Journal of Nonlinear Mathematical Physics Volume 8, Number 4 (2001), 446–457 Letter

A Finite Dimensional Analog of the Krein Formula

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Received September 19, 2000; Revised June 29, 2001; Accepted July 3, 2001

Abstract

I offer a simple and useful formula for the resolvent of a small rank perturbation of large matrices. I discuss applications of this formula, in particular, to analytical and numerical solving of difference boundary value problems. I present examples connected with such problems for the difference Laplacian and estimate numerical efficiency of the corresponding algorithms.

1 Introduction

Wide application of various versions of M G Krein's formula resulted lately in a marked progress of the theory of boundary value problems for equations of mathematical physics. In its initial form, this formula connects the resolvents of two different selfadjoint extensions of a given symmetric operator with finite defect indices in the Hilbert space [1]. Using this connection one can find, in particular, the exact solution of the Schrödinger equation with the point-wise potentials, construct the correct theory of the boundary problems for the Laplace operator on graphs [2, 3], obtain an approximate expression for the resolvent and the exponent of the second order differential operator in the domains of $\mathbb{R}^n$ in terms of the parametrix of the operator in the whole space [8, 9], etc. The formula for the resolvent of the boundary value problems, which appears in such a way, can also be used for the construction of numerical algorithms.

It is interesting to find a direct analog of M G Krein's formula for the difference equations (and here such an analog is offered). This is important for construction of numerical algorithms when we reduce the initial differential equation to some finite dimensional (i.e., matrix) problem. If we had a discrete analog of M G Krein's formula, we would have been able to develop effective numerical algorithms for solution of the difference boundary value problems.

I begin with a very simple matrix relation. For reasons which I will try to explain in what follows, I refer to it as *finite dimensional analog of the Krein formula*. After brief discussion, I present a few examples, showing the simplest applications of this relation. In particular, I give a short description of the algorithm for solving general boundary value problem for difference Laplacian in two-dimensional rectangular domain with (perhaps) small defects such as holes or cuts.

2 Low dimensional perturbations of the matrix and their resolvents

We start with a very simple question from Linear Algebra. Though rather important, as I will try to illustrate, it did not attract attention of researchers, at least, I could not find it in the literature. Even Prasolov's encyclopedia of nice problems in Linear Algebra [14] missed it. Namely, suppose A is a square matrix and we had computed its inverse A^{-1}. And — such a terrible but common disaster! — we observe that the typist has typed *one* of matrix elements of the initial matrix wrong! Must we redo the whole work (this is pretty expensive for large matrices!) or there is a cheaper possibility to obtain the correct answer?

I will show an almost obvious way to answer the last half of the question in affirmative: there is a cheaper way. I assume, of course, that all the matrices we are going to invert are indeed invertible.

Thus, suppose we have to solve the equation

$$(A+B)x = f, \tag{2.1}$$

where $x, f \in \mathbb{C}^n$, f is a known vector and x is an unknown one. If we can easily solve the "unperturbed" equation $Ax = g$ for any right hand side g (this means exactly that A^{-1} is known), we can re-write (2.1) in the form

$$\left(E + A^{-1}B\right) x = A^{-1} f. \tag{2.2}$$

We introduce new unknown vector z from the relation $z = Bx$. Then, if we multiply (2.2) by B from the left, we obtain the equation for z:

$$\left(E + BA^{-1}\right) z = BA^{-1} f. \tag{2.3}$$

The following evident assertion holds.

Proposition 2.1. *If both matrices A and $A+B$ are invertible, then $E + BA^{-1}$ is also invertible.*

Proof. Assume the contrary. This means that the homogeneous equation

$$\left(E + BA^{-1}\right) z = 0 \tag{2.4}$$

has a nontrivial solution $z_0 \neq 0$. Since A is invertible by hypothesis, the vector $x_0 = A^{-1} z_0$ exists and is nonzero. Then having substituted $z_0 = Ax_0$ into (2.4) we obtain

$$(A+B)x_0 = 0,$$

in contradiction with the fact that $A+B$ is invertible. ■

We can, therefore, solve equation (2.3) and write

$$z = \left(E + BA^{-1}\right)^{-1} BA^{-1} f. \tag{2.5}$$

It is clear that equation (2.2) can be now rewritten as

$$x = A^{-1}f - A^{-1}z. \tag{2.6}$$

Now, we can substitute expression (2.5) for vector z into (2.6) and finally obtain the solution of (2.1) in the following strange form:

$$x = A^{-1}f - A^{-1}\left(E + BA^{-1}\right)^{-1}BA^{-1}f. \tag{2.7}$$

We can also rewrite this formula as an operator relation:

$$\boxed{(A+B)^{-1} = A^{-1} - A^{-1}\left(E + BA^{-1}\right)^{-1}BA^{-1}} \tag{2.8}$$

which I refer in what follows as the *finite dimensional analog of Krein's formula.*

Obviously, all the above is meaningless for *generic* matrices A and B, because the calculation of the inverse matrix for $E + BA^{-1}$ is of the same complexity as that of the initial one. The situation changes dramatically if the *rank of B is small.*

The word "small" means in this context that the ratio $\frac{\operatorname{rk} B}{\operatorname{rk} A}$ is much smaller than 1. In this case the calculation of the inverse matrix of $E + BA^{-1}$ becomes simple.

Example 2.1. Let V be an n-dimensional vector space, $e \in V$ and $f \in V^*$. Let $B = e \otimes f$ be the linear operator in V of rank 1. Then it is easy to see that

$$\left(E + BA^{-1}\right)^{-1} = E - \left(1 + f\left(A^{-1}e\right)\right)^{-1} e \otimes \left(A^{-1}\right)^{*} f, \tag{2.9}$$

and, therefore, we need only about n^2 operations to calculate matrix $\left(E + BA^{-1}\right)^{-1}$ instead of about n^3 in the general case.

One can obtain the estimate $m^3 + mn^2$ for complexity of such a calculation when $\operatorname{rk} B = m$. This value is much smaller than n^3 provided $\frac{m}{n} \ll 1$. So we can consider relation (2.8) as a version of perturbation theory in which the ratio $\frac{\operatorname{rk} B}{\operatorname{rk} A}$ of the rank of perturbation to the rank of the unperturbed operator plays the role of small parameter.

We will see in what follows that in some important cases the calculation of the unperturbed resolvent may turn out to be incredibly simple, in distinction with direct calculation of the perturbed one, and in these cases application of formula (2.8) becomes very effective.

Before demonstrating possible applications of formula (2.8) in computational mathematics, let me briefly explain the reason to baptize a very simple relation from Linear Algebra with a famous name. As it is mentioned in Introduction, the "actual" Krein formula [1] connects the resolvents of two different self-adjoint extensions A_1 and A_2 of a given symmetric operator A_0 in an *infinite dimensional* Hilbert space H.

Unfortunately, it is very difficult, or, perhaps, even impossible, to read any operator sense into the difference $A_1 - A_2$ of such extensions, because, as a rule, this difference vanishes on the intersection of their domains. Such and similar difficulties, however, had never been an obstacle for physicists, and they eagerly used Dirac's δ-function as a potential of "point-wise interaction" in the Schrödinger equation, see, e.g., [4]. Certain arguments which I skip convinced me that "point-wise" perturbations of differential operators are in some sense *perturbations of finite rank* and, due to this fact, the corresponding problems have exact solutions.

Note in this connection that, although the actual Krein formula presupposes finiteness of defect indices of the initial symmetric operator, it may by used as well in the case of infinite indices (there are a number of papers on this topic, see e.g., [6, 7] and references therein). Such a case arises, e.g., if we consider different boundary value problems for given symmetric partial differential operator [8].

In what follows we consider a problem of calculating resolvents of extensions (see formal definition in the next section) of *difference* operators based on formula (2.8).

It seems that the corresponding relation is a finite dimensional analog of the relation for differential operators.

This impression is not an illusion. Indeed, it is possible to consider (in some well-defined sense) the difference operators as approximation of differential ones, and then one can prove that *in the case of finite defects* (e.g., for ordinary differential operators) our formulas converge to the corresponding formulas for differential operators (private communication of E Gordon and S Albeverio; together with them we intend to explain this in detail elsewhere). Such convergence plays a crucial role both for goals of numerical analysis and as an instrument for investigation of infinite dimensional operators via their finite dimensional approximations (see, e.g., [5]). Unfortunately, rigorous results about convergence of finite-dimensional approximations of operators requires for proofs a nonelementary technique which is out of frame of this work. Nevertheless, I consider (briefly and without proof) at the end of Section 3 the simplest example of convergence of finite dimensional Krein formula for difference approximations of the Schrödinger operator with δ-potential on the unit circle to the usual Krein's formula for the resolvent of this operator.

3 Boundary-value problem for the difference operators

Difference approximations of boundary value problems for differential operators are a base for the numerical solving of such problems. Here I just introduce a convenient for our nearest goals language for formal description of "abstract" difference boundary value problems. I could not find an appropriate analog of such a language in the literature. Hopefully, the following examples make it clear why this language is useful and convenient.

For any set $\mathcal{M}$, let $C(\mathcal{M})$ be the space of all complex-valued functions on $\mathcal{M}$.

A linear map $A : C(\mathcal{M}) \longrightarrow C(\mathcal{M})$ will be called a *formal difference operator* in $C(\mathcal{M})$ if for each $x \in \mathcal{M}$ there exist a *finite* set $\gamma_A(x) \subset \mathcal{M}$ and function $a_x \in C(\gamma_A(x))$ such that

$$(Af)(x) = \sum_{y \in \gamma_A(x)} a_x(y) f(y), \qquad f \in C(\mathcal{M}), \tag{3.1}$$

Let Ω be a subset of $\mathcal{M}$. The point $z \in \Omega$ is an *inner point of the set* Ω *with respect to the map* A, if $\gamma_A(z) \subseteq \Omega$.

The point $z \in \Omega$ is a *boundary point of the set* Ω *with respect to the map* A, if $\gamma_A(z) \backslash \Omega \neq \emptyset$. The *boundary of* Ω *with respect to* A is the set $\partial_A \Omega$ of all boundary points of Ω. Define the set $b_A \Omega$ of *exterior points of* Ω *with respect to* A to be

$$b_A \Omega = \bigcup_{x \in \partial_A \Omega} (\gamma_A(x) \cap \overline{\Omega}), \qquad \text{where} \quad \overline{\Omega} = \mathcal{M} \setminus \Omega.$$

Note that in the "difference" case the sets $\partial_A\Omega$ and $b_A\Omega$ do indeed depend on the map A in contrast with the continuous situation. Observe that

$$\bigcup_{x\in\Omega}\gamma_A(x)=\Omega\cup b_A\Omega.$$

Define the map $A_\Omega : C(\Omega\cup b_A\Omega)\longrightarrow C(\Omega)$ by formula (3.1) for any point $x\in\Omega$.

Let $L : C(\Omega)\longrightarrow C(\Omega\cup b_A\Omega)$ be a linear map such that $(Lf)(x)=f(x)$ for all $x\in\Omega$. The operator L will be called an *extension operator* for the map A.

We say that the operator $A_L : C(\Omega)\longrightarrow C(\Omega)$ is an L-extension of the formal difference operator A if

$$(A_Lf)(x)=(A_\Omega Lf)(x),\qquad x\in\Omega. \tag{3.2}$$

Note, that the in case described, the extension operators play the role of boundary conditions for differential operators. I hope that this will be clear from the examples of this section.

In what follows I suppose that the set Ω is finite. Let for $\lambda\in\mathbb{C}$ the operator $R_L(\lambda)$ be the resolvent of the L-extension of the operator A such that $R_L(\lambda)=(A_L-\lambda E)^{-1}$.

We show that formula (2.8) establishes a simple algebraic connection between resolvents $R_L(\lambda)$ and $R_K(\lambda)$ of two different extensions of the formal difference operator A corresponding to two extension operators L and K. (In what follows I assume that λ is a common resolvent point for both A_L and A_K.) To obtain such a connection, note first that definition (3.2) implies

$$A_K=A_L+D_{LK},\qquad \text{where}\quad D_{LK}=A_\Omega(K-L). \tag{3.3}$$

Now, let us replace matrix A^{-1} in (2.8) with $R_L(\lambda)$, matrix $(A+B)^{-1}$ with $R_K(\lambda)$, and B with D_{LK}. Then we see that

$$R_K(\lambda)=R_L(\lambda)-R_L(\lambda)(E+D_{LK}R_L(\lambda))^{-1}D_{LK}R_L(\lambda). \tag{3.4}$$

Observe that all the inverse operators in this formula exist by the hypothesis.

What do we gain from this formula? Note first of all, that it is easy to see that

$$\operatorname{rk}D_{LK}\le\#(\partial_A\Omega),\qquad\text{and}\qquad \operatorname{rk}(A_K-\lambda)=\operatorname{rk}(A_L-\lambda E)=\#(\Omega).$$

It is remarkable that as a rule (see examples in what follows) $\#(\partial_A\Omega)\ll\#(\Omega)$, and we are in the situation discussed in Section 1. Examples also show that the complexity of calculation of the resolvent for different extensions may be essentially different.

Example 3.1. Resolvent of the one-dimensional difference Laplacian. Although it seems that this example has no practical meaning, it makes very clear all previous abstract constructions and has all essential features of practically important Example 3.2.

Let $\mathcal{M}=\mathbb{Z}$ the set of integers, N a positive integer, and $\Omega=\{0,1,\dots,N-1\}$. The one-dimensional difference Laplacian is the formal difference operator $\Delta : C(\mathcal{M})\longrightarrow C(\mathcal{M})$ defined by the relation

$$(\Delta f)(x)=f(x+1)-2f(x)+f(x-1),\qquad\text{where}\quad x\in\mathcal{M}\quad\text{and}\quad f\in C(\mathcal{M}).$$

Note that operator Δ differs from the usual difference approximation of the differential expression $\frac{d^2}{dx^2}$ on the uniform grid in $\mathbb{R}^1$ by a factor only.

In the case considered $\partial_\Delta\Omega = \{0, N-1\}$ and $b_\Delta\Omega = \{-1, N\}$.

We list all the extensions for Δ. Let $\hat{l} : C(\Omega) \longrightarrow C(\{-1, N\})$, i.e., $\hat{l}$ is represented by a $2 \times N$ complex matrix. Then for $f \in C(\Omega)$ set

$$(Lf)(x) = f(x), \quad x \in \Omega, \qquad (Lf)(y) = (\hat{l}f)(y), \quad y \in b_\Delta\Omega.$$

Clearly, any extension operator for Δ must be of such form.

Among all L-extensions of Δ there exists an exceptional one, for which the corresponding resolvent has an "almost explicit" expression. This is the so-called *periodic extension*, defined by extension operator L_0 such that

$$\left(\hat{l}_0 f\right)(-1) = f(N-1), \qquad \left(\hat{l}_0 f\right)(N) = f(0). \tag{3.5}$$

The exceptional role of this extension (denoted in what follows by Δ_0 instead of Δ_{L_0} for brevity) is the consequence of the fact that it can be diagonalized by Discrete Fourier Transformation (DFT), i.e.,

$$\Delta_0 = F\Lambda F^*, \tag{3.6}$$

where Λ is the multiplication operator (i.e., the diagonal matrix)

$$(\Lambda f)(x) = -4\sin^2\frac{\pi x}{N} f(x), \qquad x \in \Omega,$$

and the unitary DFT operator F is defined by the relation

$$(Ff)(x) = \frac{1}{\sqrt{N}} \sum_{y\in\Omega} e^{-i\frac{2\pi xy}{N}} f(y).$$

Formula (3.6) immediately implies the equality

$$R_0(\lambda) = F(\Lambda - \lambda E)^{-1} F^*, \tag{3.7}$$

and this is what we meant under the *explicit formula for resolvent*.

It is well known that there exists an abnormally effective numerical method (called Fast Fourier Transformation, or FFT) for application of DFT to the vector. It requires only $\sim N\log N$ arithmetic operations instead of $\sim N^2$ for the general $N \times N$ matrices [10]. This fact crucially reduces the complexity of computation of operator (3.7).

Is there an algorithm which allows one to calculate the resolvent of an *arbitrary* extension of Δ with the same complexity as for Δ_0? Formula (3.4) gives a positive answer to this question. It only suffices to show that the computation of matrix $(E + D_{0K}R_0(\lambda))^{-1}$ is not a problem. Indeed, due to the fact that $(D_{0K}f)(x) = 0$ for $f \in C(\Omega)$ and $x \in \Omega \setminus \partial_\Delta\Omega$, to solve the equation

$$(E + D_{0K}R_0(\lambda))f = g, \tag{3.8}$$

we only have to find $f(0)$ and $f(N-1)$. We denote by δ_x the function from $C(\Omega)$ defined by

$$\delta_x(y) = \begin{cases} 0, & \text{for } x \neq y, \\ 1, & \text{for } x = y. \end{cases}$$

Since $f(x) = g(x)$ for $x \in \Omega \setminus \partial_\Delta \Omega$, we can re-write equation (3.8) in the form

$$\begin{aligned}
&(1 + (D_{0K} R_0(\lambda)\delta_0)(0))f(0) + (D_{0K} R_0(\lambda)\delta_{N-1})(0) f(N-1) \\
&\qquad = g(0) - (D_{0K} R_0(\lambda)(g - g(0)\delta_0 - g(N-1)\delta_{N-1}))(0), \\
&(D_{0K} R_0(\lambda)\delta_0)(N-1) f(0) + (1 + (D_{0K} R_0(\lambda)\delta_{N-1})(N-1)) f(N-1) \\
&\qquad = g(N-1) - (D_{0K} R_0(\lambda)(g - g(0)\delta_0 - g(N-1)\delta_{N-1}))(N-1).
\end{aligned}$$

This is a system of two linear equations for two unknowns, which is solvable due to Proposition 2.1. So, to calculate the resolvent $R_K(\lambda)$, we only have to know how to calculate $R_0(\lambda)$ and how to invert 2×2-matrices ...

This example is, as have already been said, of no practical importance, because there exists another (not DFT-based) algorithm for inverting the general three-diagonal matrix of complexity $\sim N$ (so called *sweep method*, see, e.g., [10]). For most often used types of boundary conditions (i.e., extension operators K), the matrix of Δ_K is of this kind, and the sweep method becomes preferable. For example, the Dirichlet problem corresponds to the extension defined by the map $\hat{l}$ of the form

$$\left(\hat{l} f\right)(-1) = -f(0), \qquad \left(\hat{l} f\right)(N) = -f(N-1).$$

and this leads to a three-diagonal matrix.

Note, however, that if for an extension operator K the matrix of Δ_K is not three-diagonal (as is the case, e.g., for Δ_0), one can use the Dirichlet extension as the "initial" one and solve the problem for the K-extension using only $\sim N$ arithmetic operations! The reason for using DFT in this example becomes clear from the following example.

Example 3.2. The boundary value problem of third kind for Laplacian in two-dimensional rectangle. We consider now the boundary value problems for the two-dimensional difference Laplacian. Let $\mathcal{M} = \mathbb{Z}^2$, N and M positive integers, and $\Omega = \{0, \ldots, N-1\} \times \{0, \ldots, M-1\}$. The formal two-dimensional Laplace operator which we denote by the same symbol $\Delta : C(\mathcal{M}) \longrightarrow C(\mathcal{M})$ is given by the formula:

$$\begin{aligned}
&(\Delta f)(x, y) = f(x+1, y) + f(x-1, y) + f(x, y+1) + f(x, y-1) - 4f(x, y), \\
&\text{for} \quad x, y \in \mathcal{M} \quad \text{and} \quad f \in C(\mathcal{M}).
\end{aligned}$$

Clearly, the set of the boundary points with respect to the operator Δ is

$$\partial_\Delta \Omega = (\{0, N-1\} \times \{0, \ldots, M-1\}) \cup (\{0, \ldots, N-1\} \times \{0, M-1\}),$$

so that $\#(\partial_\Delta \Omega) = 2(N + M - 2)$. We see once more that $\#(\partial_\Delta \Omega) \ll \#(\Omega) = MN$. Hence, there exists a good chance for applying Krein's formula. To actually apply it, we first describe the set $b_\Delta \Omega$.

The next geometric proposition is almost evident and we omit proof.

Proposition 3.1. *1) For $p = (x, y) \in \mathbb{Z}^2$ set $|p| = |x| + |y|$. Then $p \in b_\Delta\Omega$ if and only if there exists (and then it is unique) $\varepsilon(p) \in \mathbb{Z}^2$ such that $|\varepsilon(p)| = 1$ and $p + \varepsilon(p) \in \Omega$.*
2) $\#(b_\Delta\Omega) = 2N + 2M$.

We will not describe all extension operators for Δ (though possible, this is not interesting), instead we will consider several distinguished cases. First of all, as in the one-dimensional case considered in Example 3.1, there exists a remarkable periodic extension defined by extension operator L of the form

$$(Lf)(x, y) = f(x \bmod N,\ y \bmod M), \qquad (x, y) \in b_\Delta\Omega.$$

The corresponding operator will be denoted again by Δ_0 and it has the same characteristic property, namely, may be diagonalized by a *two-dimensional* DFT [10]. Therefore, one needs $\sim MN \log MN$ arithmetic operations for calculating the resolvent $R_0(\lambda)$ instead of about $(MN)^3$ to invert the general linear operator in $C(\Omega)$.

Among other extensions of two-dimensional difference Laplacian, I consider only the ones corresponding to *local boundary conditions* for the differential Laplace operator. These extensions are defined by the family of extension operators K of the form

$$(Kf)(p) = k(p)f(p + \varepsilon(p)), \qquad p \in b_\Delta\Omega, \tag{3.9}$$

where $k \in C(b_\Delta\Omega)$ and $\varepsilon(p)$ is defined in Proposition 3.1.

We now consider again the relation (3.8). It is easy to see that, as in the one-dimensional case, this equation can be transformed to a linear equation for function $f \in C(\partial_\Delta\Omega)$ and we need $\sim (2M + 2N - 2)^3$ arithmetic operations to solve it. For M, N large enough, the inequality $(2M + 2N - 2)^3 \ll (MN)^3$ holds, and we obtain the algorithm for solving the third kind boundary value problem for two-dimensional difference Laplacian with complexity $\sim (2M + 2N - 2)^3 + MN \log MN$ arithmetic operations. Moreover, if one has to repeatedly solve this problem for different right hand sides, it suffices to calculate matrix $(E + D_{0K}R_0(\lambda))^{-1}$ only once and then we need only $\sim (2M + 2N - 2)^2 + MN \log MN$ arithmetic operations for each right hand side. Asymptotically, this complexity is the same as that for the periodic Laplacian.

Note that in contrast with the one-dimensional case, the direct (i.e., non-iterational) methods for calculation of $R_K(\lambda)$ exist only for exceptional extension operators even from family (3.9), see [10]. It makes Example 3.2 important in practical applications.

Example 3.3. The Laplacian in the two-dimensional rectangle with a hole. Let $\mathcal{M}$, Δ and Ω be the same as in Example 3.2 and p an inner point of Ω. Let $\Omega_p = \Omega \setminus \{p\}$. It is clear that $p \in b_\Delta\Omega_p$ and $b_\Delta\Omega_p = \{p\} \cup b_\Delta\Omega$. We consider the extension operator K_p of the form (3.9) and suppose in addition that

$$(K_p f)(p) = \sum_{\{\varepsilon : |\varepsilon| = 1\}} \alpha(\varepsilon) f(p + \varepsilon), \qquad \alpha(\varepsilon) \in \mathbb{C}.$$

Note that in this formula $p + \varepsilon \in \Omega_p$ for all ε due to our hypotheses.

It is easy to see that operator Δ_{K_p} is exactly a rank 1 perturbation of $\Delta_K|_{C(\Omega_p)}$, where we consider the space $C(\Omega_p)$ as a subspace in $C(\Omega)$ consisting of functions f such that

$f(p) = 0$. So the resolvent of Δ_{K_p} can be calculated with the same efficiency as that of Δ_K! This is indeed remarkable, because one can consider operator Δ_{K_p} as the difference approximation of the differential Schrödinger operator with point-wise potential [4], and we see that the difference case can be investigated with the help of the introduced finite dimensional analog of the Krein formula in the same manner as differential operators with point-wise potentials are investigated by means of the "actual" Krein formula.

It is clear that in the same way one can construct resolvents for Laplacian in rectangle with more complicated defects (like holes containing more than one point, cuts, etc). Our approach is efficient provided $\#(b_A\Omega) \ll \#(\Omega)$ and we know an effective algorithm for calculating resolvent of at least one extension.

Example 3.4. The point-wise potentials in one-dimensional case and convergence. The aim of this example is to demonstrate that in simplest case application of formula (2.8) to the difference approximation of differential operator leads to the expression for resolvent which term-by-term converges to one obtained by applying the "actual" Krein's formula to initial differential operator.

Let A be the Laplace operator $-\frac{d^2}{dx^2}$ in $L_2([0, 2\pi])$ with periodic boundary conditions. It is evident that its resolvent is of form

$$(R_A(\lambda)\varphi)(x) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \varphi_m e^{imx} \frac{1}{m^2 - \lambda}, \tag{3.10}$$

where

$$\varphi_m = \int_0^{2\pi} \varphi(x) e^{-imx} dx. \tag{3.11}$$

Following Krein, consider the one-parametric family of self-adjoint extensions of the restriction of A onto the space of smooth functions vanishes in the neighborhood of the endpoints of the interval $[0, 2\pi]$, such that the resolvents of operators from the family are of the form

$$\begin{aligned}(R_{A_\mu}(\lambda)\varphi)(x) = {} & \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \varphi_m e^{imx} \frac{1}{m^2 - \lambda} \\ & - \mu \frac{\left(\frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \frac{\varphi_m}{m^2 - \lambda}\right) \left(\frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \frac{e^{imx}}{m^2 - \lambda}\right)}{1 + \frac{\mu}{2\pi} \sum_{m=-\infty}^{\infty} \frac{1}{m^2 - \lambda}},\end{aligned} \tag{3.12}$$

where μ is a parameter of family. (Note that all series in this expression converge either in $L_2([0, 2\pi])$ or in $\mathbb{C}$ when $\varphi \in L_2([0, 2\pi])$.) It is well-known (see, e.g., [4], where a number of similar examples are considered), that for each real μ the operator R_{A_μ} is indeed the resolvent of a self-adjoint operator A_μ in $L_2([0, 2\pi])$. This A_μ is usually called the *Schrödinger operator with δ-potential* (parameter μ plays the role of a coupling constant)[1].

[1]Of course, formula (3.12) does not give *all* possible extension of symmetric operator considered, but the family described suffices for our goals.

Let now M be a positive integer and let operator A_M in $L_2(\{0,\dots,2M-1\}])$ be of the form

$$(A_M f)_j = -\frac{1}{h^2}(f_{j-1} - 2f_j + f_{j+1}), \qquad j \in \{0,\dots,2M-1\}, \tag{3.13}$$

where $h = \pi/M$ and the "exterior" values of f are defined by "periodic boundary conditions" (3.5). It is easy to see that the resolvent of operator A_M is, due to relation (3.7), of the form

$$(R_{A_M}(\lambda)\varphi)_j = \frac{1}{2\pi}\sum_{m=-M+1}^{M} \widehat{\varphi}_m \mathrm{e}^{\mathrm{i}hmj}\frac{1}{\dfrac{4}{h^2}\sin^2\dfrac{hm}{2} - \lambda}, \tag{3.14}$$

where

$$\widehat{\varphi}_m = h\sum_{m=0}^{2M-1} \varphi_j \mathrm{e}^{-\mathrm{i}hmj}. \tag{3.15}$$

Observe that relations (3.10), (3.11) and (3.14), (3.15) are of similar form. Moreover, setting $T_h : C([0,2\pi]) \to L_2(\{0,\dots,2M-1\})$ as

$$(T_h f)_j = f(hj), \qquad j \in \{0,\dots,2M-1\}, \tag{3.16}$$

one can see that that the relations

$$\lim_{M\to\infty} \|T_h A f - A_M T_h f\|_h = 0, \qquad \lim_{M\to\infty} \|T_h R_A(\lambda) f - R_{A_M}(\lambda) T_h f\|_h = 0 \tag{3.17}$$

hold for every sufficiently smooth periodic function $f \in L_2([0,2\pi])$ if the norm in $L_2(\{0,\dots, 2M-1\})$ is

$$\|\varphi\|_h^2 = h\sum_{j=0}^{2M-1} |\varphi_j|^2.$$

This means exactly that the family of finite dimensional operators $A_M,\ M \in \mathbb{N}$, *approximates* the operator A [10, 5], or, in another words, A_M tends to A when $M \longrightarrow \infty$.

Let now $\widehat{\delta}_0^M$ be the operator in $L_2(\{0,\dots,2M-1\})$ given by the formula

$$(\widehat{\delta}_0^M f)_j = \frac{1}{h} f_0 \delta_{0j}. \tag{3.18}$$

It is easy to make use of (2.8) in order to calculate the resolvent of $A_{M\mu} \equiv A_M + \mu\widehat{\delta}_0^M$ (cf. also with Example 2.1 and relation (2.9)). In this way we obtain an expression for the resolvent of $A_{M\mu}$:

$$(R_{A_{M\mu}}(\lambda)\varphi)_j = \frac{1}{2\pi}\sum_{m=-M+1}^{M} \varphi_m \mathrm{e}^{\mathrm{i}hmj}\frac{1}{\dfrac{4}{h^2}\sin^2\dfrac{hm}{2} - \lambda}$$
$$- \mu\frac{\left(\dfrac{1}{2\pi}\displaystyle\sum_{m=-M+1}^{M} \frac{\varphi_m}{\frac{4}{h^2}\sin^2\frac{hm}{2} - \lambda}\right)\left(\dfrac{1}{2\pi}\displaystyle\sum_{m=-M+1}^{M} \frac{\mathrm{e}^{\mathrm{i}hmj}}{\frac{4}{h^2}\sin^2\frac{hm}{2} - \lambda}\right)}{1 + \dfrac{\mu}{2\pi}\displaystyle\sum_{m=-M+1}^{M} \frac{1}{\frac{4}{h^2}\sin^2\frac{hm}{2} - \lambda}}. \tag{3.19}$$

We compare now relations (3.12) and (3.19). It is easy to see that for resolvents R_{A_μ} and $R_{A_{M\mu}}$ a relation like (3.17) holds. Moreover, one can see also that *each term* in the left hand side of (3.19) converges to the corresponding term in (3.12). Hence, one can assert that in this sense the finite dimensional Krein formula converges to the "natural" Krein formula for the resolvent R_{A_μ}. This fact gives an additional argument in favor of the name "finite dimensional analog of Krein formula" for relation (2.8).

4 Concluding remarks

The finite dimensional analog of Krein's formula proved to be a useful instrument for investigation of difference equations both analytically and numerically. Moreover, it gives us a new approach to study *differential* problems (and, more generally, other "continuous" extensions) by reducing them to the corresponding difference (or, more generally, other finite dimensional) approximations. In this connection it is interesting that, in contrast with the "actual" Krein formula, our algebraic relation does not require operators involved to be Hermitian.

The method proposed for solving difference boundary value problem is applicable to a wide class of equations, in particular, in the case of complicated multi-point boundary conditions for one-dimensional equations, for the rectangular two-dimensional domains with cuts and some other "small" defects, for some cases of variable coefficient of difference operators, etc. It is clear that in every specific case one needs to adapt the general algorithm described in Sections 2, 3, but this general scheme is, nevertheless, useful for construction of particular numerical procedures.

The described method for solving of boundary value problems was successfully used in [11].

A similar approach exists also for constructing other than resolvent functions of difference operators. This is needed, e.g., in *initial boundary value problems*, see [12]. Certain moments of the method proposed for solving difference boundary value problems were announced in [13].

Acknowledgements

I am thankful to M Antonets, E Gordon and I Nefedov for helpful discussions and D Leites for hospitality and support.

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