} & \longleftrightarrow & \frac{Z^1_f(\mathfrak{U}, \mathbb{G}_P)}{C^0(\mathfrak{U}, \mathbb{G}_P)} \end{array} \quad . \tag{4.8}$$

Here a map

$$r: \quad C^0_f(\mathfrak{U}, \mathfrak{S}_P) \longrightarrow Z^1_f(\mathfrak{U}, \mathbb{G}_P) \tag{4.9a}$$

is given by the formula $r(\{\psi_\alpha\}) = \{\psi^{-1}_\alpha f_{\alpha\beta} \psi_\beta\} = \hat{f}$, π and p are projections

$$\pi: \quad Z^0(\mathfrak{U}, \mathcal{A}_P) \longrightarrow Z^0(\mathfrak{U}, \mathcal{A}_P)/Z^0(\mathfrak{U}, \mathfrak{S}_P)\,, \tag{4.9b}$$

$$p: \quad Z^1_f(\mathfrak{U}, \mathbb{G}_P) \longrightarrow C^0(\mathfrak{U}, \mathbb{G}_P) \backslash Z^1_f(\mathfrak{U}, \mathbb{G}_P), \tag{4.9c}$$

and a map δ^0 is the projection (4.4). Recall that the group $Z^0(\mathfrak{U}, \mathfrak{S}_P)$ acts on the spaces $C^0_f(\mathfrak{U}, \mathfrak{S}_P) \ni \{\psi^{-1}_\alpha\}$ and $Z^0(\mathfrak{U}, \mathcal{A}_P)$ on the right, and the group $C^0(\mathfrak{U}, \mathbb{G}_P)$ acts on the spaces $C^0_f(\mathfrak{U}, \mathfrak{S}_P) \ni \{\psi^{-1}_\alpha\}$ and $Z^1_f(\mathfrak{U}, \mathbb{G}_P)$ on the left.

Using a bijection of the moduli spaces of solutions to eqs.(4.2), (1.1a) and (4.6), we identify the points $\pi \circ \delta^0(\psi), \pi(A), p(\hat{f})$ and $p \circ r(\psi)$, where $\psi \in C^0_f(\mathfrak{U}, \mathfrak{S}_P), A \in Z^0(\mathfrak{U}, \mathcal{A}_P), \hat{f} \in Z^1_f(\mathfrak{U}, \mathbb{G}_P)$. Then we have

$$\begin{aligned} &\pi \circ \delta^0(\psi) = \pi(A) = p(\hat{f}) = p \circ r(\psi) \Longleftrightarrow \\ &\hat{f} = s \circ \pi(A) = r \circ \varphi(A), \quad A = \sigma \circ p(\hat{f}) = \delta^0 \circ \eta(\hat{f}), \quad \psi = \varphi(A) = \eta(\hat{f}), \end{aligned} \tag{4.10}$$

where φ, η, σ and s are some (local) sections of the fibrations (4.4), (4.9a), (4.9b) and (4.9c), respectively. Let us emphasize that just the ambiguity of the choice of sections of the fibrations (4.4) and (4.9) leads to the ambiguity of finding $\hat{f}$ for a given A and A for a given $\hat{f}$. As usual, one can remove this ambiguity by choosing some *special* sections φ, η, σ and s (gauge fixing).

5 Symmetry transformations and deformations of bundles

In Sect.4 we have described the transformation $r\circ\varphi : (f, \psi d\psi^{-1}) \mapsto (f, \hat{f})$. If we now define a map $T_1 : \hat{f} \mapsto \tilde{f}$ preserving the set $Z^1_f(\mathfrak{U}, \mathbb{G}_P)$, then after using the maps $\sigma \circ p$ or $\delta^0 \circ \eta$, we obtain a new flat connection $\tilde{A}$. Below we describe this method of constructing flat connections on P based on deformations $T_1 : \hat{f} \mapsto \tilde{f}$ of locally constant bundles. Namely, starting from the Čech approach, in Sect.5.1 we describe groups acting on the space of transition functions of a locally constant bundle. Then, using solutions of the functional matrix equations, we proceed in Sect.5.2 to the de Rham description and define dressing transformations acting on the space of solutions to the field equations of Chern-Simons and topological BF theories. Finally, in Sect.5.3 we discuss a special class of solutions of the functional matrix equations describing deformations of flat bundles.

5.1 Cohomological groups acting on solution spaces

To begin with, we shall consider groups acting on the set $C^1(\mathfrak{U}, \mathbb{G}_P)$ of 1-cochains. One of such groups - the group $C^0(\mathfrak{U}, \mathbb{G}_P)$ - and its left action T_0 on $C^1(\mathfrak{U}, \mathbb{G}_P)$, $Z^1(\mathfrak{U}, \mathbb{G}_P)$ and $Z^1_f(\mathfrak{U}, \mathbb{G}_P)$ have been considered in Sections 2.2 and 4.3. The action T_0 of the group $C^0(\mathfrak{U}, \mathbb{G}_P)$ gives trivial (gauge) transformations of the transition functions $\hat{f} = \{\hat{f}_{\alpha\beta}\}$, and the space of orbits of the group $C^0(\mathfrak{U}, \mathbb{G}_P)$ in $Z^1_f(\mathfrak{U}, \mathbb{G}_P)$ is the moduli space $\mathcal{M}_P \simeq C^0(\mathfrak{U}, \mathbb{G}_P)\backslash Z^1_f(\mathfrak{U}, \mathbb{G}_P) = \operatorname{Ker}\rho \subset H^1(X, \mathbb{G}_P)$. In other words, the group $C^0(\mathfrak{U}, \mathbb{G}_P)$ is the stability subgroup of the point $[\hat{f}] \in \mathcal{M}_P$.

To obtain a nontrivial map of the moduli space $\mathcal{M}_P$ onto itself, we consider the following action of the group $C^1(\mathfrak{U}, \mathbb{G}_P)$ on itself:

$$T_1(h,\ .\ ):\ \chi \mapsto T_1(h, \chi), \quad T_1(h, \chi)_{\alpha\beta} := h_{\alpha\beta}\chi_{\alpha\beta}(h^{(\alpha)}_{\beta\alpha})^{-1}, \tag{5.1}$$

where $h, \chi \in C^1(\mathfrak{U}, \mathbb{G}_P)$. Recall that $h^{(\alpha)}_{\beta\alpha} \neq h^{-1}_{\alpha\beta}$ if $h \notin Z^1(\mathfrak{U}, \mathbb{G}_P)$. From the definition (5.1) it is easy to see that $T_1(g, T_1(h, \chi)) = T_1(gh, \chi)$ for $g, h, \chi \in C^1(\mathfrak{U}, \mathbb{G}_P)$. Moreover, starting from the element $\chi \in C^1(\mathfrak{U}, \mathbb{G}_P)$, one can obtain any other element of $C^1(\mathfrak{U}, \mathbb{G}_P)$, i.e. the action T_1 is transitive.

We are interested in subgroups of the group $C^1(\mathfrak{U}, \mathbb{G}_P)$ preserving the set $Z^1_f(\mathfrak{U}, \mathbb{G}_P) \subset Z^1(\mathfrak{U}, \mathbb{G}_P) \subset C^1(\mathfrak{U}, \mathbb{G}_P)$. To describe them, we fix $\hat{f}f^{-1} = \{\hat{f}_{\alpha\beta}f^{-1}_{\alpha\beta}\} = \{\psi^{-1}_\alpha f_{\alpha\beta}\psi_\beta f^{-1}_{\alpha\beta}\} \in Z^1_f(\mathfrak{U}, \mathbb{G}_P)$ and consider an orbit of the point $\chi = \hat{f}f^{-1}$ under the action T_1 of the group $C^1(\mathfrak{U}, \mathbb{G}_P)$. We want to find an intersection of this orbit with the set $Z^1_f(\mathfrak{U}, \mathbb{G}_P)$, i.e. the set of all $h \in C^1(\mathfrak{U}, \mathbb{G}_P)$ such that

(i) $\tilde{f}f^{-1} = T_1(h, \hat{f}f^{-1}) \in Z^1(\mathfrak{U}, \mathbb{G}_P)$, i.e. the transformation $T_1(h,\ .\ ) : \hat{f} \mapsto \tilde{f} = h_{\alpha\beta}\hat{f}_{\alpha\beta}h^{-1}_{\beta\alpha}$ preserves eqs.(4.6a),

(ii) $\tilde{f}f^{-1} = T_1(h, \hat{f}f^{-1}) \in Z^1_f(\mathfrak{U}, \mathbb{G}_P) \subset Z^1(\mathfrak{U}, \mathbb{G}_P)$, i.e. the transformation $T_1(h, \ . \ ) : \hat{f} \mapsto \tilde{f} = h_{\alpha\beta}\hat{f}_{\alpha\beta}h^{-1}_{\beta\alpha}$ preserves not only eqs.(4.6a), but also eqs.(4.6b).

The condition (i) means that the transformation $T_1(h, \ . \ )$ maps $Z^1(\mathfrak{U}, \mathbb{G}_P)$ onto itself, which is realized, of course, not for any $h \in C^1(\mathfrak{U}, \mathbb{G}_P)$. This condition imposes severe constraints on h which are equivalent to the following nonlinear functional equations:

$$h_{\alpha\beta}\ \hat{f}_{\alpha\beta}\ h^{-1}_{\beta\alpha}\ h_{\beta\gamma}\ \hat{f}_{\beta\gamma}\ h^{-1}_{\gamma\beta}\ h_{\gamma\alpha}\ \hat{f}_{\gamma\alpha}\ h^{-1}_{\alpha\gamma} = 1 \quad \text{on} \quad U_\alpha \cap U_\beta \cap U_\gamma \neq \varnothing. \tag{5.2a}$$

Notice that eqs.(5.2a) are satisfied trivially if $h \in C^0(\mathfrak{U}, \mathbb{G}_P)$, since in this case $h_{\alpha\beta} = h_{\alpha|\beta} := h_{\alpha|U_\alpha\cap U_\beta}$, $h_{\beta\alpha} = h_{\beta|\alpha} := h_{\beta|U_\alpha\cap U_\beta}$.

By introducing $\chi_{\alpha\beta} = \tilde{f}_{\alpha\beta}f^{-1}_{\alpha\beta}$ and using formulae (2.5), we can rewrite eqs.(5.2a) in the form

$$\chi^{(\alpha)}_{\alpha\beta}\chi^{(\alpha)}_{\beta\gamma}\chi^{(\alpha)}_{\gamma\alpha} = 1 \quad \text{on} \quad U_\alpha \cap U_\beta \cap U_\gamma \neq \varnothing. \tag{5.2b}$$

Formulae (5.2) mean that $\chi = \{\chi_{\alpha\beta}\} = \{\chi^{(\alpha)}_{\alpha\beta}\}$ is a 1-cocycle, $\chi \in Z^1(\mathfrak{U}, \mathbb{G}_P)$.

The condition (ii) means that for the 1-cocycle $\tilde{f}$ one can find a splitting like (4.6b), i.e. there exists a 0-cochain $\tilde{\psi} = \{\tilde{\psi}_\alpha\} \in C^0(\mathfrak{U}, \mathfrak{G}_P)$ such that

$$\tilde{f}_{\alpha\beta} = \tilde{\psi}^{-1}_\alpha f_{\alpha\beta}\tilde{\psi}_\beta. \tag{5.3}$$

This imposes additional restrictions on $h \in C^1(\mathfrak{U}, \mathbb{G}_P)$, and one of their possible resolvings is the following. Recall that for the sheaf $\mathfrak{s}_{\hat{P}}$ of smooth sections of the bundle ad$\hat{P}$ we have $H^1(X, \mathfrak{s}_{\hat{P}}) = 0$ [7]. Therefore for *small* deformations $\hat{f} \mapsto \tilde{f}$ of the cocycle $\hat{f}$ there always exists $\tilde{\psi} = \{\tilde{\psi}_\alpha\} \in C^0(\mathfrak{U}, \mathfrak{G}_P)$ such that (5.3) is satisfied. As such, we shall consider elements $h \in C^1(\mathfrak{U}, \mathbb{G}_P)$ that are close to the identity and satisfying eqs.(5.2). In other words, we consider a *local group* (a neighbourhood of the identity in $C^1(\mathfrak{U}, \mathbb{G}_P)$) and take only those its elements h which satisfy the conditions (5.2). If, in addition, we change the "initial" point $\hat{f} \in Z^1_f(\mathfrak{U}, \mathbb{G}_P)$ for the action T_1, then in the described way we shall obtain all elements of the group of bijections of the set $Z^1_f(\mathfrak{U}, \mathbb{G}_P)$.

5.2 Dressing transformations

In Sect.4 we have described maps $A \mapsto \hat{f}$ and $\hat{f} \mapsto A$, where $A = \psi d\psi^{-1} \in Z^0(\mathfrak{U}, \mathcal{A}_P)$, $\hat{f} = \{\hat{f}_{\alpha\beta}\} = \{\psi^{-1}_\alpha f_{\alpha\beta}\psi_\beta\} \in Z^1_f(\mathfrak{U}, \mathbb{G}_P)$, $\psi = \{\psi_\alpha\} \in C^0_f(\mathfrak{U}, \mathfrak{G}_P)$. In Sect.5.1 we have described the map $T_1(h, \ . \ ) : \hat{f} \mapsto \tilde{f} = \{h_{\alpha\beta}\hat{f}_{\alpha\beta}h^{-1}_{\beta\alpha}\}$, where $\tilde{f} \in Z^1_f(\mathfrak{U}, \mathbb{G}_P)$ if elements $h = \{h_{\alpha\beta}\} \in C^1(\mathfrak{U}, \mathbb{G}_P)$ satisfy some conditions. Then, having found $\tilde{\psi} = \{\tilde{\psi}_\alpha\}$ by formula (5.3), we introduce

$$\tilde{A}^{(\alpha)} := -(d\tilde{\psi}_\alpha)\tilde{\psi}^{-1}_\alpha. \tag{5.4}$$

From the equations $d\tilde{f}_{\alpha\beta} = 0$ and eqs.(5.3) it follows that $\tilde{\psi}_\alpha$'s satisfy eqs.(4.2), and $\tilde{A}^{(\alpha)}$'s satisfy eqs.(4.1b), i.e.

$$\tilde{A}^{(\alpha)} = f_{\alpha\beta}\tilde{A}^{(\beta)}f^{-1}_{\alpha\beta}, \tag{5.5a}$$

$$(d\tilde{\psi}_\alpha)\tilde{\psi}_\alpha^{-1} = f_{\alpha\beta}(d\tilde{\psi}_\beta)\tilde{\psi}_\beta^{-1}f_{\alpha\beta}^{-1} \tag{5.5b}$$

on $U_\alpha \cap U_\beta \neq \varnothing$. Thus, if we take a flat connection $A = \{A^{(\alpha)}\}$ and carry out the sequence of transformations

$$(f, A) \overset{r\circ\varphi}{\mapsto} (f, \hat{f}) \overset{T_1}{\mapsto} (f, \tilde{f}) \overset{\delta^0\circ\eta}{\longmapsto} (f, \tilde{A}), \tag{5.6}$$

we obtain a new flat connection $\tilde{A} = \{\tilde{A}^{(\alpha)}\}$. Notice that by virtue of commutativity of the diagram (4.8), one can consider the map $s\circ\pi$ instead of $r\circ\varphi$ and the map $\sigma\circ p$ instead of $\delta^0\circ\eta$.

It is not difficult to verify that

$$\tilde{A} = \{\tilde{A}^{(\alpha)}\} = \{\phi_\alpha A^{(\alpha)}\phi_\alpha^{-1} + \phi_\alpha d\phi_\alpha^{-1}\} = \phi A\phi^{-1} + \phi d\phi^{-1}, \tag{5.7a}$$

where

$$\phi := \tilde{\psi}\psi^{-1} = \{\tilde{\psi}_\alpha\psi_\alpha^{-1}\} = \{\phi_\alpha\} \in C^0(\mathfrak{U}, \mathfrak{G}_P). \tag{5.7b}$$

Formally, (5.7) looks like a gauge transformation. But actually the transformation $\mathrm{Ad}_\phi : A \mapsto \tilde{A}$, defined by (5.7), consists of the sequence (5.6) of transformations and is not a gauge transformation, since $\phi_\alpha \neq f_{\alpha\beta}\phi_\beta f_{\alpha\beta}^{-1}$ on $U_\alpha \cap U_\beta \neq \varnothing$. Recall that for gauge transformations $\mathrm{Ad}_g : A \mapsto A^g = gAg^{-1} + gdg^{-1}$ one has $g_\alpha = f_{\alpha\beta}g_\beta f_{\alpha\beta}^{-1}$, i.e. $g = \{g_\alpha\}$ is a *global* section of the bundle $\mathrm{Int}P$, and $\phi = \{\phi_\alpha\}$ is a collection of *local* sections $\phi_\alpha : U_\alpha \to G$ of the bundle $\mathrm{Int}P$ which are constructed by the algorithm described above.

From formulae (3.13), (4.3) and (5.7b) it follows that the dressing transformation Ad_ϕ acts on any solution B of eqs.(1.1b) by the formula

$$\mathrm{Ad}_\phi : \ B \mapsto \ \tilde{B} = \{\tilde{B}^{(\alpha)}\} := \{\phi_\alpha B^{(\alpha)}\phi_\alpha^{-1}\} = \phi B\phi^{-1}, \tag{5.7c}$$

where $\phi = \{\phi_\alpha\}$ is defined in (5.7b). The field $\tilde{B}$ is a new solution of eqs.(1.1b). As is shown above, the transformation Ad_ϕ is not a gauge transformation.

Let us emphasize that $\phi = \{\phi_\alpha\}$ depends on $h \in C^1(\mathfrak{U}, \mathbb{G}_P)$ defining the transformation $T_1(h, \ . \ ) : \hat{f} \mapsto \tilde{f}$, and taking this into account we shall write $\phi \equiv \phi(h) = \{\phi_\alpha(h)\}$. The transformations (5.7) will be called the *dressing transformations*. In this terminology we follow the papers [12, 13, 14], where the analogous transformations were used for constructing solutions of integrable equations. It is not difficult to show that these transformations form a group.

Notice that the maps (5.6) are connected with maps between the bundles (P, f), $(\hat{P}, \hat{f})$ and $(\tilde{P}, \tilde{f})$. All these bundles are diffeomorphic but not isomorphic as locally constant bundles. A diffeomorphism of P onto $\hat{P}$ is defined by a 0-cochain $\psi = \{\psi_\alpha\} \in C_f^0(\mathfrak{U}, \mathfrak{G}_P)$, and a diffeomorphism of P onto $\tilde{P}$ is defined by $\tilde{\psi} = \{\tilde{\psi}_\alpha\} \in C_f^0(\mathfrak{U}, \mathfrak{G}_P)$. Moreover, the bundles $\hat{P}$ and $\tilde{P}$ become isomorphic as locally constant bundles after the restriction to $U_\alpha \cap U_\beta \neq \varnothing$: $\hat{P}|_{U_\alpha\cap U_\beta} \simeq \tilde{P}|_{U_\alpha\cap U_\beta}$, but these isomorphisms are different for $U_\alpha \cap U_\beta$ with different $\alpha, \beta \in I$. In other words, $h_{\alpha\beta} : \hat{P}|_{U_\alpha\cap U_\beta} \to \tilde{P}|_{U_\alpha\cap U_\beta}$ define *local* isomorphisms that do not extend up to the isomorphism of $\hat{P}$ and $\tilde{P}$ as locally constant bundles over the whole X.

5.3 A special cohomological symmetry group

Matrices $h = \{h_{\alpha\beta}\}$, defining the dressing transformations

$$\mathrm{Ad}_{\phi(h)} : \ A \mapsto A^h := \phi(h)A\phi(h)^{-1} + \phi(h)d\phi(h)^{-1}, \tag{5.8a}$$

$$\mathrm{Ad}_{\phi(h)} : \ B \mapsto B^h := \phi(h)B\phi(h)^{-1}, \tag{5.8b}$$

must satisfy the nonlinear functional equations (5.2) which are not so easy to solve. However, there exists an important class of solutions to these equations, which can be described explicitly. The merit of these solutions is the fact that they do not depend on the choice of cocycles f and $\hat{f}$.

Notice that eqs.(5.2) with all indices written down have the form

$$h_{\alpha\beta|\gamma}\ \hat{f}_{\alpha\beta|\gamma}\ h^{-1}_{\beta\alpha|\gamma}\ h_{\beta\gamma|\alpha}\ \hat{f}_{\beta\gamma|\alpha}\ h^{-1}_{\gamma\beta|\alpha}\ h_{\gamma\alpha|\beta}\ \hat{f}_{\gamma\alpha|\beta}\ h^{-1}_{\alpha\gamma|\beta} = 1, \tag{5.9}$$

where $h_{\alpha\beta|\gamma}$ means the restriction of $h_{\alpha\beta}$ defined on $U_\alpha \cap U_\beta$ to an open set $U_\alpha \cap U_\beta \cap U_\gamma$ and analogously for all other matrices. It is not difficult to verify that a collection $h = \{h_{\alpha\beta}\} \in C^1(\mathfrak{U}, \mathbb{G}_P)$ of matrices such that

$$h_{\alpha\beta|\gamma} = h_{\alpha\gamma|\beta}, \quad h_{\beta\alpha|\gamma} = h_{\beta\gamma|\alpha}, \quad h_{\gamma\beta|\alpha} = h_{\gamma\alpha|\beta} \tag{5.10}$$

satisfies eqs.(5.9) for any choice of 1-cocycles $f = \{f_{\alpha\beta}\}$ and $\hat{f} = \{\hat{f}_{\alpha\beta}\}$.

The constraints (5.10) are not very severe. They simply mean that sections $h_{\alpha\beta}$ of the sheaf $\mathbb{G}_P$ over $U_\alpha \cap U_\beta \neq \varnothing$ can be extended to sections of the sheaf $\mathbb{G}_P$ over the open set

$$\mathcal{U} = \bigcup_{\alpha,\beta\in I} U_\alpha \cap U_\beta, \tag{5.11}$$

where the summation is carried out in all $\alpha, \beta \in I$ for which $U_\alpha \cap U_\beta \neq \varnothing$. In other words, from (5.10) it follows that there exists a locally constant section $h_{\mathcal{U}} : \mathcal{U} \to G$ of the bundle Int$P$ such that $h_{\alpha\beta} = h_{\mathcal{U}|U_\alpha\cap U_\beta}$ [7]. In this case we can identify $h = \{h_{\alpha\beta}\} = \{h_{\mathcal{U}|U_\alpha\cap U_\beta}\}$ and $h_{\mathcal{U}}$. Such h form a subgroup

$$\bar{C}^1(\mathfrak{U}, \mathbb{G}_P) := \{h \in C^1(\mathfrak{U}, \mathbb{G}_P) : h_{\alpha\beta|\gamma} = h_{\alpha\gamma|\beta} \text{ on } U_\alpha \cap U_\beta \cap U_\gamma \neq \varnothing\} \tag{5.12}$$

of the group $C^1(\mathfrak{U}, \mathbb{G}_P)$.

Now let us consider $\bar{C}^1(\mathfrak{U}, \mathbb{G}_P)$ as a *local* group, i.e. let us take h from the neighbourhood of the identity in $\bar{C}^1(\mathfrak{U}, \mathbb{G}_P)$. For such h there always exists $\tilde{\psi} \in C^0(\mathfrak{U}, \mathfrak{G}_P)$ such that (5.3) is satisfied and one can introduce $\phi(h)$ by formula (5.7b). Notice that this map $\phi : \bar{C}^1(\mathfrak{U}, \mathbb{G}_P) \to C^0(\mathfrak{U}, \mathfrak{G}_P)$ is a homomorphism. Using $\phi(h)$, we can introduce a new flat connection $A^h = \mathrm{Ad}_{\phi(h)}A = \phi(h)A\phi(h)^{-1} + \phi(h)d\phi(h)^{-1}$ and a new solution $B^h = \mathrm{Ad}_{\phi(h)}B = \phi(h)B\phi(h)^{-1}$ of eqs.(1.1b) for any $h \in \bar{C}^1(\mathfrak{U}, \mathbb{G}_P)$.

6 Conclusion

In this paper, the group-theoretic analysis of the moduli space of solutions of Chern-Simons and topological BF theories and nonlocal symmetries of their equations of motion has been undertaken. We have formulated a method of constructing flat connections based on a

correspondence between the non-Abelian de Rham and Čech cohomologies. The group of all symmetries of the field equations of Chern-Simons and topological BF theories and the special cohomological symmetry group have been described. The described dressing symmetries can be lifted up to symmetries of quantum Chern-Simons and topological BF theories. It would be desirable to describe and analyze representations of these symmetry groups.

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