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Least Action Principle for an Integrable Shallow Water Equation

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Abstract

For an integrable shallow water equation we describe a geometrical approach showing that any two nearby fluid configurations are successive states of a unique flow minimizing the kinetic energy.

1 Introduction

The description of hydrodynamical flows by means of geodesics, initiated by Arnold [1], allows one to apply the methods of Riemannian geometry to the study of these flows. The geometric approach consists mainly in formulating facts for the infinite-dimensional case of fluid flows using results from classical finite-dimensional Riemannian geometry cf. [2]. For the periodic Camassa-Holm [3] equation, arising in the study of shallow water waves, a rigorous geometric study can be pursued [6]. As a result, one obtains, for instance, a proof of the fact that a state of the system can be transformed to any nearby state by a unique flow. Of all possible paths joining these two states, the system selects the one of minimal kinetic energy - the Least Action Principle holds. In the next section we highlight some aspects of the Camassa-Holm model and in Section 3 we present the geometric approach to the Least Action Principle (the full details of which can be found in [6]).

2 The Camassa-Holm equation

The spatially periodic equation

$$u_t - u_{txx} + 3uu_x = 2u_xu_{xx} + uu_{xxx}, \tag{2.1}$$

in dimensionless space-time variables (x, t) is a model for the unidirectional propagation of two-dimensional shallow water waves over a flat bottom [3], with $u(t, x)$ representing the horizontal component of the velocity or, equivalently, the water's free surface. The

interest in periodic shallow water waves is motivated by the observation that, as a matter of common experience, the waves in a channel are approximately periodic. The well-posedness issue (solutions exist, are unique and depend continuously on the data), a perequisite to the usefulness of (2.1) as a model equation for real wave phenomena, is settled in [9]. Equation (2.1) was already noticed in [8] as an abstract equation with infinitely many conservation laws. This aspect is related to the fact that (2.1) is an integrable infinite-dimensional Hamiltonian system cf. [5]: the equation can be converted into a sequence of linear ordinary differential equations whose flows have constant speed. Let us also note that equation (2.1) admits peaked travelling wave solutions that are solitons (two travelling waves reconstitute their shape and size after interacting with each other) cf. [3], and that the only way a singularity can develop in a classical solution to (2.1) is in the form of wave breaking [4]: the solution remains uniformly bounded while its slope becomes unbounded at a finite time.

3 The geometric approach

From the Lagrangian viewpoint, each state of the system (2.1) is described by a diffeomorphism of the ambient space, representing the rearrangement of the particles with respect to their initial positions. The motion of the system is therefore a path in the diffeomorphism group. Since a particle on the water's free surface will always stay on the surface and (2.1) describes two-dimensional waves (no motion takes place in the y-direction), we may regard the motion of (2.1) as that of a one-dimensional membrane. In other words, the configuration space of (2.1) can be reduced to the group $\mathcal{D}$ of smooth orientation preserving diffeomorphisms of the circle. Let $\varphi(t,\cdot)$ be a path in the diffeomorphism group, starting at the identity $\varphi(0,x)=x$, $x\in\mathbb{S}$ ($\mathbb{S}$ being the unit circle), representing the evolution of an initial state for (2.1). The material velocity field is $(t,x)\mapsto\varphi_t(t,x)$ while the spatial velocity field is $u(t,x)=\varphi_t(t,\varphi^{-1}(t,x))$. In terms of u, we have the Eulerian description (from the viewpoint of a fixed observer) while φ_t represents the Lagrangian viewpoint (the motion as seen by following each particle): for a fluid particle initially located at x, $\varphi_t(t,x)$ is its velocity at time t, while $u(t,\varphi(t,x))$ is the velocity at the location $\varphi(t,x)$.

In Lagrangian mechanics, the motion of a mechanical system is a critical point of a certain functional (the Action), defined on all the paths in configuration space having fixed endpoints. For a system with no external forces (inertial system), this Action is precisely the kinetic energy. To the order of approximation to which (2.1) was obtained from the governing equations for water waves, $u(t,x)$ represents the horizontal velocity component and $u_x(t,x)$ is the vertical velocity component at the free surface cf. [10]. Hence, the kinetic energy on the free surface at instant t is

$$\frac{1}{2}\int_{\mathbb{S}}\Big(u^2(t,x)+u_x^2(t,x)\Big)\,dx. \tag{3.1}$$

The fact that there is no preferred initial state in the configuration space of (2.1) suggests a right invariance property: observe that if we replace the path $t\mapsto\varphi(t,\cdot)$ by $t\mapsto\varphi(t,\varphi_0(\cdot))$ for a fixed time-independent $\varphi_0\in\mathcal{D}$, then the spatial velocity u is unchanged. Hence, the kinetic energy of a state $\varphi\in\mathcal{D}$ of the system (2.1) is given by

$$K(\varphi,\varphi_t)=\frac{1}{2}\int_{\mathbb{S}}\Big\{(\varphi_t\circ\varphi^{-1})^2+\left[\partial_x\left(\varphi_t\circ\varphi^{-1}\right)\right]^2\Big\}\,dx. \tag{3.2}$$

In conclusion, the motion is described by the critical points of the Action

$$\int_0^T K(\varphi,\varphi_t)\,dt = \frac{1}{2}\int_0^T \int_{\mathbb{S}} \Big\{(\varphi_t\circ\varphi^{-1})^2 + [\partial_x\,(\varphi_t\circ\varphi^{-1})]^2\Big\}\,dx\,dt, \tag{3.3}$$

over the path $\{\varphi(t,\cdot);\, t\in[0,T]\}$ in $\mathcal{D}$. Note that the quadratic functional (3.2) defines a right-invariant Riemannian metric on the Lie group $\mathcal{D}$ and the geodesic flow of this metric corresponds to the critical points of the Action. The geodesic curve $t\mapsto\varphi(t,\cdot)$ in $\mathcal{D}$ starting at the identity $\varphi(0,\cdot)=Id$ in the direction $u_0\in C^\infty(\mathbb{S})$ satisfies the equation

$$\begin{cases}\varphi_t = v\\ v_t = P(\varphi,v),\end{cases} \tag{3.4}$$

where the operator P is given by

$$P(\varphi,v) = -\Big\{\partial_x\,(1-\partial_x^2)^{-1}\Big((v\circ\varphi^{-1})^2 + \frac{1}{2}\,[(v\circ\varphi^{-1})_x]^2\Big)\Big\}\circ\varphi. \tag{3.5}$$

In terms of the Eulerian velocity $u=\varphi_t\circ\varphi^{-1}$, the geodesic equation (3.4) is precisely the Camassa-Holm equation (2.1) as pointed out in [11]. This resembles the fact that the Euler equation in fluid mechanics is an expression of the geodesic flow in the group of incompressible diffeomorphisms cf. [1], [7].

It is possible to re-express certain ideal fluid flows, as well as some geophysical fluid flows, as geodesic flows on their configuration spaces cf. [2] but the results obtained are formal in character. It turns out that for the Camassa-Holm equation a rigorous geometric study can be pursued. Without going into the details of the intricate analysis made in [6], let us present the results obtained. A study of the geodesic equation (3.4) yields the local existence of the geodesic flow on $\mathcal{D}$. Interestingly, some geodesics are defined globally in time, while on certain geodesic paths the diffeomorphisms flatten out in finite time. Pursuing the analysis, one can prove that a geodesic on $\mathcal{D}$ is locally the shortest path between two nearby elements of $\mathcal{D}$ - the Least Action Principle holds for the Camassa-Holm equation (2.1). This result has a direct physical interpretation and illustrates the power of the geometric approach. It simply means that a state of the system (2.1) is transformed to another nearby state by going through a uniquely determined flow that minimizes the kinetic energy. To the best of our knowledge, this is the first example where the Riemannian geometric approach can be pursued rigorously to obtain physically relevant information. Previous studies introduce enlarged configuration spaces [7] where the geodesics are only formal objects aimed to provide insight into the actual configuration space; however, the passage from these intermediate spaces to the configuration space remains an open question. Other approaches circumvent the analytical difficulties encountered in working with the configuration space by defining generalized flows (where particles may split and collide) but the physical meaning of these generalized flows is not understood cf. [2].

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