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Random Groups in the Optical Waveguides Theory

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Abstract

We propose a new approach to the mathematical description of light propagation in a single-mode fiber light-guide (SMFLG) with random inhomogeneities. We investigate statistics of complex amplitudes of the electric field of light wave by methods of the random group theory. We have analyzed the behavior of the coherence matrix of a monochromatic light wave and the polarization degree of a nonmonochromatic radiation in SMFLG with random inhomogeneities as the length of the fiber tends to infinity; in particular, we prove that limit polarization degree is equal to zero.

1 Introduction

To be able to analyze statistical parameters of light propagation in a single-mode fiber light-guide (SMFLG) with random inhomogeneities is vital in connection with broad application of SMFLG in promising directions of industrial development such as optical communication [1, 2] and improvement of sensors of various physical parameters [3, 4, 5, 6, 7, 8].

Existence of random inhomogeneities in real optical fibers leads to appearance of coupling of orthogonal polarization modes, it is accompanied by exchange of energy between modes. In turn, the mode coupling causes random variations of a polarization degree of radiation in SMFLG and, as a result, undesirable phenomena in devices and communication lines developed on the base of SMFLG [9, 10, 11, 12]. For a rather long fiber even small random inhomogeneities strongly affect radiation characteristics and make their theoretical analysis quite complicated.

To investigate radiation characteristics in SMFLG with random inhomogeneities physicists usually apply methods of perturbation theory [13, 14, 15, 16, 17]. These methods are, however, essentially restricted both in the magnitude of fluctuations and the length of the fiber. This forces one to interpretate the results of such calculations for sufficiently long fibers. As an illustration we point at the discussion about the value (zero or not zero?)

of the limit polarization degree in infinite fibers (see, e.g., [13, 14, 15, 16, 18, 19]). It was this very problem that initially prompted us to develop the methods used in this paper. In particular, we have proved that the limit polarization degree is zero (see Corollary 2.4 in Section 2).

Here we propose a new approach to the mathematical description and investigation of light propagation in SMFLG with random inhomogeneities. Our methods provide with a correct analytical investigation of statistical characteristics of complex amplitudes of an electric field of light wave for arbitrarily long fibers. Note, nevertheless, that the mathematical model (equation (2.1)) we use is only applicable for real physical devices with not too strong fluctuations, because it does not take into account the *reflection* of light on inhomogeneities. The results, obtained in the framework of our model, present, however, the exact solution of *mathematical* problems formulated.

Our model and methods give a convenient basis for numerical simulation of different optical fiber devises (see, e.g., [19, 20, 21, 22, 23]).

Three pillars our approach is based on are: the random process theory, as an instrument for correct mathematical description of the statistics of inhomogeneities in fiber, the differential equation with random coefficients as mathematical model of wave propagation in random media and random groups theory as (and it seems to be quite unusual) an instrument for exact solving these equations. Of similar approaches to similar problems we only know [24].

On the structure of the paper. In Section 2 we introduce notions we need, formulate the problems we intend to solve and our main results. We also formulate two theorems concerning the random groups theory itself. Both these theorems are needed to solve our problems; one of them is only a reformulation in the form convenient to us of one of the main results of the random groups theory [25, 26], while the other one seems to be new, although very close to some known results from the Markov process theory [27]. Among "physically meaningful" results we state that the limit polarization degree is exactly zero. All the proofs, both of physical and pure mathematical nature, are collected in Section 3 divided into subsections. Our calculations are sometimes quite complicated and sometimes simple but rather long so we separated the assertions from their proofs. We included more or less detailed proofs because we believe that our methods may be also of interest themselves. In proofs we omit some technical details when they do not contain in our opinion any nontrivial idea.

2 Basic definitions, equations and main results

We consider the propagation problem of a nonmonochromatic light wave in a single-mode fiber light-guide (SMFLG) with varying directions of anisotropic axes and without losses.

Let z be the lengthwise coordinate in fiber, β a parameter depending on wavelength and physical parameters of fiber, $\vec{E}(\beta, z) = \begin{pmatrix} E_x(\beta, z) \\ E_y(\beta, z) \end{pmatrix}$ the complex amplitude vector of the two orthogonal components of the electric field of a monochromatic light wave at point z, where x- and y-axes coincide with directions of anisotropic axes in this point. The propagation of a monochromatic light wave in SMFLG is described by the following equation for vector $\vec{E}(\beta, z)$ [19, 28]:

$$\frac{\partial \vec{E}}{\partial z} = X_\beta\left(\Theta(z)\right)\vec{E}, \qquad X_\beta\left(\Theta(z)\right) = \begin{pmatrix} \dfrac{\mathrm{i}\beta}{2} & \Theta(z) \\ -\Theta(z) & -\dfrac{\mathrm{i}\beta}{2} \end{pmatrix}, \tag{2.1}$$

where $\Theta(z)$ is the rotation velocity (twist) of anisotropic axes, i.e, the derivatives of the directions of anisotropic axes with respect to z.

The real optical fibers have random inhomogeneities occasioned by uncontrollable peculiarities in the preparation process of fiber and fiber devices. We will assume in what follows that the function $\Theta(z)$ is a realization of a random process. Observe that though in real fibers there appears not only fluctuation of anisotropic axes's directions, we disregard all other kinds of random inhomogeneities. The reason is that their influence on light propagation in fiber is not so essential as that of the considered one, see, e.g., [28, 29, 30].

Our main goal is to investigate the statistical characteristics of the solutions of equation (2.1). To do so, we need a rigorous mathematics description of the process $\Theta(z)$. It seems that the statistical characteristics we are interested in do not depend crucially on the details of this description. Therefore, we will consider a sufficiently simple mathematical model of such process, slightly more general than suggested in [19]. This model, in our mind, is close enough to description of the real structure of random inhomogeneities in fibers, and, on the other hand, allows us to obtain rigorous results. We give some additional arguments in favour of our model in what follows. Note also that this model appears to be very useful in simulation of light propagation in random SMLFG [19, 20, 21, 22, 23].

Let $\{\Theta_k\}_{k=0}^{\infty}$ and $\{l_k\}_{k=0}^{\infty}$ be two sequences of independent random values such that all Θ_k are distributed with the same probability measure μ_Θ on $\mathbb{R}$ and l_k are also equivalently distributed with density ρ_l supported on $\mathbb{R}_+$. Let the random function $\Theta(z)$ be defined by the relation

$$\Theta(z) = \Theta_k, \qquad \text{where} \quad \sum_{j=1}^{k-1} l_j \leqslant z < \sum_{j=1}^{k} l_j, \quad \text{and} \quad k = 1, 2, \dots . \tag{2.2}$$

The random process (2.2) is a simple and convenient mathematical model, its particular cases are often used in different applications of the probability theory [31]. As it was shown in [19] this model with specific μ_Θ and ρ_l is physically justified and results obtained in the frameworks of this model are supported by empirical observations.

In what follows we denote by angular brackets $\langle\,\cdot\,\rangle$ the mean value of a random value.

We will say that the *regular twist* of the fiber is *absent* if the distribution of Θ is symmetric with respect to zero, in particular, $\langle\Theta^{2n+1}\rangle = 0$ for all non-negative integers n.

Let us now introduce parameters of the light field in fiber, the parameters which are most interesting from a physical point of view.

The coherence matrix of a *monochromatic* light wave is defined by the relation

$$J(\beta, z) = \vec{E}(\beta, z)\vec{E}^{\dagger}(\beta, z), \tag{2.3}$$

where $\dagger$ denotes the Hermitean conjugation. The mean coherence matrix $\langle J(\beta, z)\rangle$ is an important characteristic of the monochromatic wave, because the coupling (energy exchange) of polarization modes that occurs in SMFLG with random inhomogeneities is described by the diagonal elements of this matrix.

In what follows we will need *the function* $B(\beta)$ *of spectral density of nonmonochromatic radiation in fiber* by the relation

$$B(\beta) = \operatorname{tr} J(\beta, z). \tag{2.4}$$

Note that due to equation (2.1) $B(\beta)$ is indeed independent on z.

A *incoherent nonmonochromatic* radiation in SMFLG is characterized by its *polarization degree* $p(z)$ [32], which is

$$p(z) = \sqrt{1 - \frac{4 \det J(z)}{\operatorname{tr}^2 J(z)}},$$

where $J(z)$ is the *coherence matrix of a nonmonochromatic radiation*,

$$J(z) = \int J(\beta, z)\, d\beta. \tag{2.5}$$

The *mean square polarization degree* is then equal to $\langle p^2(z)\rangle$. The study of this value is one of the main goals of our work.

Earlier, in the frameworks of perturbation theory it was shown (see, e.g., [33]) that, in the absence of the regular twist and if $\vec{E}(\beta, 0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, the *averaged intensities of the eigenmodes* in SMFLG as $z \to \infty$ are of the form

$$\langle J_{11}(\beta, z)\rangle = \frac{1}{2}\left(1 + \mathrm{e}^{-2h(\beta)z}\right), \qquad \langle J_{22}(\beta, z)\rangle = \frac{1}{2}\left(1 - \mathrm{e}^{-2h(\beta)z}\right), \tag{2.6}$$

where

$$h(\beta) = \lim_{z\to\infty} \frac{1}{z} \left\langle \left| \int_0^z \Theta(z) \mathrm{e}^{\mathrm{i}\beta z}\, dz \right|^2 \right\rangle \tag{2.7}$$

is the *h-parameter* used usually to characterize the coupling between polarization modes [9, 30, 33, 34]. As we will see in what follows, the relation like (2.6) is valid in our model, but the formula for h-parameter is slightly different from (2.7) and coincides with it only when the random twist is small enough.

We wish to analyze the behavior of these characteristics both as $z \to \infty$ and at finite z. It is easy to see that to calculate $\langle J(\beta, z)\rangle$ and $\langle p^2(z)\rangle$ it is sufficient to know the distribution of random vector $\vec{E}(\beta, z)$ and also joint distribution of the vectors $\vec{E}(\beta_1, z)$ and $\vec{E}(\beta_2, z)$ at different β_1 and β_2. We don't need other statistical parameters of vectors $\vec{E}(\beta, z)$ and so we will investigate only the mentioned ones.

Denote by $N(z)$ the nonnegative integer N such that

$$N(z) = \min\left\{ N \mid \sum_{j=1}^{N} l_j > z \right\}. \tag{2.8}$$

Note that $N(z)$, as a random integer function of z, depends on the realization of the random process $\Theta(z)$. Now, the solution of equation (2.1) with function $\Theta(z)$ of the form (2.2) can be expressed as

$$\begin{aligned}\vec{E}(\beta,z) &= U(\beta,z)\vec{E}_0(\beta)\\ &= M_\beta\left(z-\sum_{k=1}^{N(z)-1} l_k, \Theta_{N(z)}\right) M_\beta\left(l_{N(z)-1},\Theta_{N(z)-1}\right)\cdots M_\beta\left(l_1,\Theta_1\right)\vec{E}_0(\beta),\end{aligned}\tag{2.9}$$

where $\vec{E}_0(\beta)$ is a complex amplitude of light field at the entry point of fiber, and $M_\beta\,(l,\Theta) = \exp\left(l\,X_\beta(\Theta)\right)$ is *Jones matrix* [35] of the fiber section of length l and fixed axis twist Θ,

$$M_\beta(l,\Theta)=\begin{pmatrix}\cos\left(\dfrac{l\beta_\theta}{2}\right)+\mathrm{i}\dfrac{\beta}{\beta_\theta}\sin\left(\dfrac{l\beta_\theta}{2}\right) & \dfrac{2\Theta}{\beta_\theta}\sin\left(\dfrac{l\beta_\theta}{2}\right)\\ -\dfrac{2\Theta}{\beta_\theta}\sin\left(\dfrac{l\beta_\theta}{2}\right) & \cos\left(\dfrac{l\beta_\theta}{2}\right)-\mathrm{i}\dfrac{\beta}{\beta_\theta}\sin\left(\dfrac{l\beta_\theta}{2}\right)\end{pmatrix}.\tag{2.10}$$

Here we set $\beta_\theta=\sqrt{\beta^2+4\Theta^2}$.

Random matrices $M_\beta(l,\Theta)$ generate a subgroup G of the group $SU(2)$ of 2×2 unitary matrices. Thus, the analysis of the statistics of $\vec{E}(\beta,z)$ reduces to the analysis of statistics of the products of random matrices from some group, i.e., to a problem from the theory of random groups.

In what follows we will need functions $m_{\beta k}(l,\Theta)$, $k\in\{0,1,3\}$ such that

$$M_\beta(l,\Theta)=\begin{pmatrix} m_{\beta 0}(l,\Theta)+\mathrm{i}m_{\beta 1}(l,\Theta) & m_{\beta 3}(l,\Theta)\\ -m_{\beta 3}(l,\Theta) & m_{\beta 0}(l,\Theta)-\mathrm{i}m_{\beta 1}(l,\Theta)\end{pmatrix}.\tag{2.11}$$

Together with the random vector $\vec{E}(\beta,z)$ which describes the complex amplitude of the electric field at point z, we consider also a random vector $\vec{E}_N(\beta)$ which describes the field at the output of the fiber that consists of precisely N random sections. The vector $\vec{E}_N(\beta)$ is defined by a formula similar to (2.9) but where the matrix-valued function $U(\beta,z)$ is replaced with the matrix-valued function

$$U_N(\beta)=M_\beta\left(l_N,\,\Theta_N\right)M_\beta\left(l_{N-1},\,\Theta_{N-1}\right)\cdots M_\beta\left(l_1,\,\Theta_1\right).\tag{2.12}$$

We will see that it is possible to use this “discrete semigroup” instead of “continuous” one from (2.9) in order to investigate the statistical properties of electric field in the random fiber. This is useful because there exist powerful results about the random matrix products in random group theory.

Let us formulate now some useful theorems based on the random group theory. Let $\{l_j\}_{j=1}^\infty$ be a sequence of independent equivalently distributed on $\mathbb{R}_+$ random values and $\{c_j\}_{j=1}^\infty$ be a sequence of independent equivalently distributed random vectors with values in $\mathbb{R}^n$ and absolutely continuous with respect to Lebesgue measure probability distributions. Let also $Y:\mathbb{R}^n\longrightarrow\mathfrak{g}$ be smooth enough and almost everywhere (with respect to the measure μ_c generated by random vector c) nonconstant function on $\mathbb{R}^n$ with values in the Lie algebra $\mathfrak{g}$ of the orthogonal or unitary group acting in the m-dimensional space, i.e., with values in the space of $m \times m$ skew-symmetric or skew-Hermitean matrices, respectively. Define the matrix-valued function $A : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow G$, where $G = SO(m)$ or $G = SU(m)$, by setting

$$A(z, q) = \exp\left(z\, Y(q)\right).$$

The matrix-valued function $A(z, q)$ generates a *random semigroup*, i.e., the random matrix-valued function $U(z_1, z_2)$, where $z_1, z_2 \in \mathbb{R}_+$, of the form

$$U(z_2, z_1) = A\left(z_2 - \sum_{j=1}^{N(z_2)-1} l_j, c_{N(z_2)}\right) \times A\left(l_{N(z_2)-1}, c_{N(z_2)-1}\right) \cdots A\left(l_{N(z_1)+1}, c_{N(z_1)+1}\right) A\left(\sum_{j=1}^{N(z_1)} l_j - z_1, c_{N(z_1)}\right),$$

where $N(z)$ is defined by (2.8) and $z_2 \geqslant z_1 \geqslant 0$. (Hereafter we assume $l_0 = 0$.) The term "continuous random semigroup" is related with the fact that a solution of the stochastic differential equation (2.1) is of such form, and also with the obvious fact that the equality $U(z_3, z_2)U(z_2, z_1) = U(z_3, z_1)$ holds for $z_3 \geqslant z_2 \geqslant z_1 \geqslant 0$. Below we deal with the matrix-valued function $U(z) \equiv U(z, 0)$. We note that the defined above random semigroup is stationary in the sense that the statistical characteristics of the matrix $U(z_1, z_2)$ depend only on $z_2 - z_1$.

Together with the *continuous* matrix semigroup, we will consider the *discrete* random semigroup U_{NK}, $N, K \in \mathbb{Z}_+$ which is defined for integer $N \geqslant K \geqslant 0$ by the equality

$$U_{NK} = A(l_N, c_N)A(l_{N-1}, c_{N-1}) \cdots A(l_{K+1}, c_{K+1})A(l_K, c_K).$$

As in the continuous case the term "semigroup" stems from the relation $U_{NK}U_{KL} = U_{NL}$ valid for $N \geqslant K \geqslant L \geqslant 0$, the stationarity condition is realized and we will again be interested in the statistical properties of the family $U_N \equiv U_{N0}$.

We denote by $\widehat{G}$ the minimal closed subgroup in G which contains all matrices $\{A(t, q) \mid (t, q) \in \mathbb{R}_+ \times \operatorname{supp} \mu_c\}$. By definition, $\widehat{G}$ is a non-discrete subgroup in the compact simple group G.

This theorem is a specialization of "the central limit theorem" for compact stochastic groups for the considered case (see, for example, [25, 26]).

Theorem 2.1. *Assume that the family of matrices $\{A(z, q) \mid (z, q) \in \mathbb{R}_+ \times \operatorname{supp} \mu_c\}$ is not contained in any conjugacy class with respect to any normal subgroup of $\widehat{G}$. Then the probability distribution of the random matrix U_N tends to the Haar's measure on $\widehat{G}$ as $N \to \infty$*[1].

The following theorem includes several statements which make it possible to calculate limit mean values of different linear (with respect to the semigroups introduced) quantities.

[1]The condition concerning the conjugacy classes is present in the central limit theorem [25, 26] for arbitrary compact stochastic group. This condition is certainly realized if $\widehat{G}$ is simple: no normal subgroups. As we will see below, if $\widehat{G}$ is semisimple this condition is equivalent to the statement that the family $\{A(z, q) \mid (z, q) \in \mathbb{R}_+ \times \operatorname{supp} \mu_c\}$ does not have any common eigenvector corresponding to the eigenvalue different from 1.

Formally, this theorem does not depend on Theorem 2.1 but in fact its hypothesis are equivalent to the ones of Theorem 2.1 and its conclusions are corollaries of Theorem 2.1.

Theorem 2.2. *Let P_0 be an orthogonal projection on a common eigenspace corresponding to the common zero eigenvalue of the family $\{Y(q) \mid q \in \operatorname{supp} \mu_c\}$. Suppose that the characteristic function $\hat{\rho}_l(\lambda)$, $\lambda \in \mathbb{C}$, of measure ρ_l, defined for* $\operatorname{Re} \lambda \leqslant 0$ *by the relation*

$$\hat{\rho}_l(\lambda) = \int_0^\infty \rho_l(z) \mathrm{e}^{-\lambda z}\, dz = \langle \mathrm{e}^{-\lambda l} \rangle, \tag{2.13}$$

is holomorphic in the closed left half-plane and has a meromorphic extension to the whole complex plane. Then

1) the limit mean of matrices U_N as $N \to \infty$ exists and $\lim\limits_{N\to\infty} \langle U_N \rangle = P_0$;

2) the limit mean of matrices $U(z)$ as $z \to \infty$ exists and $\lim\limits_{z\to\infty} \langle U(z) \rangle = P_0$.

This theorem allows us to find some limit values using discrete semigroup U_N instead of continuous $U(z)$. As we see below, this is sometimes more simple to do.

The following statements are obtained by applying the limit theorems on random matrix semigroups in order to derive statistical properties of a radiation in SMFLG when the fiber length tends to infinity.

Corollary 2.1. *The limit distribution of $\vec{E}_N(\beta)$ as $N \to \infty$ is the uniform distribution on the three-dimensional sphere.*

Corollary 2.2. *Let $\beta_1 \neq \pm\beta_2$. Then the vectors $\vec{E}_N(\beta_1)$ and $\vec{E}_N(\beta_2)$ are independent as $N \to \infty$.*

Corollary 2.3. $\lim\limits_{z\to\infty} \langle J(\beta, z) \rangle = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$

Corollary 2.4. $\lim\limits_{z\to\infty} \langle p^2(z) \rangle = 0.$

These results deal with the limit properties of light waves in fibers. The result of Corollary 2.3 is well known for fibers without regular twist (see, e.g., [33]). Corollary 2.4 seems to be most meaningful for physicists. Note in this connection that the problem of determination of the limit polarization degree is widely discussed in the literature up to now, see, e.g., [13, 14, 15, 16, 18, 19].

It is very interesting also to find out the dependence of the mean coherence matrix and polarization degree on the length-wise coordinate z. Some particular results in this direction are listed bellow. We emphasize that all these results have an asymptotic nature, i.e., are valid approximately for *sufficiently large z*.

First, some preliminary notices.

As mentioned above, the problems connected with continuous semigroup $U(z)$ are often essentially more complicated than the ones connected with U_N. This is why in what follows we deal with U_N instead of $U(z)$ and intend now to explain some reasons for possibility to perform such replacement. In fact, we wish that there were no essential differences between the asymptotic behavior of $\langle U_N \xi \rangle$ and $\langle U(N\langle l \rangle)\xi \rangle$ as $N \to \infty$. We did not try to prove or even to formulate this assertion in rigorous mathematical manner, but we base our wishes on various numerical experiments which confirmed this "fact" [20]. We think that it is possible to make a *theorem* from this "intuitive" assertion, but its proof must be quite difficult. The reader can anticipate this from presented in Section 3 proofs of asymptotic assertions for the discrete semigroup; they appear to be long enough even in this relatively simple case.

Proposition 2.1. *Suppose that in a fiber the regular twist is absent and* $2\langle m_{\beta 3}^2(l,\Theta)\rangle < 1$, *where* $m_{\beta j}$ *is defined in (2.11). Then the diagonal elements of the "mean coherence matrix"* $\left\langle U_N(\beta)J(\beta,0)U_N^\dagger(\beta)\right\rangle$ *at large* N *fulfill relation (2.6) with* $z = N\langle l\rangle$ *and* h*-parameter of the form*

$$h(\beta) = -\frac{1}{2\langle l\rangle}\ln\left(1 - \left\langle \frac{8\Theta^2}{\beta^2+4\Theta^2}\sin^2\left(\frac{l}{2}\sqrt{\beta^2+4\Theta^2}\right)\right\rangle\right). \tag{2.14}$$

It is interesting to compare relation (2.14) with (2.7). To do so, we must obtain an explicit formula for the "classical" h-parameter. The following assertion holds.

Proposition 2.2.

$$\lim_{z\to\infty}\frac{1}{z}\left\langle\left|\int_0^z \Theta(z)e^{i\beta z}\,dz\right|^2\right\rangle = \frac{4\langle\Theta^2\rangle}{\beta^2\langle l\rangle}\left\langle\sin^2\frac{l\beta}{2}\right\rangle. \tag{2.15}$$

To compare the different expressions for h-parameter, i.e., (2.14) and (2.15), we consider the case when $\rho_l(z) = \frac{1}{\langle l\rangle}\exp\left(-\frac{z}{\langle l\rangle}\right)$. Suppose that $\operatorname{supp}\mu_\Theta \subset [-\Theta_{\max},\Theta_{\max}]$ with $\Theta_{\max} \ll \sqrt{\beta^2+\langle l\rangle^{-2}}$. Then it is easy to see that the ratio of expressions (2.14) and (2.15) is of the form $1 + O\left(\frac{4\Theta_{\max}^2}{\langle l\rangle^{-2}+\beta^2}\right)$. Thus, the conditions of applicability of expression (2.7) appear to be quite rigidly restricted.

Proposition 2.3. *The asymptotic of the mean square of the polarization degree is of the form*

$$\langle p_N^2\rangle = \frac{1}{3}\sqrt{\frac{2\pi}{N}}\int\frac{\widetilde{B}^2(\beta)}{\sqrt{f(\beta)}}\,d\beta + O\left(\frac{1}{N^{3/2}}\right), \tag{2.16}$$

where $\widetilde{B}(\beta) = \frac{\operatorname{tr}J_0(\beta)}{\operatorname{tr}J_N} = \frac{B(\beta)}{\int B(\beta)\,d\beta}$ *is a normalized spectral function,* $B(\beta)$ *is defined in (2.4),* $f(\beta)$ *is a rational function of the averaged polynomials in* $m_{\beta j}$ *and their derivatives with respect to* β *(for details see proof in Section 3). In particular, if the regular twist is absent, then*

$$f(\beta) = \frac{8}{3}\sum_{k\in\{0,1,3\}}\left\langle\left(\frac{\partial m_{\beta j}(l,\Theta)}{\partial\beta}\right)^2\right\rangle + \frac{32}{3}\frac{\left\langle\frac{\partial m_{\beta 1}(l,\Theta)}{\partial\beta}m_{\beta 0}(l,\Theta) - m_{\beta 1}(l,\Theta)\frac{\partial m_{\beta 0}(l,\Theta)}{\partial\beta}\right\rangle^2}{\langle m_{\beta 3}^2(l,\Theta)\rangle}.$$

Observe, that beside the power series on N, the full asymptotic expansion for $\langle p_N^2 \rangle$ contains also a finite sum of terms of the form $a\mathrm{e}^{-\alpha N}$. (One can see it from the proof of this Proposition.) Unfortunately, we cannot explicitly estimate the values a and α in these terms using our approach, hence, we cannot estimate the accuracy of asymptotic (2.16) as function of N.

3 Proofs of basic assertions

3.1 Proof of Theorem 2.2

To prove heading 1), introduce matrix $S = \langle A \rangle$, the mean of the matrix family $\{A(z,q) \mid (z,q) \in \mathbb{R}_+ \times \operatorname{supp} \mu_c\}$, i.e.,

$$S = \int \rho_l(z) \int \exp\left(z\, Y(q)\right)\, d\mu_c(q)\, dz.$$

Since the families $\{l_j\}_{j=1}^{\infty}$ and $\{c_j\}_{j=1}^{\infty}$ are independent, we immediately see that

$$\langle U_N \rangle = S^N.$$

Obviously, $SP_0 = P_0$, therefore, $S^N P_0 = P_0$ and $\lim\limits_{N\to\infty} S^N P_0 = P_0$.

Let us show now that $\lim\limits_{N\to\infty} S^N (E - P_0) = 0$, wherefrom the statement required. To this end, consider an arbitrary nonzero vector $x \in \mathbb{C}^m$ such that $P_0 x = 0$. The unitarity of A implies that $P_0 S x = 0$, too. Indeed, if $Ax = x$, then $x = A^\dagger x$ and, therefore, $P_0 = S^\dagger P_0$. Conjugating both sides of this equality we see that $P_0 = P_0 S$. It is clear that absolute values of all eigenvalues of the matrix S do not exceed 1 since

$$\|S\| = \|\langle A(z,q) \rangle\| \leqslant \langle \|A(z,q)\| \rangle = 1.$$

Now we will need a lemma.

Lemma 3.1. *(i) If the absolute value of the eigenvalue η of S is equal to* 1*, then $\eta = 1$;*

(ii) the vector ε is an eigenvector with eigenvalue 1 *of S if and only if it is a common eigenvector of the matrix family $\{A(z,q) \mid (z,q) \in \mathbb{R}_+ \times \operatorname{supp} \mu_c\}$ with eigenvalue* 1*.*

Proof. Let $\eta \neq 1$ be the eigenvalue of the matrix S and ε_η be the corresponding normalized eigenvector. Then there exists $q_0 \in \operatorname{supp} \mu_c$ such that

$$\frac{d}{dz} \exp\left(z\, Y(q_0)\right) \varepsilon_\eta \neq 0, \tag{3.1}$$

because otherwise for any $q \in \operatorname{supp} \mu_c$ we would have had

$$0 = \left\| \frac{d}{dz} \exp\left(z\, Y(q)\right) \varepsilon_\eta \right\| = \left\| \exp\left(z\, Y(q)\right) Y(q) \varepsilon_\eta \right\| = \left\| Y(q) \varepsilon_\eta \right\|,$$

i.e., $Y(q)\varepsilon_\eta = 0$ and $\eta = 1$ in contradiction with assumption. By continuity of Y the inequality (3.1) holds in some neighborhood Ω_{q_0} of positive μ_c-measure of the point q_0.

There are two possibilities: either ε_η is a common eigenvector of the matrix family $\{Y(q) \mid q \in \operatorname{supp}\mu_c\}$ or there exists a q_0 such that the vector $\exp(z\,Y(q_0))\,\varepsilon_\eta$ is noncollinear to ε_η for all nonzero z.

Consider the first possibility. Since Y is skew-Hermitian or skew-orthogonal, all its eigenvalues are imaginary. Let $\mathrm{i}\chi(q)$, where $\chi(q) \not\equiv 0$, be the eigenvalue of $Y(q)$ corresponding to the eigenvector ε_η. From the condition $\chi(q_0) \neq 0$ for some q_0 and a continuity of Y it follows that we can find a neighborhood Ω_{q_0} so that the inequality $|\chi(q)| > \frac{1}{2}|\chi(q_0)|$ holds for any $q \in \Omega_{q_0}$. Let $\hat{\rho}_l(\omega)$ be a characteristic function of the distribution ρ_l, i.e.,

$$\hat{\rho}_l(\omega) = \int \rho_l(z)\,\mathrm{e}^{\mathrm{i}z\omega}\,dz.$$

It is known that $|\hat{\rho}_l(\omega)| \leqslant 1$ for all ω and $|\hat{\rho}_l(\omega)| < 1$ for $\omega \neq 0$ if ρ_l is piecewise continuous function. So, $|\hat{\rho}_l(\chi(q))| < 1-\delta$ for all $q \in \Omega_{q_0}$ and some positive δ. Then

$$\begin{aligned}|\eta| = |(S\varepsilon_\eta, \varepsilon_\eta)| &= \left|\int d\mu_c(q) \int \mathrm{e}^{\mathrm{i}z\chi(q)}\rho_l(z)\,dz\right| = \left|\int \hat{\rho}_l(\chi(q))\,d\mu_c(q)\right| \\ &\leqslant \int |\hat{\rho}_l(\chi(q))|\;d\mu_c(q) = \left(\int_{\Omega_{q_0}} + \int_{\mathbb{R}^n\setminus\Omega_{q_0}}\right)|\hat{\rho}_l(\chi(q))|\,d\mu_c(q) \leqslant 1-\delta\,\mu_c(\,\Omega_{q_0}) < 1,\end{aligned}$$

as was required.

The second possibility is considered in a similar way. Let $z_0 > 0$ be such that $\rho_l(z_0) \neq 0$. Then, by assumption, the vector $\exp(z_0\,Y(q_0))\,\varepsilon_\eta$ is noncollinear to ε_η and for some $\delta > 0$ we have $|(\exp(z_0\,Y(q_0))\,\varepsilon_\eta,\,\varepsilon_\eta)| = 1-2\delta$. It follows from the continuity of Y that $|(\exp(z\,Y(q))\,\varepsilon_\eta,\,\varepsilon_\eta)| = 1-\delta$ for all $(z,q) \in [z_0-\Delta,\,z_0+\Delta]\times\Omega_{q_0}$. Thus,

$$\begin{aligned}|\eta| = |(S\varepsilon_\eta, \varepsilon_\eta)| &= \left|\int d\mu_c(q)\int \rho_l(t)\,(\exp(z\,Y(q))\,\varepsilon_\eta,\,\varepsilon_\eta)\,dz\right| \\ &\leqslant 1-\delta\,\mu_c(\,\Omega_{q_0})\,\mu_l\,([z_0-\Delta,\,z_0+\Delta]) < 1.\end{aligned}$$

This completes the proof of Lemma 3.1. ■

These estimates and the theorem on the Jordan form of matrices imply that there exists an $\nu < 1$ such that the inequality

$$\|Sx\| \leqslant \nu\|x\|$$

holds for all x such that $P_0x = 0$. Thus, inequality $\|S^Nx\| \leqslant \nu^N\|x\|$ and S-invariance of the projection P_0 implies now that

$$\lim_{N\to\infty} S^N = P_0.$$

This completes the proof of the first heading of Theorem 2.2.

Let us prove heading 2). The independence of the random values $\{l_j\}_{j=1}^\infty$, $\{c_j\}_{j=1}^\infty$ implies that

$$\begin{aligned}\langle U(t)\rangle \equiv V(t) = \sum_{k=1}^\infty \int \prod_{j=1}^k d\mu_c(q_j) \int_{\sum_{j=1}^{k-1} l_j \leqslant z} \prod_{j=1}^{k-1} \rho_l(l_j)\,dl_j \\ \times \prod_{j=1}^{k-1} \mathrm{e}^{l_jY(q_j)}\mathrm{e}^{\left(z-\sum_{j=1}^{k-1} l_j\right)Y(q_k)}\Phi_l\left(z-\sum_{j=1}^{k-1} l_j\right),\end{aligned}$$

where

$$\Phi_l(z) = \int_z^\infty \rho_l(s)\,ds. \tag{3.2}$$

Introduce the matrix

$$L(z) = \int \mathrm{e}^{z\,Y(q)}\,d\mu_c(q),$$

and calculate the Laplace transform $\widehat{V}(\lambda)$ of the matrix-valued function $V(t)$,

$$\widehat{V}(\lambda) = \int_0^\infty V(z)\,\mathrm{e}^{-\lambda z}\,dz.$$

To do this, we use the convolution theorem for the Laplace transforms. Thus, setting

$$\widehat{L}_\rho(\lambda) = \int_0^\infty \rho_l(z)L(z)\,\mathrm{e}^{-\lambda z}\,dz, \qquad \widehat{L}_\Phi(\lambda) = \int_0^\infty \Phi_l(z)L(z)\,\mathrm{e}^{-\lambda z}\,dz,$$

we obtain the expression for $\widehat{V}(\lambda)$ of the form

$$\widehat{V}(\lambda) = \sum_{k=1}^\infty \Big(\widehat{L}_\rho(\lambda)\Big)^{k-1}\,\widehat{L}_\Phi(\lambda) = \Big(E - \widehat{L}_\rho(\lambda)\Big)^{-1}\,\widehat{L}_\Phi(\lambda).$$

Note that by the definition $\|\widehat{L}_\rho(\lambda)\| < 1$ for $\operatorname{Re}\lambda > 0$; hence, all the singularities of the matrix-valued function in last formula are only poles lying in left half-plane and only $\lambda = 0$ is a pole with the zero imaginary part. (Recall that as follows from our hypotheses about density ρ_l, the matrices $\widehat{L}_\rho(\lambda)$ and $\widehat{L}_\Phi(\lambda)$ are meromorphic in the whole plane and regular in the closed right half-plane.)

We now calculate the limit of $V(z)$ as $z \to \infty$ via the inverse Laplace transformation. For $a > 0$ we have

$$\lim_{z\to\infty} V(z) = \lim_{z\to\infty} \frac{1}{2\pi\mathrm{i}} \int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} \widehat{V}(\lambda)\,\mathrm{e}^{\lambda z}\,d\lambda = \operatorname*{Res}_{\lambda=0} \widehat{V}(\lambda).$$

To calculate the residue in this formula, observe that by hypotheses the matrix $L(z)$ has a block structure, $L(z) = P_0 \oplus L^\perp(z)$; hence, $\widehat{L}_\rho(\lambda) = \hat{\rho}_l(\lambda)P_0 \oplus \widehat{L}^\perp_\rho(\lambda)$, where $\hat{\rho}_l(\lambda)$ is defined by (2.13). Thus,

$$\widehat{V}(\lambda) = (1 - \hat{\rho}_l(\lambda))^{-1}\,P_0\widehat{L}_\Phi(\lambda) \oplus \widehat{V}^\perp(\lambda).$$

Only the first term in this expression has a pole at $\lambda = 0$, and the corresponding residue is

$$\operatorname*{Res}_{\lambda=0} \widehat{V}(\lambda) = \frac{P_0\widehat{L}_\Phi(0)}{-\dfrac{d\hat{\rho}_l(\lambda)}{d\lambda}\Big|_{\lambda=0}}.$$

It is evident that

$$-\frac{d\hat{\rho}_l(\lambda)}{d\lambda}\Big|_{\lambda=0} = \int_0^\infty z\rho_l(z)\,dz = \langle l\rangle,$$

and since $P_0L(z) = P_0$, we obtain

$$P_0\widehat{L}_\Phi(0) = P_0\int_0^\infty \Phi_l(z)\,dz = \langle l\rangle P_0.$$

Thus, $\operatorname*{Res}_{\lambda=0} \widehat{V}(\lambda) = P_0$ which completes the proof.

3.2 Proof of Corollary 2.1

The absolute value of vector $\vec{E}_N(\beta)$ does not depend on N due to (2.1). Thus, a distribution of the vector $\vec{E}_N(\beta)$ has support on the three-dimensional sphere of radius $|\vec{E}_0(\beta)|$. The limit distribution of the vector $\vec{E}_N(\beta) = U_N(\beta)\vec{E}_0(\beta)$ as $N \to \infty$ for an arbitrary initial vector $\vec{E}_0(\beta)$, is determined uniquely by the limit distribution of the random matrix $U_N(\beta)$.

The random matrices $M_\beta(l,\Theta)$ defined by expression (2.10) are elements of the group $SU(2)$. Let us show that the subgroup $\mathcal{M} \subset SU(2)$ generated by the matrix family $\{M_\beta(l,\Theta) \mid (l,\Theta) \in \mathbb{R}_+ \times \operatorname{supp} \mu_\Theta\}$ coincides with $SU(2)$. Then, by Theorem 2.1, the limit distribution of $U_N(\beta)$ defined by (2.12), is the Haar measure on $SU(2)$. The matrices M_β depend on two independent random parameters: l and Θ. Thus, $\dim(\mathcal{M}) \geqslant 2$. Since $\dim(SU(2)) = 3$, either $\dim(\mathcal{M}) = 2$ or $\dim(\mathcal{M}) = 3$. In the last case $\mathcal{M} = SU(2)$. It is known [36], that there is no two-dimensional subgroups in $SU(2)$. We are done.

Thus, the limit distribution of the random matrix $U_N(\beta)$ as $N \to \infty$ is the Haar measure on $SU(2)$; hence, the limit distribution of the vector $\vec{E}_N(\beta) = U_N(\beta)\vec{E}_0(\beta)$ is uniform on the three-dimensional sphere with radius $|\vec{E}_0(\beta)|$.

3.3 Proof of Corollary 2.2

We will show that the limit joint distribution of the matrices $U_N(\beta_1)$ and $U_N(\beta_2)$ is the Haar measure on the group $SU(2) \times SU(2)$. This means that these matrices and therefore the vectors $\vec{E}_N(\beta_1)$ and $\vec{E}_N(\beta_2)$ for $\beta_1 \neq \pm\beta_2$ became statistically independent as $N \to \infty$.

Let G be a subgroup of $SU(2) \times SU(2)$. The elements of G are pairs of matrices (g_1, g_2), $g_{1,2} \in SU(2)$. Denote by $\mathrm{Pr}_i : G \longrightarrow SU(2)$, $i = 1, 2$, the homomorphic projections, i.e., $\mathrm{Pr}_i\left((g_1, g_2)\right) = g_i$. The following statement is known from the theory of semisimple compact groups (see, for example, [36]).

Lemma 3.2. *Any automorphisms of the group $SU(2)$ is of the form $\mathrm{Aut}\,(g) = hf(g)h^\dagger$, where h is some fixed element from $SU(2)$ and $f(g)$ is equal to either g or the complex conjugate matrix $\overline{g}$.*

As a corollary we obtain

Lemma 3.3. *If the subgroup $G \subseteq SU(2) \times SU(2)$ is such that both Pr_1 and Pr_2 are epimorphisms, then either $G = SU(2) \times SU(2)$ or $G = \{(g, \mathrm{Aut}\,(g)) \mid g \in SU(2)\}$, where Aut is an automorphism of the group $SU(2)$.*

Proof. Since $\mathrm{Pr}_i(G) = SU(2)$, then $e \in \mathrm{Pr}_i(G)$ where e is the unit element of $SU(2)$. Denote: $G_e = \mathrm{Pr}_1^{-1}(e)$. Then G_e is a normal subgroup in $SU(2)$ as the kernel of the homomorphism Pr_1.

Set now ${G_e}' = \{g \in SU(2) \mid (e, g) \in G_e\} \subset SU(2)$. It is clearly that ${G_e}'$ is a subgroup; let us show that it is a normal one. Indeed, $ghg^{-1} \in SU(2)$ for any $h \in {G_e}' \subset SU(2)$ and given $g \in SU(2)$ because $SU(2)$ is a group. Since Pr_2 is a surjection, there exists $\tilde{g}$ such that $(\tilde{g},\ g) \in G$ and since G_e is normal, we see that

$$(\tilde{g},\ g)(e, h)(\tilde{g},\ g)^{-1} = \left(e, ghg^{-1}\right) \in G_e.$$

Hence, $ghg^{-1} \in {G_e}'$ for any $g \in SU(2)$, so ${G_e}'$ is a normal subgroup.

But $SU(2)$ is a simple group, so either ${G_e}' = SU(2)$ or ${G_e}' = e$.

Consider the first case. Let $g_1 \in SU(2)$. Then since Pr_2 is onto, there exists $g' \in SU(2)$ such that $(g_1, g') \in G$. Let now $g_2 \in SU(2)$; then $\bigl(e, g_2(g')^{-1}\bigr) \in G_e \subset G$, since $g_2(g')^{-1} \in SU(2) = {G_e}'$. Therefore

$$\bigl(e, g_2(g')^{-1}\bigr) \cdot (g_1, g') = (g_1, g_2) \in G.$$

This means that $G = SU(2) \times SU(2)$.

Now, consider the case when ${G_e}' = e$. In this case for each $g \in SU(2)$ there is only one $g' \in SU(2)$ such that $(g, g') \in G$. Indeed, let there exist $g'' \in SU(2)$ such that $g' \neq g''$ and $(g, g'') \in G$. Then $(g, g')^{-1} \cdot (g, g'') \in G$ since G is a group. On the other hand

$$(g, g')^{-1} \cdot (g, g'') = \bigl(g_1^{-1}, (g')^{-1}\bigr) \cdot (g_1, g'') = \bigl(e, (g')^{-1} g''\bigr) \in G_e.$$

Therefore, $(g')^{-1} g'' \in {G_e}' = e$ and $g' = g''$.

Hence, there exists a map $f : SU(2) \longrightarrow SU(2)$ such that $f(g) = g'$ if and only if $(g, g') \in G$. The map f is a homomorphism. Indeed, let $(g, g') \in G$ and $(h, h') \in G$. Then $(gh, g'h') \in G$ because of G is a group and by the definition of f we see that $f(gh) = g'h' = f(g)f(h)$.

Let us show that f is an injective homomorphism, i.e., if $g_1 \neq g_2$, then $f(g_1) \neq f(g_2)$. The kernel of f is a normal subgroup of $SU(2)$. This means that either $\ker f = e$ and in this case f is an injective homomorphism, or $\ker f = SU(2)$. In the last case the image of $SU(2)$ is the single element e which is impossible because $\mathrm{Pr}_2(G) = SU(2)$. Since Pr_2 is onto, f is a surjective homomorphism. So f is both an injective and surjective, hence, f is an automorphism and $G = \{(g, f(g)) \mid g \in SU(2)\}$. Proof of Lemma 3.3 is completed. ■

Let now $G \subseteq SU(2) \times SU(2)$ be the group generated by pairs of matrices $(M_{\beta_1}(l, \Theta)$, $M_{\beta_2}(l, \Theta))$. Thus, by Theorem 2.1, the limit distribution of the matrix pair $(U_N(\beta_1)$, $U_N(\beta_2))$ as $N \to \infty$ is the Haar measure on G. Let us show that in our case $G = SU(2) \times SU(2)$. Since random matrices $M_\beta(l, \Theta)$ generated the whole $SU(2)$, then G satisfies to the hypothesis of Lemma 3.3. Assume that G is different from $SU(2) \times SU(2)$. This means that there is an automorphism Ψ of the group $SU(2)$ such that $G = \{(g, \Psi(g)) \mid g \in SU(2)\}$. In this case the Haar measure on G is the image of the Haar measure on $SU(2)$ with respect to the natural isomorphism $g \to (g, \Psi(g))$ of groups $SU(2)$ and G. In accordance with Lemma 3.2, there exists a matrix $h \in SU(2)$ such that $g_2 = h f(g_1) h^\dagger$ for any pair $(g_1, g_2) \in G$.

Assume that $f(g) = g$. For a given complex matrix R we have

$$\Bigl\langle g_1 R g_2^\dagger \Bigr\rangle = \Bigl\langle g_1 R h g_1^\dagger h^\dagger \Bigr\rangle = \left(\int_{SU(2)} g R h g^\dagger \, dg \right) h^\dagger,$$

where $\langle \cdot \rangle$ denotes the averaging over Haar measure on $SU(2)$. Any element $g \in SU(2)$ may be uniquely represented in the form

$$g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \tag{3.3}$$

where a and b are complex numbers such that $|a|^2+|b|^2=1$. Introduce the angles φ, ψ and ϑ from the relations

$$a=\mathrm{e}^{\mathrm{i}\varphi}\cos\vartheta, \qquad b=\mathrm{e}^{\mathrm{i}\psi}\sin\vartheta, \qquad \vartheta\in[0,\pi/2], \qquad \varphi,\psi\in[0,2\pi];$$

and denote the elements of the matrix Rh by r_{mn}. Then

$$gRhg^{\dagger}=\begin{pmatrix} a\bar{a}r_{11}+a\bar{b}r_{12}+\bar{a}br_{21}+b\bar{b}r_{22} & -abr_{11}+a^2r_{12}-b^2r_{21}+abr_{22} \\ -\bar{a}\bar{b}r_{11}-\bar{b}^2r_{12}+\bar{a}^2r_{21}+\bar{a}\bar{b}r_{22} & b\bar{b}r_{11}-a\bar{b}r_{12}-\bar{a}br_{21}+a\bar{a}r_{22} \end{pmatrix}.$$

We need now to calculate

$$\left\langle a^{k_1}\bar{a}^{k_2}b^{k_3}\bar{b}^{k_4}\right\rangle=\left\langle(\cos\vartheta)^{k_1+k_2}(\sin\vartheta)^{k_3+k_4}\mathrm{e}^{\mathrm{i}(k_1-k_2)\varphi}\mathrm{e}^{\mathrm{i}(k_3-k_4)\psi}\right\rangle.$$

The invariant normalized measure on $SU(2)$ is $dg=\frac{1}{4\pi^2}\sin 2\vartheta\,d\vartheta\,d\varphi\,d\psi$, therefore,

$$\left\langle a^{k_1}\bar{a}^{k_2}b^{k_3}\bar{b}^{k_4}\right\rangle=\begin{cases} 0 & \text{if } k_1\neq k_2 \quad \text{or} \quad k_3\neq k_4, \\ \dfrac{k_1!\,k_3!}{(k_1+k_3+1)!} & \text{if } k_1=k_2 \quad \text{and} \quad k_3=k_4. \end{cases} \tag{3.4}$$

Using (3.4) we see that $\left\langle gRhg^{\dagger}\right\rangle=\frac{1}{2}\operatorname{tr}(Rh)E$, where E is the unit matrix. Therefore, for $R=h^{-1}$ we have

$$\left\langle g_1h^{-1}g_2^{\dagger}\right\rangle=h^{\dagger}. \tag{3.5}$$

Now we find the mean value of the product $g_1Rg_2^{\dagger}$ in a different way. In the space $\mathrm{Mat}(2;\mathbb{C})$ of complex 2×2-matrices, consider a linear operator V of the form

$$V(R)=M_{\beta_1}(l,\Theta)RM_{\beta_2}^{\dagger}(l,\Theta). \tag{3.6}$$

Then $\left\langle U_N(\beta_1)RU_N^{\dagger}(\beta_2)\right\rangle=\langle V\rangle^N(R)$ and $\left\langle g_1Rg_2^{\dagger}\right\rangle=\lim\limits_{N\to\infty}\langle V\rangle^N(R)$. It is easy to see that V preserves the inner product in the space of complex matrices given by the formula $(R_1,R_2)=\operatorname{tr}(R_1R_2^{\dagger})$. Therefore, V is a unitary operator. Consider the matrix-valued function $R(l)=M_{\beta_1}(l,\Theta)RM_{\beta_2}^{\dagger}(l,\Theta)$. Since $M_\beta(l,\Theta)=\exp(l\,X_\beta(\Theta))$, function $R(l)$ satisfies the differential equation

$$\frac{dR(l)}{dl}=X_{\beta_1}(\Theta)R(l)+R(l)X_{\beta_2}^{\dagger}(\Theta)=Y(\Theta)R(l)$$

with the initial condition $R(0)=R$. This means that operator (3.6) is of the form $V=\exp(l\,Y(\Theta))$.

The matrix $\langle V\rangle$ satisfies the hypothesis of Theorem 2.2, so the limit of the expression $\langle V\rangle^N(R)$ as $N\to\infty$ exists and is equal to $P_0(R)$, where P_0 is the projection on the common kernel of the operators $Y(\Theta)$, or equivalently, the common eigenspace of the operator family (3.6), corresponding to the eigenvalue 1. Let us show that the operator family (3.6) does not have the common eigenvalue 1 when $\beta_1\neq\pm\beta_2$. It is obvious that if a matrix R is an eigenvector of operator (3.6) with eigenvalue 1 for any pair $(l,\Theta)\in\mathbb{R}_+\times\operatorname{supp}\mu_\Theta$, then the identity

$$M_{\beta_1}(l,\Theta)RM_{\beta_2}^{\dagger}(l,\Theta)=R \tag{3.7}$$

holds. Differentiating both parts of (3.7) with respect to l and using the relation $M_\beta(l,\Theta) = \exp\left(l\,X_\beta(\Theta)\right)$, we see that

$$X_{\beta_1}(\Theta)R + RX^\dagger_{\beta_2}(\Theta) = 0.$$

The matrix $X_\beta(\Theta)$ defined by formula (2.1) can be expressed as

$$X_\beta(\Theta) = \frac{\mathrm{i}\beta}{2}\,\sigma_1 + \Theta\,\sigma_2,$$

where $\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then

$$\left(\frac{\mathrm{i}\beta_1}{2}\,\sigma_1 + \Theta\,\sigma_2\right) R - R\left(\frac{\mathrm{i}\beta_2}{2}\,\sigma_1 + \Theta\,\sigma_2\right) = 0. \tag{3.8}$$

Differentiating (3.8) by Θ we obtain

$$\sigma_2 R - R\sigma_2 = 0. \tag{3.9}$$

If we substitute (3.9) into (3.8) we see that

$$\beta_1\sigma_1 R - \beta_2 R\sigma_1 = 0. \tag{3.10}$$

It follows from relations (3.9) and (3.10) that the elements r_{ij} of the matrix R satisfy the following system of equations

$$\begin{cases} r_{11} - r_{22} = 0, \\ r_{12} + r_{21} = 0, \\ (\beta_1 - \beta_2)r_{11} = 0, \\ (\beta_1 + \beta_2)r_{12} = 0. \end{cases}$$

It is obvious that this system has a nontrivial solution only if $\beta_1 = \pm\beta_2$ and, therefore, 1 is not a common eigenvalue of the operator family (3.6) when $\beta_1 \neq \pm\beta_2$.

Thus, it follows from heading 1) of Theorem 2.2 that

$$\left\langle g_1 R g_2^\dagger \right\rangle = \lim_{N\to\infty} \left\langle U_N(\beta_1) R U_N^\dagger(\beta_2) \right\rangle = 0$$

for $\beta_1 \neq \pm\beta_2$ and any complex matrix R in contradiction with (3.5).

The case when the automorphism defining G contains complex conjugation can be considered in similar way.

So the joint distribution of matrices $U_\infty(\beta_1)$ and $U_\infty(\beta_2)$ is the Haar measure on $SU(2)\times SU(2)$, i.e., matrices $U_\infty(\beta_1)$ and $U_\infty(\beta_2)$ as well as vectors $\vec{E}_\infty(\beta_1)$, $\vec{E}_\infty(\beta_2)$ are independent for different $\beta_1 \neq \pm\beta_2$.

3.4 Proofs of Corollaries 2.3 and 2.4

Observe that, due to equation (2.1), *the total energy of radiation*, being equal to $\int B(\beta)\,d\beta$, where $B(\beta)$ is defined by (2.4), is preserved during propagation of the radiation in SMFLG, so we may assume that $\int B(\beta)\,d\beta = 1$.

Taking into account that $\vec{E}_N(\beta) = U_N(\beta)\vec{E}_0(\beta)$ we rewrite expression (2.5) for the coherence matrix at the output of the fiber that consists of N random sections as

$$J_N = \int U_N(\beta)\vec{E}_0(\beta)\vec{E}_0^\dagger(\beta)U_N^\dagger(\beta)\,d\beta. \tag{3.11}$$

Since the trace of the product of two matrices does not depend on the order of factors, and the matrix $U_N(\beta)$ is unitary, we deduce from (3.11) that

$$\operatorname{tr} J_N = \int \operatorname{tr}\left(\vec{E}_0^\dagger(\beta)U_N^\dagger(\beta)U_N(\beta)\vec{E}_0(\beta)\right)\,d\beta = \int B(\beta)\,d\beta = 1. \tag{3.12}$$

Thus, the trace of the coherence matrix at any point of the fiber is equal to 1, irrespectively of the concrete structure of inhomogeneities in the fiber.

To calculate the limit mean value of the coherence matrix, consider the bilinear form

$$(J_N\vec{s},\vec{r}\,) = \int \left(\vec{s},\vec{E}_N(\beta)\right)\left(\overline{\vec{r},\vec{E}_N(\beta)}\right)d\beta,$$

and find the limit mean value for this form for arbitrary vectors $\vec{s}$ and $\vec{r}$ from the space of two-dimensional complex vectors $\mathbb{C}^2$. Since the distribution of the vector $\vec{E}_N(\beta)$ on the three-dimensional sphere of radius $B(\beta)$ as $N\to\infty$ is uniform (Corollary 2.1), we have

$$\langle (J_\infty\vec{s},\vec{r}\,)\rangle = \int B(\beta)\,d\beta \int_{S^3} (\vec{s},\vec{\varsigma}\,)(\overline{\vec{r},\vec{\varsigma}}\,)\,d\vec{\varsigma} = f(\vec{s},\vec{r}\,),$$

where S^3 is the unit three-dimensional sphere, and f is a function of two vectors-arguments. It is easy to show that f is invariant with respect to an arbitrary rotation specified by the unitary matrix V, i.e., $f(V\vec{s},V\vec{r}\,) = f(\vec{s},\vec{r}\,)$. Additionally, this function is linear in the first argument and antilinear in the second one. We will show now that $f(\vec{s},\vec{r}\,) = \alpha(\vec{s},\vec{r}\,)$, where $\alpha = \text{const}$. Fixing the first argument of f and using linearity, we see that $f(\vec{s},\vec{r}\,) = (g(\vec{s}\,),\vec{r}\,)$, where $g(\cdot)$ is a linear function on a finite dimensional vector space. Hence, there exists a matrix G such that $g(\vec{s}\,) = G\vec{s}$. Thus, $f(\vec{s},\vec{r}\,) = (G\vec{s},\vec{r}\,)$. Using the invariance of f we have $(GV\vec{s},V\vec{r}\,) = (V^\dagger GV\vec{s},\vec{r}\,) = (G\vec{s},\vec{r}\,)$, i.e., $G = V^\dagger GV$ for any unitary matrix V. Due to the Schur lemma [36] all eigenvalues of the matrix G with this property coincide, i.e., $G = \alpha E$, where E is the unit matrix. Thus, $\langle (J_\infty\vec{s},\vec{r}\,)\rangle = \alpha(\vec{s},\vec{r}\,)$, where α is a constant. To find this constant, we calculate the limit mean value of the trace of the coherence matrix. Using the formula

$$\operatorname{tr} J_N = (J_N\vec{e}_1,\vec{e}_1) + (J_N\vec{e}_2,\vec{e}_2),$$

where $\{\vec{e}_1,\vec{e}_2\}$ is an orthonormal basis in $\mathbb{C}^2$, we obtain: $\langle \operatorname{tr} J_\infty\rangle = 2\alpha$. But, on the other hand, since (3.12) we deduce that $\operatorname{tr} J_\infty = \operatorname{tr} J_0 = 1$. Therefore, $\alpha = \frac{1}{2}$ and

$$\langle (J_\infty\vec{s},\vec{r}\,)\rangle = \frac{1}{2}(\vec{s},\vec{r}\,). \tag{3.13}$$

It follows from (3.13) that $\langle J_\infty\rangle = \frac{1}{2}\,E$. From the second part of Theorem 2.2 it follows also that $\lim\limits_{z\to\infty}\langle J(z)\rangle = \frac{1}{2}\,E$. This is exactly the assertion of Corollary 2.3.

Let us calculate now the limit mean value of the square of polarization degree. The square of the polarization degree at the output of the fiber that consists of N random sections is described by the formula

$$p_N^2 = 1 - \frac{4\det J_N}{\operatorname{tr}^2 J_N}. \tag{3.14}$$

Therefore, taking into account (3.12), to find the limit mean square of the degree of polarization using formula (3.14), we have to find the limit mean value of $\det J_N$. To this end, we introduce an orthonormal basis $\{\vec{e}_1,\,\vec{e}_2\}$ in $\mathbb{C}^2$. Then

$$\det J_N = (J_N\vec{e}_1,\vec{e}_1)(J_N\vec{e}_2,\vec{e}_2) - |(J_N\vec{e}_2,\vec{e}_1)|^2. \tag{3.15}$$

Clearly, to calculate $\langle\det J_N\rangle$, one should know how to find $\langle(J_N\vec{s},\vec{r}\,)(J_N\vec{u},\vec{v}\,)\rangle$ for arbitrary vectors $\vec{s}$, $\vec{r}$, $\vec{u}$ and $\vec{v}$. We have

$$(J_N\vec{s},\vec{r}\,)(J_N\vec{u},\vec{v}\,) = \iint \left(\vec{s},\vec{E}_N(\beta_1)\right)\left(\overline{\vec{r},\vec{E}_N(\beta_1)}\right)\left(\vec{u},\vec{E}_N(\beta_2)\right)\left(\overline{\vec{v},\vec{E}_N(\beta_2)}\right) d\beta_1\, d\beta_2.$$

Since the vectors $\vec{E}_N(\beta_1)$ and $\vec{E}_N(\beta_2)$ are independent as $N\to\infty$ (Corollary 2.2), and their limit distributions are uniform (Corollary 2.1), we obtain

$$\begin{aligned}\langle(J_\infty\vec{s},\vec{r}\,)(J_\infty\vec{u},\vec{v}\,)\rangle &= \iint B(\beta_1)B(\beta_2)\,d\beta_1\,d\beta_2 \int_{S^3}\int_{S^3}(\vec{s},\vec{\varsigma}\,)(\overline{\vec{r},\vec{\varsigma}}\,)(\vec{u},\vec{\tau}\,)(\overline{\vec{v},\vec{\tau}}\,)\,d\vec{\varsigma}\,d\vec{\tau}\\ &= \left(\int B(\beta)\,d\beta\right)^2 \int_{S^3}(\vec{s},\vec{\varsigma}\,)(\overline{\vec{r},\vec{\varsigma}}\,)\,d\vec{\varsigma}\int_{S^3}(\vec{u},\vec{\varsigma}\,)(\overline{\vec{v},\vec{\varsigma}}\,)\,d\vec{\varsigma} = \frac{1}{4}(\vec{s},\vec{r}\,)(\vec{u},\vec{v}\,).\end{aligned}$$

Applying this result to relation (3.15) and taking into account that vectors $\vec{e}_1$ and $\vec{e}_2$ are orthonormal, we see that $\langle\det J_\infty\rangle = \frac{1}{4}$. Thus, it follows from (3.14) that $\langle p_\infty^2\rangle = 0$, and since $\langle p_\infty\rangle \leqslant \sqrt{\langle p_\infty^2\rangle}$, we also have $\langle p_\infty\rangle = 0$.

From the second part of Theorem 2.2 it follows that $\lim\limits_{z\to\infty}\langle p^2(z)\rangle = 0$.

3.5 Proof of Proposition 2.1

We wish to estimate the dependence of the diagonal components of $J_N(\beta)$ on N in the case when the distribution of Θ is symmetric with respect to zero, i.e., $\langle\Theta^{2k+1}\rangle = 0$ for all integers k.

Clearly, the matrix $J_N(\beta) = \vec{E}_N(\beta)\vec{E}_N^{\,\dagger}(\beta)$ is Hermitian. According to our model, the vector $\vec{E}_k$ at the endpoint of the k-th fiber section results from the vector $\vec{E}_{k-1}$ at the endpoint of $(k-1)$-st fiber section by multiplying by the unitary matrix $M_\beta(l_k,\Theta_k)$ (see (2.10)). Hence,

$$J_k(\beta) = M_\beta(l_k,\Theta_k)J_{k-1}(\beta)M_\beta^\dagger(l_k,\Theta_k) \equiv \widehat{M}(\beta;\,l_k,\Theta_k)(J_{k-1}(\beta)). \tag{3.16}$$

Now, consider the introduced linear operator in the space $\mathcal{H}$ of Hermitian 2×2-matrices. It is easy to see that the operator family (3.16) preserves the usual inner product $(A,B) \equiv$

$\operatorname{tr}(AB^\dagger)$ in the space of complex matrices and, therefore, it is unitary. Observe also that operators from this family belong to the tensor product of the standard two-dimensional representation of $SU(2)$ in $\mathbb{C}^2$ and its Hermitean conjugate; hence, have a unique common eigenvector $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ with common eigenvalue 1. One can see also from (3.16) that *if the distribution of* Θ *is symmetric with respect to zero*, the operator $\langle \widehat{M}(\beta) \rangle$ has another, orthogonal to previous one, eigenvector $\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then the subspace of the diagonal matrices is invariant with respect to the operator $\langle \widehat{M}(\beta) \rangle$. Hence, we obtain the relation

$$\operatorname{diag}(\langle J_N(\beta) \rangle) = \frac{1}{2}\,(\operatorname{diag}(J_0(\beta)),\,\sigma_0)\,\sigma_0 + \frac{1}{2}\,\eta_1^N\,(\operatorname{diag}(J_0(\beta)),\,\sigma_1)\,\sigma_1, \tag{3.17}$$

where η_1 is the eigenvalue of the operator $\langle \widehat{M}(\beta) \rangle$ corresponding to the eigenvector σ_1. Clearly,

$$\eta_1 = \frac{1}{2}\left(\langle \widehat{M}(\beta) \rangle \sigma_1, \sigma_1\right) = \frac{1}{2}\left\langle \operatorname{tr} M_\beta(l,\Theta)\sigma_1 M_\beta^\dagger(l,\Theta)\sigma_1 \right\rangle = 1 - 2\left\langle m_{\beta 3}^2(l,\Theta) \right\rangle.$$

It follows now from (3.17) that

$$\begin{aligned}
\langle J_{11}(\beta,N) \rangle &= \frac{\operatorname{tr} J_0}{2}\left(1 + \frac{J_{11}^0 - J_{22}^0}{\operatorname{tr} J_0}\,\eta_1^N\right),\\
\langle J_{22}(\beta,N) \rangle &= \frac{\operatorname{tr} J_0}{2}\left(1 - \frac{J_{11}^0 - J_{22}^0}{\operatorname{tr} J_0}\,\eta_1^N\right).
\end{aligned} \tag{3.18}$$

If $\eta_1 > 0$, this relation can be re-written in the form similar to (2.6):

$$\begin{aligned}
\langle J_{11}(\beta,z) \rangle &= \frac{\operatorname{tr} J_0}{2}\left(1 + \frac{J_{11}^0 - J_{22}^0}{\operatorname{tr} J_0}\,\mathrm{e}^{-2hz}\right),\\
\langle J_{22}(\beta,z) \rangle &= \frac{\operatorname{tr} J_0}{2}\left(1 - \frac{J_{11}^0 - J_{22}^0}{\operatorname{tr} J_0}\,\mathrm{e}^{-2hz}\right),
\end{aligned} \tag{3.19}$$

where $z = N\langle l \rangle$ and $h = -\frac{1}{2\langle l \rangle}\ln \eta_1$.

3.6 Proof of Proposition 2.2

As in the proof of Theorem 2.2, to calculate the value of the h-parameter, we use the Laplace transformation. Let $N(z)$ be defined in (2.8), $z_k = \sum\limits_{j=1}^{k} l_j$ and $\Phi_l(z)$ be defined in (3.2). We deduce from (2.7), due to independence of random values l_k, Θ_k, that

$$\begin{aligned}
h(\beta) &= \lim_{z\to\infty} \frac{1}{z}\left\langle \sum_{k=0}^{N(z)-1} \int_{z_k}^{z_{k+1}}\int_{z_k}^{z_{k+1}} \Theta_{k+1}^2 \mathrm{e}^{\mathrm{i}\beta(t-s)}\,dt\,ds + \int_{z_{N(z)}}^{z}\int_{z_{N(z)}}^{z} \Theta_{N(z)+1}^2 \mathrm{e}^{\mathrm{i}\beta(t-s)}\,dt\,ds \right\rangle\\
&= \frac{4\langle \Theta^2 \rangle}{\beta^2} \lim_{z\to\infty} \frac{1}{z} \sum_{k=0}^{\infty} \int_{z_k \leqslant z} \prod_{j=1}^{k} \rho_l(l_j)\,dl_j\,\Phi_l(z - z_k)\left(\sum_{j=1}^{k} \sin^2 \frac{l_j\beta}{2} + \sin^2 \frac{(z-z_k)\beta}{2}\right).
\end{aligned}$$

Using now the convolution theorem for the Laplace transformation, we can express the last expression as

$$\begin{aligned} h(\beta) &= \frac{4\langle\Theta^2\rangle}{\beta^2} \lim_{z\to\infty} \frac{1}{2\pi\mathrm{i}\, z} \int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} \left(\frac{\widehat{\Psi}_l(\lambda)}{1-\hat{\rho}_l(\lambda)} + \frac{\widehat{\Phi}_l(\lambda)\hat{\psi}_l(\lambda)}{(1-\hat{\rho}_l(\lambda))^2} \right) d\lambda \\ &= \frac{4\langle\Theta^2\rangle}{\beta^2} \lim_{z\to\infty} \frac{1}{2\pi\mathrm{i}} \int_{a-\mathrm{i}\infty}^{a-\mathrm{i}\infty} \lambda \left(\frac{\widehat{\Psi}_l(\lambda)}{1-\hat{\rho}_l(\lambda)} + \frac{\widehat{\Phi}_l(\lambda)\hat{\psi}_l(\lambda)}{(1-\hat{\rho}_l(\lambda))^2} \right) \mathrm{e}^{\lambda z}\, d\lambda \\ &= \frac{4\langle\Theta^2\rangle}{\beta^2} \operatorname*{Res}_{\lambda=0} \left(\frac{\lambda\widehat{\Psi}_l(\lambda)}{1-\hat{\rho}_l(\lambda)} + \frac{\lambda\widehat{\Phi}_l(\lambda)\hat{\psi}_l(\lambda)}{(1-\hat{\rho}_l(\lambda))^2} \right), \end{aligned}$$

where $\hat{\rho}_l(\lambda)$ is defined in (2.13) and

$$\begin{aligned} \widehat{\Phi}_l(\lambda) &= \int_0^\infty \Phi_l(z)\mathrm{e}^{-\lambda z}\, dz = \frac{1}{\lambda}(1-\hat{\rho}_l(\lambda)), \\ \widehat{\Psi}_l(\lambda) &= \int_0^\infty \Phi_l(z) \sin^2\frac{z\beta}{2}\, \mathrm{e}^{-\lambda z}\, dz, \\ \hat{\psi}_l(\lambda) &= \int_0^\infty \rho_l(z) \sin^2\frac{z\beta}{2}\, \mathrm{e}^{-\lambda z}\, dz. \end{aligned}$$

We see that $\widehat{\Psi}_l(\lambda)$, $\hat{\psi}_l(\lambda)$ and $\frac{\lambda}{1-\hat{\rho}_l(\lambda)}$ have finite limits as $\lambda \to 0$. Hence,

$$\operatorname*{Res}_{\lambda=0} \left(\frac{\lambda\widehat{\Psi}_l(\lambda)}{1-\hat{\rho}_l(\lambda)} + \frac{\lambda\widehat{\Phi}_l(\lambda)\hat{\psi}_l(\lambda)}{(1-\hat{\rho}_l(\lambda))^2} \right) = \frac{\hat{\psi}_l(0)}{-\left.\dfrac{d\hat{\rho}_l(\lambda)}{d\lambda}\right|_{\lambda=0}} = \frac{1}{\langle l\rangle} \left\langle \sin^2\frac{l\beta}{2} \right\rangle.$$

Thus, we obtain relation (2.15).

3.7 Proof of Proposition 2.3

In this section we consider an asymptotic expansion of the mean square of the polarization degree at large N. We can only find the principal term of this asymptotic.

Note that simultaneously with the calculation of the asymptotic expansion of the mean square of the polarization degree we obtain another proof of the relation $\langle p_\infty^2\rangle = 0$.

Our problem reduces to the asymptotic estimate of the mean value of the determinant of the coherence matrix. To do this, it suffices to calculate the mean value the tensor product $J_N \otimes J_N$, because the determinant of 2×2-matrix is a *linear* function of the elements of its tensor square. Further, since J_N is an integral over the parameter β of matrices $J_N(\beta)$, the mentioned tensor product is the integral over the pair (β_1, β_2) of the tensor product $J_N(\beta_1) \otimes J_N(\beta_2)$. It follows from (3.16) that there exists a linear operator $\widehat{M}(\beta_1, \beta_2;\, l, \Theta)$ in $\mathcal{H}^2 \equiv \mathcal{H} \otimes \mathcal{H}$ such that

$$J_N(\beta_1) \otimes J_N(\beta_2) = \widehat{M}(\beta_1, \beta_2;\, l_N, \Theta_N)\,(J_{N-1}(\beta_1) \otimes J_{N-1}(\beta_2))\,. \tag{3.20}$$

Note that the operator family $\widehat{M}(\beta_1, \beta_2;\, l, \Theta)$ is unitary because it preserves the inner product $(A_1 \otimes B_1, A_2 \otimes B_2) \equiv (A_1, A_2)(B_1, B_2)$ in $\mathcal{H}^2$.

Hereafter we denote by $\{\sigma_j,\ j\in\{0,1,2,3\}\}$ the usual Pauli basis in the space $\mathcal{H}$ of Hermitian 2×2-matrices:

$$\sigma_0=\begin{pmatrix}1&0\\0&1\end{pmatrix},\qquad \sigma_1=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\qquad \sigma_2=\begin{pmatrix}0&1\\1&0\end{pmatrix},\qquad \sigma_3=\begin{pmatrix}0&i\\-i&0\end{pmatrix}.$$

We also denote by $s_j=\frac{\sigma_j}{\sqrt{2}}$ the corresponding orthonormal basis. Remember that the inner product in $\mathcal{H}$ is defined as $(A,B)=\operatorname{tr}AB$. For the set $\{x_\alpha\}$ of vectors in a given vector space we denote by $\mathcal{L}\left(\{x_\alpha\}\right)$ its linear envelope.

Let us now list some facts from the representation theory of the group $SU(2)$.

F1) Let π be a representation of the group $SU(2)$ in the space $\mathcal{H}$ defined by the relation $\pi(g)A=gAg^\dagger$, $A\in\mathcal{H}$. Clearly, this representation is reducible and its decomposition into irreducible components is of the form $\mathcal{H}=H_0\oplus H_3$, where $H_0=\mathcal{L}(\sigma_0)$ and $H_3=\mathcal{L}\left(\{\sigma_1,\sigma_2,\sigma_3\}\right)$.

F2) Let $\pi^{\otimes 2}$ be a representation of $SU(2)\times SU(2)$ in $\mathcal{H}^2$ such that

$$\pi^{\otimes 2}(g_1,g_2)(A\otimes B)=\pi(g_1)A\otimes\pi(g_2)B.$$

Then from the decomposition above it follows that

$$\mathcal{H}^2=H_0\otimes H_0\oplus H_0\otimes H_3\oplus H_3\otimes H_0\oplus H_3\otimes H_3\equiv H^2_{00}\oplus H^2_{03}\oplus H^2_{30}\oplus H^2_{33}$$

and this is the decomposition of $\pi^{\otimes 2}$ into irreducible components.

F3) Denote by π^2 the representation of $SU(2)$ in $\mathcal{H}^2$, obtained by restricting $\pi^{\otimes 2}$ onto the image of the diagonal embedding of $SU(2)$ into $SU(2)\times SU(2)$. Then the components H^2_{00}, H^2_{03} and H^2_{30} are still irreducible, but the component H^2_{33} can be further decomposed. Indeed, consider the natural action of the group $\mathbb{Z}_2=\{\bar{0},\bar{1}\}$ on $\mathcal{H}^2$ by twist: $\bar{1}(A\otimes B)=B\otimes A$. Then $H^2_{33}=H^2_{33s}\oplus H^2_{33a}$ is the direct sum of the spaces of symmetric and skew-symmetric tensors under this action. The space H^2_{33s} can be further decomposed into the direct sum

$$H^2_{33s}=\mathcal{L}\left(\left\{\sum_{i=1}^{3}\sigma_i\otimes\sigma_i\right\}\right)\oplus\mathcal{L}\left(\left\{\sum_{i=1}^{3}\sigma_i\otimes\sigma_i\right\}\right)^{\perp}\equiv H^2_1\oplus H^2_5.$$

These are all irreducible components of π^2.

Thus, the complete decomposition of π^2 in $\mathcal{H}^2$ consists of *two* one-dimensional components, spanned by vectors $\varepsilon_0=s_0\otimes s_0$ and $\varepsilon_1=\sum\limits_{i=1}^{3}s_i\otimes s_i$, *three* three-dimensional components in the spaces H^2_{03}, H^2_{30} and H^2_{33a} and *one* five-dimensional component in H^2_5.

We will also need some relations between vectors and operators in $\mathcal{H}^2$.

R1) Let $T=\varepsilon_0+\varepsilon_1$. Then for $A,B\in\mathcal{H}$, one has $(A\otimes B,T)=\operatorname{tr}AB$.

R2) $\Delta=\frac{1}{2}\left(\varepsilon_0-\varepsilon_1\right)$. Then for $A\in\mathcal{H}$, one has $(A\otimes A,\Delta)=\det A$.

R3) Let $g:\mathbb{R}\longrightarrow SU(2)$ be a smooth function, and $g_2:\mathbb{R}^2\longrightarrow SU(2)\times SU(2)$ be such that $g_2(x,y)=(g(x),g(y))$. Let further D be a differential operator in $C^\infty(\mathbb{R}^2)$ of the form $D=\frac{\partial}{\partial x}-\frac{\partial}{\partial y}$. (Here we denote by $C^\infty(\mathbb{R}^2)$ the space of smooth functions with values in an arbitrary finite dimensional vector space.) The operator $D_0:C^\infty(\mathbb{R}^2)\longrightarrow C^\infty(\mathbb{R})$ is then $D|_{y=x}$, and similarly $D_0^2 = D^2|_{y=x}$. The following relations can be proven by direct, but long, calculations:

$$D_0^2\left(\pi^{\otimes 2}(g_2(x,y))\varepsilon_1,\, \varepsilon_1\right) = -4\operatorname{tr}\frac{dg^\dagger}{dx}\,\frac{dg}{dx}, \tag{3.21}$$

$$D_0\pi^{\otimes 2}(g_2(x,y))\varepsilon_1 \in H^2_{33a}. \tag{3.22}$$

Denote by $\widehat{L}(\beta_1,\beta_2)$ the mean operator $\langle\,\widehat{M}(\beta_1,\beta_2;\, l,\Theta)\rangle$ and by $F_N(\beta_1,\beta_2)$ the mean value of $J_N(\beta_1)\otimes J_N(\beta_2)$. Then it follows from (3.20) that

$$F_N(\beta_1,\beta_2) = \widehat{L}^N(\beta_1,\beta_2)F_0(\beta_1,\beta_2), \tag{3.23}$$

and we can use, as in the previous section, the spectral decomposition of the operator $\widehat{L}(\beta_1,\beta_2)$ to calculate $F_N(\beta_1,\beta_2)$.

Note that we deal here with operators in the 16-dimensional space $\mathcal{H}^2$ and there are no reasonable reasons for existence of an analytical solution of spectral problem for operator $\widehat{L}(\beta_1,\beta_2)$. This is the main obstacle for obtaining complete asymptotic decomposition of the polarization degree. We will see, nevertheless, that it is possible to obtain some analytical expressions for two major eigenvalues and corresponding eigenvectors of this operator which suffices to construct the leading term of the asymptotic of the polarization degree.

Using definitions above, we deduce:

$$\begin{aligned}\langle\det J_N\rangle &= \iint\left(F_N(\beta_1,\beta_2),\Delta\right)\,d\beta_1 d\beta_2\\ &= \iint\left(\widehat{L}^N(\beta_1,\beta_2)F_0(\beta_1,\beta_2),\Delta\right)\,d\beta_1 d\beta_2.\end{aligned} \tag{3.24}$$

The integrand in (3.24) can be written in the form

$$\begin{aligned}\left(\widehat{L}^N(\beta_1,\beta_2)F_0(\beta_1,\beta_2),\Delta\right) &= \frac{1}{2}\left(F_0(\beta_1,\beta_2),\left(\widehat{L}^\dagger(\beta_1,\beta_2)\right)^N(\varepsilon_0-\varepsilon_1)\right)\\ &= \frac{1}{2}\left(F_0(\beta_1,\beta_2),\varepsilon_0\right) - \frac{1}{2}\left(F_0(\beta_1,\beta_2),\left(\widehat{L}^\dagger(\beta_1,\beta_2)\right)^N\varepsilon_1\right).\end{aligned} \tag{3.25}$$

As follows from the definition of vector ε_0,

$$\left(F_0(\beta_1,\beta_2),\varepsilon_0\right) = \frac{1}{2}\operatorname{tr}J_0(\beta_1)\operatorname{tr}J_0(\beta_2) = \frac{1}{2}B(\beta_1)B(\beta_2). \tag{3.26}$$

Let $\{\varepsilon_i^\dagger(\beta_1,\beta_2),\, \eta_i(\beta_1,\beta_2)\}_{i=0}^{15}$ be the set of normalized eigenvectors and corresponding eigenvalues of $\widehat{L}^\dagger(\beta_1,\beta_2)$. Then the dual basis is the set $\{\varepsilon_i(\beta_1,\beta_2)\}_{i=0}^{15}$ of eigenvectors of $\widehat{L}(\beta_1,\beta_2)$. Moreover, it follows from previous discussions that the vector $\varepsilon_0^\dagger(\beta_1,\beta_2) = \varepsilon_0(\beta_1,\beta_2) = \varepsilon_0$ is orthogonal to all other eigenvectors ε_i, $\eta_0(\beta_1,\beta_2) = 1$, $\varepsilon_1^\dagger(\beta,\beta) = \varepsilon_1(\beta,\beta) = \frac{1}{\sqrt{3}}\varepsilon_1$ and $\eta_1(\beta,\beta) = 1$. Further, it follows from Theorem 2.2 that $|\eta_i(\beta_1,\beta_2)| \leqslant \nu < 1$ for $i = 2,\ldots,15$. Expanding ε_1 in terms of the eigenvectors $\varepsilon_i^\dagger$ we obtain

$$\begin{aligned}\left(\widehat{L}^\dagger(\beta_1,\beta_2)\right)^N\varepsilon_1 &= \sum_{i=0}^{15}\left(\varepsilon_1,\varepsilon_i(\beta_1,\beta_2)\right)\left(\widehat{L}^\dagger(\beta_1,\beta_2)\right)^N\varepsilon_i^\dagger(\beta_1,\beta_2)\\ &= \sum_{i=1}^{15}\eta_i^N(\beta_1,\beta_2)\left(\varepsilon_1,\varepsilon_i(\beta_1,\beta_2)\right)\,\varepsilon_i^\dagger(\beta_1,\beta_2).\end{aligned}$$

It follows from the properties of eigenvalues η_i that the relation

$$\left(\widehat{L}^\dagger(\beta_1,\beta_2)\right)^N \varepsilon_1 = \mathrm{e}^{-Nh(\beta_1,\beta_2)}\left(\varepsilon_1, \varepsilon_1(\beta_1,\beta_2)\right)\, \varepsilon_1^\dagger(\beta_1,\beta_2) + O\left(\mathrm{e}^{-\alpha N}\right), \tag{3.27}$$

holds for $\alpha = -\ln\nu > 0$ and $h(\beta_1,\beta_2) = -\ln\eta_1(\beta_1,\beta_2)$.

Substituting (3.27) into (3.25) and using (3.26) we obtain

$$\begin{aligned}\left(\widehat{L}^N(\beta_1,\beta_2)F_0(\beta_1,\beta_2),\Delta\right) &= \frac{1}{4}\,B(\beta_1)B(\beta_2)\\ &- \frac{1}{2}\,\mathrm{e}^{-Nh(\beta_1,\beta_2)}\left(\varepsilon_1,\varepsilon_1(\beta_1,\beta_2)\right)\left(F_0(\beta_1,\beta_2),\varepsilon_1^\dagger(\beta_1,\beta_2)\right) + O\left(\mathrm{e}^{-\alpha N}\right).\end{aligned} \tag{3.28}$$

Now by substituting (3.28) into (3.24), changing variables

$$\beta = \frac{1}{2}(\beta_1+\beta_2), \qquad \delta = \beta_1 - \beta_2,$$

and integrating over δ by the saddle-point method, we obtain

$$\langle \det J_N\rangle = \frac{1}{4}\left(\int B(\beta)\,d\beta\right)^2 - \frac{1}{6}\int \left(F_0(\beta,\beta),\varepsilon_1\right)\sqrt{\frac{2\pi}{Nf(\beta)}}\,d\beta + O\left(\frac{1}{N^{3/2}}\right), \tag{3.29}$$

where $f(\beta) = D_0^2 h(\beta,\beta)$. Here we use the relation

$$D_0 h(\beta,\beta) = -D_0\eta_1(\beta,\beta) = -\frac{1}{3}\,D_0\left(\widehat{L}^\dagger(\beta,\beta)\varepsilon_1,\,\varepsilon_1\right) = 0, \tag{3.30}$$

which follows from the standard perturbation theory [37] and the fact that

$$\left(\widehat{L}^\dagger(\beta_1,\beta_2)\varepsilon_1,\,\varepsilon_1\right) = \left(\widehat{L}^\dagger(\beta_2,\beta_1)\varepsilon_1,\,\varepsilon_1\right).$$

(To prove the last relation it suffices to note that, by definition,

$$\begin{aligned}&\left(\widehat{L}^\dagger(\beta_1,\beta_2)(A\otimes A),\,A\otimes A\right)\\ &\quad= \left\langle\left(M_{\beta_1}(l,\Theta)AM_{\beta_1}^\dagger(l,\Theta)\otimes M_{\beta_2}(l,\Theta)AM_{\beta_2}^\dagger(l,\Theta),\,A\otimes A\right)\right\rangle\\ &\quad= \left\langle \mathrm{tr}\,\left(M_{\beta_1}(l,\Theta)AM_{\beta_1}^\dagger(l,\Theta)A\right)\mathrm{tr}\,\left(M_{\beta_2}(l,\Theta)A\widehat{M}_{\beta_2}^\dagger(l,\Theta)A\right)\right\rangle\end{aligned}$$

for any Hermitian 2×2-matrix A.) Taking into account the relations $\int B(\beta)\,d\beta = \mathrm{tr}\,J_N$ and $\varepsilon_1 = \varepsilon_0 - 2\Delta$, we can express (3.29) in the form

$$\frac{1}{4}\,\mathrm{tr}^2 J_N - \langle\det J_N\rangle = \frac{1}{3}\int\left(\frac{1}{4}\,\mathrm{tr}^2 J_0(\beta) - \det J_0(\beta)\right)\sqrt{\frac{2\pi}{Nf(\beta)}}\,d\beta + O\left(\frac{1}{N^{3/2}}\right).$$

Using now the formula (3.14) and a normalized spectral function

$$\widetilde{B}(\beta) = \frac{\mathrm{tr}\,J_0(\beta)}{\mathrm{tr}\,J_N} = \frac{B(\beta)}{\int B(\beta)\,d\beta},$$

we obtain expression (2.16) for the polarization degree.

Our last problem is to calculate the value $f(\beta)$ in terms of parameters of $\Theta(z)$. To do this, observe that

$$D_0^2\eta_1(\beta,\beta) = D_0^2 \mathrm{e}^{-h(\beta,\beta)} = -D_0^2 h(\beta,\beta),$$

because $D_0 h(\beta,\beta) = 0$. Let now apply D_0^2 to the relation

$$\widehat{L}^\dagger(\beta_1,\beta_2)\varepsilon_1(\beta_1,\beta_2) = \eta_1(\beta_1,\beta_2)\varepsilon_1(\beta_1,\beta_2) \tag{3.31}$$

and then scalar it by ε_1. We obtain:

$$\begin{aligned} f(\beta) &= D_0^2 h(\beta,\beta) \\ &= -\frac{1}{3}\left(D_0^2\widehat{L}^\dagger(\beta,\beta)\varepsilon_1,\, \varepsilon_1\right) - \frac{2}{\sqrt{3}}\left(D_0\widehat{L}^\dagger(\beta,\beta)D_0\varepsilon_1(\beta,\beta),\, \varepsilon_1\right). \end{aligned} \tag{3.32}$$

The first term in the right hand side of (3.32) is calculated by using relation (3.21),

$$\begin{aligned} \left(D_0^2\widehat{L}^\dagger(\beta,\beta)\varepsilon_1,\, \varepsilon_1\right) &= \left\langle D_0^2\left(\pi^{\otimes 2}(M_\beta(l,\Theta), M_\beta(l,\Theta))\varepsilon_1,\, \varepsilon_1\right)\right\rangle \\ &= -4\left\langle \operatorname{tr}\frac{M_\beta^\dagger(l,\Theta)}{\partial\beta}\,\frac{M_\beta(l,\Theta)}{\partial\beta}\right\rangle. \end{aligned} \tag{3.33}$$

To calculate the second term, we need to find the vector $D_0\varepsilon_1(\beta,\beta)$. Applying D_0 to equation (3.31) and taking into account relation (3.30), we obtain for this vector the equation

$$\left(E - \widehat{L}^\dagger(\beta,\beta)\right) D_0\varepsilon_1(\beta,\beta) = \frac{1}{\sqrt{3}}\, D_0\widehat{L}^\dagger(\beta,\beta)\varepsilon_1. \tag{3.34}$$

Due to relation (3.22), the right hand side of equation (3.34) belongs to the space H^2_{33a}, which is invariant, as follows from F3), under the operator in the left hand side. The space H^2_{33a} is, by definition, orthogonal to the spaces H^2_{00} and H^2_1 (see F2) and F3)). The direct sum of these spaces is exactly the kernel of the operator $E - \widehat{L}^\dagger(\beta,\beta)$, hence, the equation (3.34) has a unique solution which is orthogonal to $H^2_{00} \oplus H^2_1$, and this solution belongs to H^2_{33a}. Since vector $\varepsilon_1(\beta_1,\beta_2)$ is normalized and orthogonal to ε_0, this solution is the one we need. So, we can rewrite equation (3.34) as an equation in the three-dimensional space H^2_{33a}. It is easy to see that the orthonormalized basis in H^2_{33a} can be chosen in the form

$$\phi_1 = \frac{s_1\otimes s_2 - s_2\otimes s_1}{\sqrt{2}}, \qquad \phi_2 = \frac{s_2\otimes s_3 - s_3\otimes s_2}{\sqrt{2}}, \qquad \phi_3 = \frac{s_3\otimes s_1 - s_1\otimes s_3}{\sqrt{2}}.$$

Introduce the 3×3-matrix $S_{ij} = \left(\left(E - \widehat{L}^\dagger(\beta,\beta)\right)\phi_j, \phi_i\right)$, $i,j \in \{1,2,3\}$, and vectors $v_i = (D_0\varepsilon_1(\beta,\beta), \phi_i)$ and $u_i = \left(D_0\widehat{L}^\dagger(\beta,\beta)\varepsilon_1, \phi_i\right)$, $i \in \{1,2,3\}$. Then equation (3.34) takes the form

$$Sv = \frac{1}{\sqrt{3}}\, u. \tag{3.35}$$

By direct calculation we see that

$$S = 2\begin{pmatrix} m_{11} & m_{13} & m_{01} \\ m_{13} & m_{33} & m_{03} \\ -m_{01} & -m_{03} & 1 - m_{00} \end{pmatrix}, \qquad u = 4\sqrt{2}\begin{pmatrix} d_{30} \\ -d_{10} \\ -d_{31} \end{pmatrix}.$$

Here

$$\begin{aligned} m_{kj} &= \langle m_{\beta k}(l,\Theta) m_{\beta j}(l,\Theta)\rangle, \\ d_{kj} &= \left\langle \frac{\partial m_{\beta k}(l,\Theta)}{\partial\beta} m_{\beta j}(l,\Theta) - m_{\beta k}(l,\Theta)\frac{\partial m_{\beta j}(l,\Theta)}{\partial\beta}\right\rangle, \\ m_{\beta k}(l,\Theta) &= \frac{\mathrm{i}^k}{2}\operatorname{tr} M_\beta(l,\Theta)\sigma_k, \qquad k,j \in \{0,1,3\}. \end{aligned}$$

We can now express the function $f(\beta)$ in terms of components of vector v and elements of matrix $M_\beta(l,\Theta)$ and its derivatives with respect to β. Using (3.32), (3.33) we finally obtain

$$f(\beta) = \frac{8}{3}\sum_{k\in\{0,1,3\}}\left\langle\left(\frac{\partial m_{\beta j}(l,\Theta)}{\partial\beta}\right)^2\right\rangle - 8\sqrt{\frac{2}{3}}\,(-v_1 d_{30} + v_2 d_{10} - v_3 d_{31})$$

This relation is essentially simplified when the regular twist is absent: u and many elements of S vanish and the solution of equation (3.35) attains a simple explicit form:

$$v = \begin{pmatrix} 0 \\ -2\sqrt{\dfrac{2}{3}}\,\dfrac{d_{10}}{m_{33}} \\ 0 \end{pmatrix}.$$

So, in this case

$$\begin{aligned} f(\beta) = \frac{8}{3}&\left\langle \frac{l^2\beta^2}{4(\beta^2+4\Theta^2)} + \frac{4\Theta^2\sin^2\dfrac{l}{2}\sqrt{\beta^2+4\Theta^2}}{(\beta^2+4\Theta^2)^2}\right\rangle \\ &+\frac{4}{3}\frac{\left\langle\dfrac{l\beta^2}{\beta^2+4\Theta^2} + \dfrac{4\Theta^2\sin l\sqrt{\beta^2+4\Theta^2}}{(\beta^2+4\Theta^2)^{3/2}}\right\rangle^2}{\left\langle\dfrac{\Theta^2}{\beta^2+4\Theta^2}\sin^2\dfrac{l}{2}\sqrt{\beta^2+4\Theta^2}\right\rangle}. \end{aligned}$$

This completes the proof of Proposition 2.3.

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