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Some Recent Results on Integrable Bilinear Equations

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Abstract

This paper shows that several integrable lattices can be transformed into coupled bilinear differential-difference equations by introducing auxiliary variables. By testing the Bäcklund transformations for this type of coupled bilinear equations, a new integrable lattice is found. By using the Bäcklund transformation, soliton solutions are obtained. By the dependent variable transformation, this new coupled bilinear equations can be reduced to a coupled extended Lotka–Voltera equation and another equation.

The purpose of this short paper is to search for new integrable systems in bilinear form. The methods used here are Hirota's method and Bäcklund transformations [1–7]. As we know, the use of the Hirota bilinear transformation in the search of exact solutions of continuous and discrete systems is now well established [1–5]. More recently, Hirota's method has been systematically used in the search for new integrable equations in both $(1+1)$ and $(2+1)$ dimensions by testing multi-soliton solutions or Bäcklund transformations (see, e.g. [8, 9]). The key points behind these ideas to obtain new integrable systems are to first generalize bilinear forms of known integrable systems and then to test the generalized bilinear forms for multi-soliton solutions or Bäcklund transformations.

In this paper, we will focus on the differential-difference case and find some new integrable systems by testing their bilinear Bäcklund transformations. In order to do so, let us first recall the bilinear forms for some known integrable differential-difference equations.

Example 1. The so-called Belov–Chaltikian lattice is given by [10]

$$b_t(n) = b(n)(b(n+1)-b(n-1))-c(n)+c(n-1), \tag{1}$$

$$c_t(n) = c(n)(b(n+2)-b(n-1)). \tag{2}$$

By the dependent variable transformation

$$b(n) = \left(\ln \frac{f\left(n+\frac{1}{2}\right)}{f\left(n-\frac{1}{2}\right)}\right)_t, \qquad c(n) = \frac{f\left(n+\frac{5}{2}\right) f\left(n-\frac{3}{2}\right)}{f\left(n+\frac{3}{2}\right) f\left(n-\frac{1}{2}\right)},$$

equations (1) and (2) are transformed into the following bilinear form [11]

$$\left(D_t^2 e^{\frac{1}{2}D_n} - D_z e^{\frac{1}{2}D_n}\right) f(n)\bullet f(n) = 0, \tag{3}$$

$$\left(D_z e^{D_n} - D_t^2 e^{D_n} + 2e^{2D_n} - 2e^{D_n}\right) f(n)\bullet f(n) = 0, \tag{4}$$

with z being an auxiliary variable, here the Hirota's bilinear differential operator $D_x^m D_t^k$, the bilinear difference operator $\exp(\delta D_n)$ and the bilinear differential-difference operator $D_x^m D_t^k \exp(\delta D_n)$ are defined by [1–5]

$$D_x^m D_t^k\, a\bullet b \equiv \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^k a(x,t)b(x',t')|_{x'=x,t'=t},$$

$$\exp(\delta D_n)a(n)\bullet b(n) \equiv \exp\left[\delta\left(\frac{\partial}{\partial n} - \frac{\partial}{\partial n'}\right)\right] a(n)b(n')|_{n'=n} = a(n+\delta)b(n-\delta),$$

$$D_x^m D_t^k \exp(\delta D_n)\; a(n)\bullet b(n)$$
$$\equiv \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^k a(n+\delta,x,t)b(n-\delta,x',t')|_{x'=x,t'=t}.$$

Example 2. Consider the Blaszak–Marciniak lattice [12]

$$a_t(n) = c(n+1) - c(n-1), \tag{5}$$

$$b_t(n) = a(n-1)c(n-1) - a(n)c(n), \tag{6}$$

$$c_t(n) = c(n)(b(n) - b(n+1)). \tag{7}$$

By the dependent variable transformation

$$a(n) = \frac{D_t^2 f(n+1)\bullet f(n+1)}{2f(n)f(n+2)}, \qquad b(n) = \left(\ln\frac{f(n)}{f(n+1)}\right)_t, \qquad c(n) = \frac{f(n)f(n+2)}{f^2(n+1)},$$

equations (5), (6) and (7) are transformed into the following bilinear form [13]

$$\left(D_t^2 - 2D_z e^{D_n}\right) f(n)\bullet f(n) = 0, \tag{8}$$

$$\left(D_z D_t - 4\sinh^2\left(\frac{1}{2}D_n\right)\right) f(n)\bullet f(n) = 0, \tag{9}$$

where z is an auxiliary variable.

Example 3. The relativistic Toda lattice of Ruijsenaars is given by [14, 15]

$$\ddot{q_n} = \dot{q_n}\left(\dot{q}_{n-1}\frac{g^2\exp(q_{n-1}-q_n)}{1+g^2\exp(q_{n-1}-q_n)} - \dot{q}_{n+1}\frac{g^2\exp(q_n-q_{n+1})}{1+g^2\exp(q_n-q_{n+1})}\right). \tag{10}$$

By the dependent variable transformation $q_n = \kappa\log\left(-g^2\right) + \log\frac{f_n}{f_{n+1}}$, (10) can be transformed into the trilinear form [16],

$$\begin{vmatrix} \dot{f}_{n-1} & f_{n-1} & f_n \\ \dot{f}_n & f_n & f_{n+1} \\ \ddot{f}_n & \dot{f}_n & \dot{f}_{n+1} \end{vmatrix} = 0, \tag{11}$$

which can be equivalently written as

$$\begin{aligned}&(D_t^2 f_n \bullet f_n) \sinh^2\left(\frac{D_n}{2}\right) f_n \bullet f_n \\ &\quad - \cosh\left(\frac{D_n}{2}\right)\left[D_t \exp\left(\frac{D_n}{2}\right) f_n \bullet f_n\right] \bullet \left[D_t \exp\left(\frac{D_n}{2}\right) f_n \bullet f_n\right] = 0.\end{aligned} \tag{12}$$

By introducing an auxiliary variable z, equation (12) can be decoupled into the bilinear form

$$\left(D_t^2 e^{\frac{1}{2}D_n} - D_z e^{\frac{1}{2}D_n}\right) f(n) \bullet f(n) = 0, \tag{13}$$

$$\left(D_z e^{D_n} - \frac{1}{2}D_t^2 e^{D_n} - \frac{1}{2}D_t^2\right) f(n) \bullet f(n) = 0. \tag{14}$$

Example 4. We now consider three new lattices. The first one is [17]

$$\begin{aligned}&u_{tt}(n+1) + u_{tt}(n) + u_{tt}(n-1) - 3u(n)(u_t(n+1) + u_t(n-1)) \\ &\quad +3u(n+1)u_t(n+1) + 3u(n-1)u_t(n-1) - \frac{1}{4}u(n+1) - \frac{1}{4}u(n-1) \\ &\quad +\frac{1}{2}u(n) + [u(n+1) - 2u(n) + u(n-1)][(u(n+1) - u(n-1))^2 \\ &\quad -(u(n+1) - u(n))(u(n) - u(n-1))] = 0.\end{aligned} \tag{15}$$

By the dependent variable transformation $u(n) = (\ln f(n))_t$, equation (15) can be transformed into the bilinear form

$$\left(D_t^2 e^{\frac{1}{2}D_n} - D_z e^{\frac{1}{2}D_n}\right) f(n) \bullet f(n) = 0, \tag{16}$$

$$\left(D_t^3 e^{\frac{1}{2}D_n} + 3D_t D_z e^{\frac{1}{2}D_n} - D_t e^{\frac{1}{2}D_n}\right) f(n) \bullet f(n) = 0. \tag{17}$$

The second lattice under consideration is given by [18]

$$\begin{aligned}&u_{ttt}(n+1) + u_{ttt}(n) + 2(u_t(n+1) - u_t(n))(u_{tt}(n+1) - u_{tt}(n)) \\ &\quad = e^{u(n+2)-2u(n+1)+u(n)} - e^{u(n+1)-2u(n)+u(n-1)}.\end{aligned} \tag{18}$$

By the dependent variable transformation $u(n) = \ln f(n)$, equation (18) can be transformed into the bilinear form

$$\left(D_t^2 e^{\frac{1}{2}D_n} - D_z e^{\frac{1}{2}D_n}\right) f(n) \bullet f(n) = 0, \tag{19}$$

$$\left(D_z D_t - 4\sinh^2 \frac{1}{2}D_n\right) f(n) \bullet f(n) = 0. \tag{20}$$

The third lattice is given by [19]

$$u_{tt}(n) + u_t(n)(u_t(n-1) - u_t(n+1)) = 4e^{u(n+2)-u(n-1)} - 4e^{u(n+1)-u(n-2)}. \tag{21}$$

By the dependent variable transformation $u(n) = \ln(f(n+1)/f(n))$, equation (21) can be transformed into the bilinear form

$$\left(D_z D_t - 2e^{D_n} + 2\right) f(n) \bullet f(n) = 0, \tag{22}$$

$$\left(D_z^3 D_t + 6D_z^2 e^{D_n} + 6D_z^2 - 2e^{D_n} + 2\right) f(n) \bullet f(n) = 0. \tag{23}$$

Based on these examples, it is quite natural for us to consider the following generalized bilinear equations

$$F_1\left(D_t, D_z, \sinh(\alpha_1 D_n), \ldots, \sinh(\alpha_l D_n)\right) f(n) \bullet f(n) = 0, \tag{24}$$

$$F_2\left(D_t, D_z, \sinh(\alpha_1 D_n), \ldots, \sinh(\alpha_l D_n)\right) f(n) \bullet f(n) = 0, \tag{25}$$

and further search for new integrable differential-difference systems of this type by testing Bäcklund transformations, where F_i $(i = 1, 2)$ are two even order polynomials in D_t, D_z, $\sinh(\alpha_1 D_n)$, $\ldots$ and $\sinh(\alpha_l D_n)$, and l is a given positive integer; the α_i, $i = 1, 2, \ldots, l$, are l different constants, and $F_i(0, 0, \ldots, 0) = 0$. We could search for new integrable systems of the type (24) and (25) via the following steps. Firstly following [9], we seek new bilinear forms F_1 and F_2 individually by testing Bäcklund transformations. If a Bäcklund transformation for $F_1 = 0$ is compatible with a Bäcklund transformation found for $F_2 = 0$, then this coupled system is also integrable. For example, from [9] we know that

$$\left(D_z D_t + A D_t \sinh(D_n) - 4\sinh^2\left(\frac{1}{2}D_n\right)\right) f(n) \bullet f(n) = 0 \tag{26}$$

has a Bäcklund transformation where A is an arbitrary constant. We also find that

$$\left(D_z D_t e^{\frac{1}{2}D_n} + B D_z e^{\frac{1}{2}D_n} + C D_t e^{\frac{1}{2}D_n}\right) f(n) \bullet f(n) = 0 \tag{27}$$

is integrable in the sense of having a Bäcklund transformation where B and C are arbitrary constants. After some trial and error, we know that (26) and (27) have compatible Bäcklund transformations if we choose $A = 2$, $B = -1/2$, $C = 0$. In this case, we have the following new coupled integrable bilinear system

$$\left(D_z D_t + 2D_t \sinh(D_n) - 4\sinh^2\left(\frac{1}{2}D_n\right)\right) f(n) \bullet f(n) = 0, \tag{28}$$

$$\left(D_z D_t e^{\frac{1}{2}D_n} - \frac{1}{2} D_z e^{\frac{1}{2}D_n}\right) f(n) \bullet f(n) = 0, \tag{29}$$

where z is an auxiliary variable. Concerning (28) and (29), we have the following result:

Proposition. *The system of bilinear Eqs. (28)–(29) has the following Bäcklund transformation:*

$$\left(D_z + \lambda^{-1} e^{-D_n} - \lambda e^{D_n} + \mu\right) f(n) \bullet g(n) = 0, \tag{30}$$

$$\left(D_t e^{-\frac{1}{2}D_n} + \lambda D_t e^{\frac{1}{2}D_n} - (\lambda - \lambda\gamma) e^{\frac{1}{2}D_n} + \gamma e^{-\frac{1}{2}D_n}\right) f(n) \bullet g(n) = 0, \tag{31}$$

$$\left(D_z e^{\frac{1}{2}D_n} - \lambda^{-1} D_z e^{-\frac{1}{2}D_n} + \lambda\omega e^{\frac{1}{2}D_n} - \omega e^{-\frac{1}{2}D_n}\right) f(n) \bullet g(n) = 0, \tag{32}$$

where λ, μ, γ and ω are arbitrary constants.

This result can be proved by using Hirota's bilinear operator identities. Due to limited space, we do not give the details of the proof. Instead we are going to construct soliton solutions of (28)–(29) by using the BT (30)–(32). Firstly, by applying the BT (30)–(32) to the trivial solution $f(n) = 1$, we can obtain the 1-soliton solution

$$g(n) = 1+\exp\left(pn + 2\sinh(p)z + \frac{1}{2}\tanh\left(\frac{1}{2}p\right)t + \eta^0\right),$$

where p and η^0 are constants, for the parameters $\lambda = 1$, $\mu = 0$, $\gamma = \frac{1}{2}$ and $\omega = 2\sinh p$. Further by applying the BT (30)–(32) to the 1-soliton solution $f(n) = 1 + \exp(\eta_1)$ we can deduce the following 2-soliton solution

$$g(n) = 1+A_1e^{\eta_1}+e^{\eta_2}+A_2e^{\eta_1+\eta_2},$$

where

$$\eta_i = p_in + 2\sinh(p_i)z + \frac{1}{2}\tanh\left(\frac{1}{2}p_i\right)t + \eta_i^0,$$

$$A_1 = \frac{\sinh(p_1) + \sinh(p_2)}{-\sinh(p_1) + \sinh(p_2)}, \qquad A_2 = -\frac{\sinh\left(\frac{1}{2}(p_1 - p_2)\right)}{\sinh\left(\frac{1}{2}(p_1 + p_2)\right)},$$

with p_i and η_i^0 constants for the set of parameters $\lambda = 1$, $\mu = 0$, $\gamma = \frac{1}{2}$ and $\omega = 2\sinh(p_2)$. Besides, by using MATHEMATICA, we can show that (28)–(29) have the 3-soliton solutions

$$f = 1+e^{\eta_1}+e^{\eta_2}+e^{\eta_3}+A_{12}e^{\eta_1+\eta_2}+A_{13}e^{\eta_1+\eta_3}+A_{23}e^{\eta_2+\eta_3}+A_{12}A_{23}A_{13}e^{\eta_1+\eta_2+\eta_3},$$

where

$$\eta_i = p_in+2\sinh(p_i)z+\frac{1}{2}\tanh\left(\frac{1}{2}p_i\right)t+\eta_i^0,$$

$$A_{ij} = \frac{\sinh\left(\frac{1}{2}(p_i - p_j)\right)(\sinh(p_i) - \sinh(p_j))}{\sinh\left(\frac{1}{2}(p_i + p_j)\right)(\sinh(p_i) + \sinh(p_j))}.$$

By the dependent variable transformation $u(n) = \ln f(n)$, $v(n) = \frac{f_z(n)}{f(n)}$, (28) and (29) are transformed into the system

$$(u_t(n+1)-u_t(n-1)-1)e^{u(n+1)+u(n-1)-2u(n)}+1+v_t(n) = 0, \tag{33}$$

$$v_t(n+1)+v_t(n)+(v(n+1)-v(n))\left(u_t(n + 1) - u_t(n) - \frac{1}{2}\right) = 0. \tag{34}$$

In particular, if we choose $v(n) = 0$ and $a(n) = e^{u(n)-u(n-1)}$, then (33)–(34) become an extended Lotka–Volterra lattice [20, 21]

$$\frac{d}{dt}(a(n)a(n+1)) = a(n)(a(n+1)-a(n)). \tag{35}$$

Therefore the system (33) and (34) is a coupled extended Lotka–Volterra equation. We now derive the z-flow. In order to do so, we view t appearing in (28)–(29) as an auxiliary variable. Setting $U(n) = \ln f(n)$, $V(n) = \frac{f_t(n)}{f(n)}$, then (28)–(29) can be transformed into the following system in nonlinear variables

$$\begin{aligned}&(V(n) - V(n+2) + 1)e^{U(n+2)+U(n)-2U(n+1)}\\&\quad+(V(n-1) - V(n+1) + 1)e^{U(n+1)+U(n-1)-2U(n)} - 2\\&\quad+\left(V(n+1) - V(n) - \frac{1}{2}\right)(U_z(n+1) - U_z(n)) = 0,\end{aligned} \tag{36}$$

$$V_z(n)+(V(n+1)-V(n-1)-1)e^{U(n+1)+U(n-1)-2U(n)}+1 = 0. \tag{37}$$

In summary, we have observed that several integrable lattices can be transformed into coupled bilinear differential-difference equations by introducing auxiliary variables. We propose a coupled generalized Hirota's bilinear equations. By testing Bäcklund transformations for such type of coupled bilinear equations, a new integrable lattice is found. By using the Bäcklund transformation, soliton solutions are obtained. By the dependent variable transformation, this new coupled bilinear equations can be reduced to a coupled extended Lotka–Voltera equation and another equation. Besides, by using the Bäcklund transformation with special Bäcklund parameters $\lambda = 1$, $\mu = 0$, $\gamma = \frac{1}{2}$, $\omega = 0$, we can obtain rational solutions for the lattices (33)–(34) and (36)–(37). For example, starting from the trivial solution $f(n) = 1$, we have the following polynomial solution of (28)–(29):

$$g(n) = n + 2z + \frac{1}{4}t.$$

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