On Chase-Like Bound-Distance Decoding Algorithms

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Abstract - For the decoding of a binary linear block code of Hamming distance \( d \) over AWGN channels, a soft-decision decoder is said to be bounded-distance (BD) decoding if its squared error-correction radius is equal to \( d \). A Chase-like algorithm outputs the best (most likely) codeword in a list of candidates generated by a conventional algebraic binary decoder whose input vectors are determined by the reliability order of the hard-decisions. Let \( \Delta(d) \) denote the smallest size of input vector sets of Chase-like algorithms which achieve BD decoding. When \( d \) approaches to infinity, the best known upper bound on \( \Delta(d) \) is \( \Delta(d) \leq (\lambda + o(1))d^{1/2} \), where \( \lambda \approx 2.414 \). In this paper, we show \( \Delta(d) \leq (\psi + o(1))d^{1/2} \), where \( \psi \approx 2.218 \).

Index Terms - Chase-like algorithm, algebraic binary decoder, bounded-distance decoding

I. Introduction

In this paper, we consider the decoding of binary linear block codes over additive white Gaussian noise (AWGN) channels. As the algorithms proposed by Chase in [1], a Chase-like algorithm outputs the best (most likely) codeword in a list of candidates generated by a conventional algebraic binary decoder whose input vectors are determined by the reliability order of the hard-decisions. A decoding algorithm is called a bounded-distance (BD) decoding if its error-correction radius reaches the maximum. It is well-known that any BD decoding is asymptotically optimal. When applied to a binary linear block code of length \( n \) and minimal Hamming distance \( d \), the original Chase algorithms [1] achieve BD decoding while the numbers of input vectors are \( C_{n/2}^{d/2} \) and \( d^{d/2}+1 \), respectively. Since the decoding complexity of a Chase-like algorithm is by and large proportional to the number of the input vectors, it is of interest to design Chase-like BD decoding algorithms with as least input vectors as possible. Let \( \Delta(d) \) denote the smallest size of input vector sets of Chase-like BD decoding algorithms. In 2003, \( \Delta(d) \leq [d/4] \) and \( \Delta(d) \leq [d/6]+1 \) were proved in [2] and [3], respectively. When the minimal Hamming distance \( d \) approaches to infinity, \( \Delta(d) \leq O(d^{1/2}) \), \( \Delta(d) \leq O(d^{1/3}) \), \( \Delta(d) \leq O(d^{1/6} \ln d) \) were shown in [4], [5], [6], respectively. The best known asymptotic upper bound on \( \Delta(d) \) is shown in [7]: \( \Delta(d) \leq (\lambda + o(1))d^{1/2} \), where \( \lambda \approx 2.414 \). In this paper, we will improve this upper bound further.

II. Preliminaries

Let \( V^n \) denote the set of binary vectors of length \( n \). For \( \bar{u} = (u_1, u_2, \ldots, u_n) \in V^n \), let \( \bar{s}(\bar{u}) = ((-1)^{u_1}, (-1)^{u_2}, \ldots, (-1)^{u_n}) \) be the bipolar vector corresponding to \( \bar{u} \). For two real vectors \( \bar{x}, \bar{y} \in R^n \), their squared Euclidean distance is defined as \( d^2(\bar{x}, \bar{y}) = (x_1-y_1)^2 + \cdots + (x_n-y_n)^2 \), where \( x_i \) and \( y_i \) are the \( i \)-th entry of \( \bar{x} \) and \( \bar{y} \), respectively. Suppose that a linear binary block code \( C \subset V^n \) of Hamming distance \( d \) is used for error control over the additive white Gaussian noise (AWGN) channel with BPSK signaling. When the transmitted codeword is \( \bar{c} = (c_1, c_2, \ldots, c_n) \in C \), the conditional density function of the received vector \( \bar{r} \in R^n \) is

\[
p(\bar{r} | \bar{c}) = 1 / (\pi N_0)^{n/2} e^{-d^2(\bar{r}, \bar{c}) / 2 N_0}.
\]

For given received vector \( \bar{r} \), a vector \( \bar{u} \in V^n \) is said to be better (or more likely) than another vector \( \bar{v} \in V^n \) if \( d^2(\bar{r}, \bar{v}) < d^2(\bar{r}, \bar{u}) \). Hence, a maximum-likelihood (ML) decoder always outputs the best codeword.

Suppose that \( \bar{r} = (r_1, r_2, \ldots, r_n) \in R^n \) is a received vector. Let \( \bar{z} = (z_1, z_2, \ldots, z_n) \in V^n \) denote the hard-decision vector defined by: \( z_i = 0 \) for \( r_i > 0 \) and \( z_i = 1 \) for \( r_i \leq 0 \). For simplicity, without loss of generality, we assume further that the entries have been permuted according to the reliability order of the hard-decisions such that

\[
|r_1| \leq |r_2| \leq \cdots |r_N|.
\]

Like in [4] to [9], we assume further that the Hamming distance \( d \) of the code is odd for simplicity. Let \( \tau = (d-1)/2 \). Assume that a conventional bounded-distance- \( \tau \) algebraic binary decoder, which outputs a codeword within Hamming distance \( \tau \) of the sum of the hard-decision vector \( \bar{z} \) and the input vector, if any, is available. For any set \( U \subset V^n \), let \( C(U) \) denote the Chase-like algorithm which outputs the best codewords in a list of candidates generated by the algebraic decoder with \( U \) as the input vector set. For a decoding algorithm \( A \) of a binary block code, its squared error-correction radius (SECR) is defined as the largest number, denoted \( p(A) \), such that \( A \) decodes correctly whenever the received vector is within squared Euclidean distance \( p(A) \) of the bipolar vector corresponding to the transmitted codeword.

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Then, a decoding algorithm $A$ achieves BD decoding if and only if $\rho(A) = d$.

For $U \subset V^n$ and positive integer $l$, let $\Omega_l(U)$ denote the set of vectors in $V^n$ which are beyond Hamming distance $l$ of each vector in $U$. $\Omega_l(U)$ is called the unchecked region of the Chase-like algorithm $C(U)$. The SEC of $C(U)$ can be computed [8] by

$$\rho(C(U)) = \min \{ d, \min_{v \in \Omega_l(U)} \sigma(v) \} \quad (1)$$

where $\sigma(v)$ is the minimal squared Euclidean distance (MSED) between the vector $s(v)$ and the vectors in

$$H_N := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : 0 \leq x_1 \leq \cdots \leq x_n \leq 1\}.$$ 

Since the size of the unchecked region $\Omega_l(U)$ is very large, it is not easy to estimate the minimum of $\sigma(v)$ over $\Omega_l(U)$ for a general input set $U$.

For $0 \leq j \leq m$ and a vector $\tilde{u} = (u_1, u_2, \ldots, u_n)$, let $\gamma_j,\tilde{u}$ denote the sub-vector $(u_{j+1}, u_{j+2}, \ldots, u_n)$ of $\tilde{u}$. Let $w_j,\tilde{u}$ denote the Hamming weight of $\gamma_j,\tilde{u}$. By convention, $w_0,\tilde{u}$ is also abbreviated as $w(\tilde{u})$. For two different vectors $\tilde{u}, \tilde{v} \in V^n$, $\tilde{u}$ is said to be smaller than $\tilde{v}$ if $w_1,\tilde{u} < w_1,\tilde{v}$ for all $0 \leq i < m$. For $\tilde{u}, \tilde{v} \in V^n$, it is proved in [8,10,11] that the MSED of $\tilde{u}$ is not smaller than that of $\tilde{v}$ if $\tilde{u}$ is smaller than $\tilde{v}$. When the nonzero entries of the input vectors are confined in the leftmost positions (the most unreliable positions), it is shown in [4,5,6,7] that there is a unique minimal vector in the unchecked region $\Omega_l(U)$.

For any binary vector $\tilde{u}$, let $\tilde{u}^j$ denote the concatenation of $j$ copies of $\tilde{u}$. For $0 \leq j \leq N$, let $\tilde{t}_j$ denotes the vector $1^j0^{N-j}$. To improve the upper bound on $\rho(A)$, we will investigate the Chase-like algorithm whose input vector set $U$ is of form

$$U_j = \{ \tilde{t}_0, \tilde{t}_d, \tilde{t}_{d+1}, \ldots, \tilde{t}_j : j \in J \} \quad (2)$$

where $J$ is a set of odd integers between $1$ and $d - 2$.

III. The Minimal Vector in $\Omega_l(U_j)$

If $U_j$ is a set of form (2), the following theorem shows that there is a unique minimal vector in $\Omega_l(U_j)$.

**Lemma 1** Let $J = (a_1, a_2, \ldots, a_m)$ be a set of odd integers with $1 \leq a_1 < a_2 < \cdots < a_m \leq d$. The set $\Omega_l(U_j)$ has a unique minimal sequence

$$\tilde{f}_j = 1^{c_0}0^{c_1}1^{c_2}0^{c_3} \cdots 1^{c_{j-1}}0^{c_j+1}10^{N-d-2} \quad (3)$$

where $c_0 = (a_1 + 1)/2$ and $c_i = (a_{i+1} - a_i)/2$ for $j = 1, \ldots, k - 1$.

**Proof:** From (3), we see

$$\gamma_a, (\tilde{f}_j) = 1^{c_j+1}0^j \quad (4)$$

$$\gamma_{a_j, a_{j+1}}(\tilde{f}_j) = 1^{j+1}0^j, 0 < j < k \quad (5)$$

$$\gamma_{a_k, N}(\tilde{f}_j) = 010^{N-d-2} \quad (6)$$

Then, we have

$$d_{\mu}(\tilde{f}_j, \tilde{t}_0) = w(\tilde{f}_j) = \sum_{j=0}^{k-1} c_j' \quad (7)$$

$$d_{\mu}(\tilde{f}_j, \tilde{t}_{d+1}) = (d - w_{d,\tilde{f}_j}(\tilde{f}_j)) + 2 = \sum_{j=0}^{k-1} c_j + 2 \quad (8)$$

$$d_{\mu}(\tilde{f}_j, 0^d10^{N-d-1}) = w_{d,\tilde{f}_j}(\tilde{f}_j) + 2 = \sum_{j=0}^{k-1} c_j + 1 \quad (9)$$

and, for $1 \leq j \leq k$,

$$d_{\mu}(\tilde{f}_j, \tilde{t}_a) = ((a_j - w_{d,\tilde{f}_j}(\tilde{f}_j)) + w_{a_j,\tilde{f}_j}(\tilde{f}_j)) = \sum_{j=0}^{k-1} c_j + 1 \quad (10)$$

Therefore, from $\sum_{j=0}^{k-1} c_j = (a_k + 1)/2 = r + 1$, we see $\tilde{f}_j \in \Omega_l(U_j)$.

Now assume that $\tilde{u}$ is an arbitrary vector in $\Omega_l(U_j)$.

For $1 \leq j \leq k$, from $d_{\mu}(\tilde{u}, \tilde{t}_j) = w_{a_j,\tilde{u}}(\tilde{u}) + w_{a_j,\tilde{u}}(\tilde{u}) \geq r + 1$ and $d_{\mu}(\tilde{u}, \tilde{t}_j) = (a_j - w_{d,\tilde{u}}(\tilde{u})) + w_{a_j,\tilde{u}}(\tilde{u}) \geq r + 1$, we see that

$$w_{a_j,\tilde{u}}(\tilde{u}) \geq r + 1 - (a_j - 1)/2 = w_{a_j,\tilde{u}}(\tilde{f}_j). \quad (11)$$

Furthermore, we can conclude that

$$w_{d+1,\tilde{u}}(\tilde{u}) \geq 1 \quad (12)$$

Assume in contrary that $w_{d+1,\tilde{u}}(\tilde{u}) = 0$. Then, from $w_{d,\tilde{u}}(\tilde{u}) = w_{a_j,\tilde{u}}(\tilde{u}) \geq 1$, we see that the $(d + 1)$-th entry of $\tilde{u}$ is equal to 1, from $d_{\mu}(\tilde{u}, \tilde{t}_j) \geq r + 1$ and $d_{\mu}(\tilde{u}, \tilde{t}_j) \geq r + 1$, we see that $w_{a_j,\tilde{u}}(\tilde{u}) = r + 1$. Therefore, $d_{\mu}(\tilde{u}, 0^d10^{N-d-1}) = w_{a_j,\tilde{u}}(\tilde{u}) - 1 = r$, contradicts to $\tilde{u} \in \Omega_l(U_j)$.

Let $i$ be an arbitrary integer with $1 \leq i < N$. If $1 \leq i < a_1$, from (4) and (7), we have

$$w_{a_1,\tilde{u}}(\tilde{u}) \geq \max\{ w_{a_1,\tilde{u}}(\tilde{u}), w_{0,\tilde{u}}(\tilde{u}) - i \} \geq \max\{ w_{a_1,\tilde{u}}(\tilde{f}_j), w_{0,\tilde{u}}(\tilde{f}_j) - i \} = w_{a_1,\tilde{u}}(\tilde{f}_j). \quad (13)$$

If $a_j \leq i < a_{j+1}$ for some $j$ with $1 \leq j < k$, from (5) and (7), we have

$$w_{a_j,\tilde{u}}(\tilde{u}) \geq \max\{ w_{a_j,\tilde{u}}(\tilde{u}), w_{a_j,\tilde{u}}(\tilde{u}) - (i - a_j) \} \geq \max\{ w_{a_j,\tilde{u}}(\tilde{f}_j), w_{a_j,\tilde{u}}(\tilde{f}_j) - (i - a_j) \} = w_{a_j,\tilde{u}}(\tilde{f}_j). \quad (14)$$

If $i \geq a_k$, from (6) and (8), we also have $w_{a_k,\tilde{u}}(\tilde{u}) \geq w_{a_k,\tilde{u}}(\tilde{f}_j)$.

Hence, $\tilde{f}_j$ is smaller than $\tilde{u}$ if $\tilde{u} \neq \tilde{f}_j$. 

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According to (1) and Lemma 1, the Chase-like algorithm $C(U_j)$ achieves BD decoding if and only if $\sigma(f_j) \geq d$.

IV. Conditions for $C(U_j)$ Achieving BD Decoding

To give conditions for $\sigma(f_j) \geq d$, we show some properties of the MSEDs at first.

Lemma 2 For any vectors $\vec{u} \in V^n$ and $\vec{v} \in V^m$ and integers $a,b$ with $a > b > 0$.

\[
\sigma(\vec{u}^{a-b} \vec{v}^{b}) \geq \sigma(\vec{u}^{a} \vec{v}^{b})
\]

Proof: Let $s = (a-b)/(a+b)$. Suppose that $x = (x_1, x_2, ..., x_{n+m}) \in H_{n+m}$ is a vector such that $d_s(x, \mathbb{1}+\epsilon x) = \sigma(\vec{u}^{a-b} \vec{v}^{b})$. Then, we have $x_n = x_{n+1} = ... = x_{n+a-b}$ and $x_{n+a-b+1} = x_{n+a-b+2} = ... = x_{m+2n} = I$, where $t$ is the number defined by

\[
t = \begin{cases} 
  x_n, & \text{if } s < x_n, \\
  x_{n+2a+1}, & \text{if } s > x_{n+2a+1}, \\
  s, & \text{if } x_n \leq s \leq x_{n+2a+1}.
\end{cases}
\]

Let $\vec{y} = (y_1, y_2, ..., y_{n+m})$ be the vector defined by

\[
y_i = \begin{cases} 
  x_i, & \text{if } i \leq m or m+2a, \\
  x_{n+2a+1}, & \text{if } m+2a+1 \leq i \leq m+2a.
\end{cases}
\]

Then,

\[
\sigma(\vec{u}^{a-b} \vec{v}^{b}) \leq d_{s}(\mathbb{1}+\epsilon x, \vec{x}) = b(1-x_i^2) + (a-b)(1-x_{n+2a+1}^2) - a(1-s)^2 - (1+\delta)^2.
\]

(10)

Let $\delta$ be the right part of the equality (10). Then, if $s < x_n$, we have

\[
\phi = (a-b)((1-x_{n+2a+1})^2 - (1-x_n^2)) \leq 0.
\]

If $s > x_{n+2a+1}$ we also have

\[
\phi = b((1-x_i^2) + (1-x_n^2) - (1-x_{n+2a+1}^2)) = 2b(x_i^2 - x_{n+2a+1}^2) \leq 0.
\]

If $x_n \leq s \leq x_{n+2a+1}$, we still have

\[
\phi = 2b((1-x_n^2) + (a-b)(1-x_{n+2a+1}^2) - a(1-s)^2 - (1+\delta)^2) \leq 2b(1+s^2) + (a-b)(1-s^2) - a(1-s)^2 - (1+\delta)^2 = 0.
\]

Hence, (9) is valid.

The following lemma can be found in [11].

Lemma 3 Let $\vec{u}$ be an arbitrary vector in $V^n$. If $w_{i+1}(\vec{u})/m = \min \{1, w(\vec{u})/m\}$ holds for all $i$ with $0 \leq i < m$, the MSED $\sigma(\vec{u})$ is given by

\[
\sigma(\vec{u}) = \begin{cases} 
  m - \frac{m}{m+2w(\vec{u})}, & \text{if } w(\vec{u})/m < 1/2, \\
  m, & \text{otherwise}.
\end{cases}
\]

To design Chase-like algorithms achieving BD decoding with a small set of input vectors, according to Lemma 2, we can only consider the Chase-like algorithms $C(U_j)$ such that the numbers defined in Lemma 1 satisfy $e_0 \geq e_1 \geq ... \geq e_{k-1}$. Furthermore, we assume that the set $j$ contains $d - 2$. Then, $e_{k-1} = 1$. Assume that

\[
h_i = 1 < h_1 < h_2 < ... < h_p
\]

are the distinct integers in the list $c_0, c_1, ..., c_{k-1}$ and, for each $0 \leq j \leq p$, $h_j$ repeats $g_j$ times in such list. Then, we have

\[
\sum_{i=1}^{g_j} h_i = r + 1.
\]

(12)

and (3) can be rewritten further as

\[
f_j = \xi - \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \x
Let $g_p = \{\tau + 1 - \sum_{i=0}^{p-1} f_i(i+1) / \lfloor p + 1 \rfloor \}$ and $i_p = \tau + 1 - \sum_{i=0}^{p-1} f_i(i+1) - g_p(p+1)$.

For $0 \leq i \leq p - 1$, let

$$g_i = \begin{cases} f_i + 1, & \text{if } \tau \leq i \\ f_i, & \text{otherwise}. \end{cases}$$

Then $h_g, g_1, g_2, \ldots, h_p, g_p$ are positive integers satisfying (11) and (12). From the left inequality of (17), we see

$$b^i(\tau+1)^i \geq \sum_{i=0}^{p-1} (i+1)^{-2a} \geq \int_0^\tau x^{-2a} dx = p^{-2a} / (2-2a),$$

and thus

$$p \leq (\tau+1)^{\lfloor 2 \rfloor}((2-2a) / b)^{\lfloor 2 \rfloor(2-2a)}. \tag{19}$$

Hence,

$$\begin{aligned}
\sum_{i=0}^{p-1} \frac{(h_{i+1} - h_i)^2}{2g_i + h_{i+1} - h_i} & = \sum_{i=0}^{p-1} \frac{1}{2b(\tau+1)^{\lfloor 2 \rfloor}} \\
& \leq \frac{1}{2b(\tau+1)^{\lfloor 2 \rfloor}} \sum_{i=0}^{p-1} (i+1)^{-2a} \\
& \leq \frac{1}{2b(\tau+1)^{\lfloor 2 \rfloor}} \left(1 + \int_0^\tau x^{-2a} dx \right) \\
& = \frac{1}{2b(\tau+1)^{\lfloor 2 \rfloor}} \left(1 + \frac{p^{2-2a}}{2a} \right) \leq \frac{p^{2-2a}}{4ab(\tau+1)^{\lfloor 2 \rfloor}} \leq \frac{p^{2-2a}}{4ab}.
\end{aligned}$$

If we choose $b$ further as

$$b = 3^{\tau-a} \min(20a) \tau^{-\lfloor 2 \rfloor(2-2a)} / 2, \tag{20}$$

Then we have

$$\sum_{i=0}^{p-1} \frac{(h_{i+1} - h_i)^2}{2g_i + h_{i+1} - h_i} \leq 5 / 3,$$

and thus according to Theorem 1, the Chase-like algorithm $C(U_j)$ achieves BD decoding.

From $g_p \leq f_p^{-1}$, (19) and (20), we have

$$\begin{aligned}
\sum_{i=0}^{p-1} g_i & \leq p + 1 + \sum_{i=0}^{p-1} b(\tau+1)^i(i+1)^{-2a} \\
& \leq p + 1 + b(\tau+1)^i \int_0^\tau (1 + (p+1)^{-2a} + \int_0^\tau x^{-2a} dx \\
& = p + 1 + b(\tau+1)^i \left(1 + (p+1)^{-2a} + \frac{p^{1-2a} - 1}{1 - 2a} \right) \\
& \leq 1 + 2b(\tau+1)^i \\
& + \left(2-2a\right)^{\frac{1}{2-2a}} b^{\frac{1}{2-2a}} + \frac{1}{1 - 2a} (2-2a)^{\frac{1}{2-2a}} b^{\frac{1}{2-2a}} (\tau+1)^{\lfloor 2 \rfloor} \\
& = 1 + 2a(\tau+1)^i + \sqrt{\frac{40a(1-a)}{3}} \left(1 + \frac{3}{20a(1 - 2a)} \right) (\tau+1)^{\lfloor 2 \rfloor}.
\end{aligned}$$

Hence, we have proved the following theorem.

**Theorem 2** When the Hamming distance $d$ of the code approaches infinity, the Chase-like algorithms can achieve BD decoding with $(\psi + o(1))d^{\lfloor 2 \rfloor}$ input vectors, where

$$\psi = \min_{0 < c < 1/2} \left(1 + \frac{3}{20a(1 - 2a)} \right) \approx 2.218.$$

**V. Conclusions**

In literature, there are many works to estimate the smallest size, denoted by $\Delta(d)$ for binary block code of Hamming distance $d$, of input vector sets of Chase-like algorithms which achieve BD decoding. Unlike most of these works, we deal with in this paper some Chase-like algorithms with an additional input vector whose nonzero entries are not confined in the most unreliable positions. With a similar method used in [7], we show that such a Chase-like algorithm has also a unique minimal vector in its unchecked region and then improve the best known upper bound on $\Delta(d)$ to: $\Delta(d) \leq (\psi + o(1))d^{\lfloor 2 \rfloor}$, where $\psi = 2.218$.

**References**


