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Painlevé Analysis and Singular Manifold Method for a (2 + 1) Dimensional Non-Linear Schrödinger Equation

P G ESTÉVEZ * *and G A HERNÁEZ*

Area de Física Teórica, Facultad de Física,
Universidad de Salamanca 37008 Salamanca, Spain
* *E-mail: pilar@sonia.usal.es*

Abstract

The real version of a (2 + 1) dimensional integrable generalization of the nonlinear Schrödinger equation is studied from the point of view of Painlevé analysis. In this way we find the Lax pair, Darboux transformations and Hirota's functions as well as solitonic and dromionic solutions from an iterative procedure.

1 Introduction

Non linear Schrödinger type equations in (2 + 1) dimensions were proposed by Calogero in [1] and then discussed by Zakharov [15].

The equation under study in this paper is:

$$\begin{aligned} &u_t - u_{xy} - 2m_y u = 0,\\ &\omega_t + \omega_{xy} + 2m_y\omega = 0,\\ &m_x + u\omega = 0. \end{aligned} \tag{1.1}$$

In physics this sort of equations arises in studying spin systems in (2 + 1) dimensions [8, 10, 9, 11].

2 Singular manifold method

2.1 Leading term analysis

In order to perform the Painlevé property [12] for equation (1.1) we need to expand the fields u, ω and m in a generalized Laurent expansion of the fields in terms of an arbitrary singularity manifold $\chi(x, y, t) = 0$. Such expansion should be of the form [14]:

$$u = \sum_{j=0}^{\infty} u_j(x, y, t)\chi^{j-\alpha}, \quad \omega = \sum_{j=0}^{\infty} \omega_j(x, y, t)\chi^{j-\beta}, \quad m = \sum_{j=0}^{\infty} m_j(x, y, t)\chi^{j-\gamma}. \tag{2.1}$$

By substituting (2.1) in (1.1), we have for the leading terms:

$$\alpha = \beta = \gamma = 1, \qquad m_0 = \chi_x, \qquad u_0\omega_0 = \chi_x^2 \tag{2.2}$$

from where we see that leading analysis is not able to determine u_0 and ω_0 independently and only gives us its product. This suggests us to write the dominant terms u_0 and ω_0 in a more general way as:

$$u_0 = A(x,y,t)\chi_x, \qquad \omega_0 = \frac{1}{A(x,y,t)}\chi_x. \tag{2.3}$$

2.2 Truncated expansions. Auto-Bäcklund transformations

Following the singular manifold method developed by Weiss [13], we truncate expansions (2.1) at the constant level. This implies that the singular manifold is not yet an arbitrary function because it is determined by the truncation condition. Due to this fact, we denote it as ϕ. We can therefore write the solutions (2.1) to equation (1.1) in the following way:

$$m' = m + \frac{\phi_x}{\phi}, \qquad u' = u + \frac{A(x,y,t)\phi_x}{\phi}, \qquad \omega' = \omega + \frac{\phi_x}{A\phi}. \tag{2.4}$$

The set of equations (2.4) are the auto-Bäcklund transformations between two solutions of (1.1).

2.3 Expression of the solutions in terms of the Singular manifold

Substituting equations (2.4) in (1.1) we obtain a polynomial in ϕ. If we require all the coefficients of this polinomial to be zero we obtain the following expressions after some algebraic manipulations (we used MAPLEV to handle the calculation):

$$\begin{aligned} u &= -\frac{A}{2}\left(v + \left(\frac{A_x}{A} + h\right)\right), & \omega &= -\frac{1}{2A}\left(v - \left(\frac{A_x}{A} + h\right)\right), \\ m_x &= \frac{1}{4}\left(\left(\frac{A_x}{A} + h\right)^2 - v^2\right), & m_y &= \frac{1}{2}\left(\frac{A_t}{A} - \frac{A_x}{A}\frac{A_y}{A} - v_y\right), \end{aligned} \tag{2.5}$$

where v, w and q are defined as:

$$v = \frac{\phi_{xx}}{\phi_x}, \qquad w = \frac{\phi_t}{\phi_x}, \qquad q = \frac{\phi_y}{\phi_x} \tag{2.6}$$

and $h = h(y,t)$ is a function which arises after performing an integration in x.

2.4 Singular manifold equations

- From the compatibility of the definitions (2.6) we obtain the generic equations:

$$\begin{aligned} &\phi_{xxt} = \phi_{txx} \Longrightarrow v_t = (w_x + vw)_x, \\ &\phi_{xxy} = \phi_{yxx} \Longrightarrow v_y = (q_x + vq)_x, \\ &\phi_{yt} = \phi_{ty} \Longrightarrow q_t = w_y + wq_x - qw_x. \end{aligned} \tag{2.7}$$

- On the other hand, if substitute (2.4) in (1.1) we obtain the equations:

$$\begin{aligned} &0 = h_t + hh_y, \qquad 0 = w + hq - \frac{A_y}{A}, \\ &0 = \left(\frac{A_x A_y}{A^2} - \frac{A_t}{A}\right)_x + \left(v_x - \frac{v^2}{2} + \frac{1}{2}\left(\frac{A_x}{A} + h\right)^2\right)_y. \end{aligned} \tag{2.8}$$

The set (2.7)–(2.8) constitutes the singular manifold equations.

3 Lax pair

3.1 Singular manifold equations as nonlinear PDE's

The singular manifold equations (2.7)–(2.8) can be considered as a system of nonlinear coupled PDE's in v, w, q and A. Using the procedure of the previous section we can perform the leading term analysis over the SME in order to obtain:

$$\begin{aligned} v &= \frac{\psi_x^+}{\psi^+} + \frac{\psi_x^-}{\psi^-}, & \frac{A_x}{A} &= \frac{\psi_x^+}{\psi^+} - \frac{\psi_x^-}{\psi^-} - h, \\ \frac{A_y}{A} &= \frac{\psi_y^+}{\psi^+} - \frac{\psi_y^-}{\psi^-} - h_y x, & \frac{A_t}{A} &= \frac{\psi_t^+}{\psi^+} - \frac{\psi_t^-}{\psi^-} - h_t x, \end{aligned} \tag{3.1}$$

where we use two singular manifolds ψ^+ and ψ^- because the Painlevé expansion has two branches [2, 4].

3.2 Eigenfunctions and the singular manifold

Integrating (3.1) we obtain the expresions of ϕ and A in terms of the eigenfunctions:

$$\phi_x = \psi^+\psi^-, \qquad \phi_t + h\phi_y = \psi^-\psi_y^+ - \psi^+\psi_y^- - h_y x\psi^+\psi^-, \qquad A = \frac{\psi^+}{\psi^-}e^{-hx}. \tag{3.2}$$

3.3 Lax pair

Using expresions (3.1) in (2.5) and in the singular manifold equations (2.7)–(2.8) we can obtain, after some calculation, the **Lax pair** for equation (1.1):

$$\begin{aligned} 0 &= \psi_x^+ + u\psi^- e^{hx}, \\ 0 &= \psi_x^- + \omega\psi^+ e^{-hx}, \\ 0 &= \psi_t^+ - m_y\psi^+ + h\psi_y^+ + u_y\psi^- e^{hx} + \frac{1}{2}h_y\psi^+, \\ 0 &= \psi_t^- + m_y\psi^- + h\psi_y^- - \omega_y\psi^+ e^{-hx} + \frac{1}{2}h_y\psi^-. \end{aligned} \tag{3.3}$$

From the compatibility condition between these equations we obtain again that the spectral parameter h must satisfy:

$$h_t + hh_y = 0 \tag{3.4}$$

and hence the spectral problem for equation (1.1) is non-isospectral.

4 Darboux transformations

Substituting (3.2) in (2.4) we obtain iterated solutions u', ω' and m':

$$u' = u + \frac{{\psi_1^+}^2 e^{-h_1 x}}{\phi_1}, \qquad \omega' = \omega + \frac{{\psi_1^-}^2 e^{h_1 x}}{\phi_1}, \qquad m' = m + \frac{\phi_{1x}}{\phi_1} \tag{4.1}$$

which will satisfy the Lax pair (3.3) with eigenfunctions ψ'^{+} and ψ'^{-} and spectral parameter h_2 and we can construct a singular manifold ϕ' through ψ'^{+} and ψ'^{-} as:

$$\phi'_x = \psi'^{+}\psi'^{-}. \tag{4.2}$$

We can consider the Lax pair (3.3) as a system of coupled non linear PDE's ([3, 6]) in ψ'^{+}, ψ'^{-}, m', u' and ω' so that the singular manifold method can be applied to the Lax pair itself and truncated expansions for ψ'^{+} and ψ'^{-} should be added to the expansions (4.1). Such expansions could be written as:

$$\psi'^{+} = \psi_2^{+} - \frac{\psi_1^{+}\Omega^{+}}{\phi_1}, \qquad \psi'^{-} = \psi_2^{-} - \frac{\psi_1^{-}\Omega^{-}}{\phi_1}, \tag{4.3}$$

where it's easy to check that Ω^{+} and Ω^{-} are given by:

$$\Omega^{+} = \frac{\psi_1^{+}\psi_2^{-}e^{-(h_1-h_2)x} - \psi_2^{+}\psi_1^{-}}{h_2 - h_1}, \qquad \Omega^{-} = \frac{\psi_1^{+}\psi_2^{-} - \psi_2^{+}\psi_1^{-}e^{(h_1-h_2)x}}{h_2 - h_1} \tag{4.4}$$

and ψ_1^{+}, ψ_1^{-} satisfy the Lax pair (3.3) with spectral parameter h_1 and ψ_2^{+} and ψ_2^{-} satisfy the same Lax pair with spectral parameter h_2.

The set of equations (4.1) and (4.3) where Ω^{+} and Ω^{-} are given by (4.4), constitute a transformation of potentials and eigenfunctions that leaves invariant the Lax pairs. Hence, (4.1) and (4.3) should be considered as a Darboux transformation [7].

5 Hirota's function

Considering (4.2) as a non linear equation in ϕ', ψ'^{+} and ψ'^{-} it's trivial to proof that we can define a truncated expansion:

$$\phi' = \phi_2 - \frac{\Omega^{+}\Omega^{-}}{\phi_1}, \tag{5.1}$$

where ϕ_2 satisfies:

$$\phi_{2x} = \psi_2^{+}\psi_2^{-}. \tag{5.2}$$

Since (5.1) defines a singular manifold for m', we can use it to build an iterated solution:

$$m'' = m + \frac{\tau_x}{\tau}, \tag{5.3}$$

where

$$\tau = \phi'\phi_1 = \phi_1\phi_2 - \Omega^{+}\Omega^{-} \tag{5.4}$$

is the Hirota's τ-function [5].

6 Solutions

6.1 Line solitons $m = -abx$, $u = a$, $\omega = b$

If we restrict ourselves to the case when h_1 and h_2 are constants, solutions of the Lax pair (3.3) are:

$$\begin{aligned} \psi_i^+ &= \exp\left[\alpha_i x + \beta_i y + \left(\frac{ab}{\alpha_i} - \alpha_i\right)\beta_i t\right], \\ \psi_i^- &= -\frac{\alpha_i}{a}\exp\left[\frac{ab}{\alpha_i}x + \beta_i y + \left(\frac{ab}{\alpha_i} - \alpha_i\right)\beta_i t\right], \end{aligned} \tag{6.1}$$

where $i = 1, 2$ and α_i and β_i are constants.

Using (3.2), (6.1) and (4.1) we obtain for the first iteration the one-soliton solution:

$$m'_x = -ab + \partial_{xx}[\ln\phi_1] \tag{6.2}$$

and for the second we have the two-soliton solution:

$$m''_x = -ab + \partial_{xx}[\ln\tau], \tag{6.3}$$

where

$$\phi_i = -\frac{\alpha_i}{a}\frac{c_i}{\alpha_i + \frac{ab}{\alpha_i}}\left(1 + F_i\right), \tag{6.4}$$

$$\tau = \phi_1\phi_2 - \Omega^+\Omega^- = \frac{\alpha_1\alpha_2}{a^2}\frac{1}{\left(\alpha_1 + \frac{ab}{\alpha_1}\right)\left(\alpha_2 + \frac{ab}{\alpha_2}\right)}\left[1 + F_1 + F_2 - A_{12}F_1F_2\right], \tag{6.5}$$

$$F_i = \exp\left[\left(\alpha_i + \frac{ab}{\alpha_i}\right)x + 2\beta_i y - 2\left(\alpha_i - \frac{ab}{\alpha_i}\right)\beta_i t + \varphi_i\right], \tag{6.6}$$

$$A_{12} = \frac{ab(\alpha_1 - \alpha_2)^2}{(\alpha_1\alpha_2 + ab)^2} \tag{6.7}$$

and we have redefined $c_i = e^{-\varphi_i}$.

6.2 Dromions $m = 0$, $u = 0$, $\omega = b$

For this seminal solutions and assuming that h_1 and h_2 are constants, non trivial solutions of the Lax pair are:

$$\psi_i^+ = K_i(y, t), \qquad \psi_i^- = \frac{b}{h_i}e^{-h_i x}K_i(y, t), \tag{6.8}$$

where K_i are x-independent functions that satisfy:

$$K_{it} + h_i K_{iy} = 0. \tag{6.9}$$

Following the same procedure as for the solitons, we have for the first and second iteration respectively:

$$m'_y = \partial_{xy}\ln[\phi_1], \tag{6.10}$$

$$m''_y = \partial_{xy} \ln[\tau], \tag{6.11}$$

where

$$\phi_i = -\frac{b}{h_i^2}\left(R_i(y,t) + K_i^2(y,t)e^{-h_i x}\right), \tag{6.12}$$

$$\tau = \phi_1\phi_2 - \Omega^+\Omega^- = \frac{b^2}{h_1^2 h_2^2}\left\{R_1R_2 + R_2K_1^2e^{-h_1x} + R_1K_2^2e^{-h_2x}\right\} \tag{6.13}$$

and R_i must satisfy:

$$R_{it} + h_iR_{iy} = 0. \tag{6.14}$$

Some particular elections for R_i and K_i that yield different dromionic configurations are for instance:

$$\text{a)}\quad R_i = 1 + e^{c_i(y-h_it)}, \qquad K_i^2 = 1 + a_ie^{c_i(y-h_it)}, \tag{6.15}$$

$$\begin{aligned}\text{b)}\quad & R_1 = 1 + e^{c_1(y-h_1t)} + e^{c_2(y-h_1t)},\\ & K_1^2 = 1 + a_1e^{c_1(y-h_1t)} + a_2e^{c_2(y-h_1t)}, \qquad R_2 = K_2 = 0.\end{aligned} \tag{6.16}$$

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