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Periodic Motions Galore: How to Modify Nonlinear Evolution Equations so that They Feature a Lot of Periodic Solutions

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Abstract

A simple trick is illustrated, whereby nonlinear evolution equations can be modified so that they feature a lot – or, in some cases, only – *periodic* solutions. Several examples (ODEs and PDEs) are exhibited.

1 Introduction and outline of results

Many natural phenomena (in physics, chemistry, biology, physiology, ...), and also several phenomena pertaining to the social sciences (for instance in economics), display a cyclic behavior. Hence, to the extent they can be described by mathematical models, they call into play evolution equations that feature *periodic* solutions. Practitioners of such mathematical modeling ("applied mathematics") are therefore likely to profit from a simple trick, whereby evolution equations can be modified in a rather neat way, so that they feature a lot of – or, in some cases, only – *periodic* solutions. Purpose and scope of this paper is to illustrate the efficacy of such a trick, by applying it to several standard evolution equations, and by tersely discussing the periodic behavior of solutions of these modified equations. As we will see, this simple trick relates periodicity in real time to analyticity in a (suitably introduced) time variable, and thereby brings to light an explicit relation among analyticity in complex time and integrable behavior of the corresponding real dynamics. A representative list of such equations reads as follows:

First-order algebraic complex ODE

$$\dot{w} - \mathrm{i}\Omega w = \alpha w^{p/q}; \tag{1.1}$$

Polynomial vector field in the plane

$$\dot{u} + \Omega v = a_1 U - a_2 V, \qquad \dot{v} - \Omega u = a_1 V + a_2 U, \tag{1.2a}$$

$$U \equiv \sum_{m=0}^{[[p/2]]} (-1)^m \binom{p}{2m} u^{p-2m} v^{2m},$$

$$V \equiv \sum_{m=0}^{[[(p-1)/2]]} (-1)^m \binom{p}{2m+1} u^{p-2m-1}\, v^{2m+1}; \tag{1.2b}$$

$$\dot{u} + \Omega v = a_1\left(u^2 - v^2\right) - 2a_2 uv, \qquad \dot{v} - \Omega u = a_2\left(u^2 - v^2\right) + 2a_1 uv; \tag{1.3}$$

Oscillator with additional cubic force

$$\ddot{x} + (\Omega/2)^2 x = a^2 x^{-3}; \tag{1.4}$$

Painlevé-type equations (complex and real versions)

$$\ddot{w} + \Omega^2 w = \left(\alpha w^2 + \gamma\right)\exp(5\mathrm{i}\Omega t); \tag{1.5}$$

$$\ddot{w} - 5\mathrm{i}\Omega\dot{w} - 6\Omega^2 w = \alpha w^2 + \gamma\exp(5\mathrm{i}\Omega t); \tag{1.6}$$

$$\ddot{w} + 5\mathrm{i}\Omega\dot{w} - 6\Omega^2 w = \alpha w^2\exp(5\mathrm{i}\Omega t) + \gamma; \tag{1.7}$$

$$\ddot{w} - 3\mathrm{i}\Omega\dot{w} - 2\Omega^2 w = \alpha w^3 + (\gamma w + \delta)\exp(3\mathrm{i}\Omega t); \tag{1.8}$$

$$\ddot{u} + \Omega^2 u = \cos(5\Omega t)\left[a_1\left(u^2 - v^2\right) - 2a_2 uv + c_1\right] - \sin(5\Omega t)\left[a_2\left(u^2 - v^2\right) + 2a_1 uv + c_2\right], \tag{1.9a}$$

$$\ddot{v} + \Omega^2 v = \sin(5\Omega t)\left[a_1\left(u^2 - v^2\right) - 2a_2 uv + c_1\right] + \cos(5\Omega t)\left[a_2\left(u^2 - v^2\right) + 2a_1 uv + c_2\right]; \tag{1.9b}$$

$$\ddot{u} + 5\Omega\dot{v} - 6\Omega^2 u = a_1\left(u^2 - v^2\right) - 2a_2 uv + c_1\cos(5\Omega t) - c_2\sin(5\Omega t), \tag{1.10a}$$

$$\ddot{v} - 5\Omega\dot{u} - 6\Omega^2 v = a_2\left(u^2 - v^2\right) + 2a_1 uv + c_2\cos(5\Omega t) + c_1\sin(5\Omega t); \tag{1.10b}$$

$$\ddot{u} - 5\Omega\dot{v} - 6\Omega^2 u = \left[a_1\left(u^2 - v^2\right) - 2a_2 uv\right]\cos(5\Omega t) - \left[a_2\left(u^2 - v^2\right) + 2a_1 uv\right]\sin(5\Omega t) + c_1, \tag{1.11a}$$

$$\ddot{v} + 5\Omega\dot{u} - 6\Omega^2 v = \left[a_1\left(u^2 - v^2\right) - 2a_2 uv\right]\sin(5\Omega t) + \left[a_2(u^2 - v^2) + 2a_1 uv\right]\cos(5\Omega t) + c_2; \tag{1.11b}$$

$$\ddot{u} + 3\Omega\dot{v} - 2\Omega^2 u = a_1 u\left(u^2 - 3v^2\right) - a_2 v\left(3u^2 - v^2\right) + (c_1 u - c_2 v + d_1)\cos(3\Omega t) - (c_2 u + c_1 v + d_2)\sin(3\Omega t), \tag{1.12a}$$

$$\ddot{v} - 3\Omega\dot{u} - 2\Omega^2 v = a_2 u\left(u^2 - 3v^2\right) + a_1 v\left(3u^2 - v^2\right) + (c_1 u - c_2 v + d_1)\sin(3\Omega t) + (c_2 u + c_1 v + d_2)\cos(3\Omega t); \tag{1.12b}$$

Autonomous second-order ODEs (complex and real versions)

$$\ddot{w} - \mathrm{i}(5/2)\Omega\dot{w} - (3/2)\Omega^2 w = \alpha w^2; \tag{1.13}$$

$$\ddot{w} - \mathrm{i}\Omega\dot{w} - 2\Omega^2 = \alpha\exp(w); \tag{1.14a}$$

$$\ddot{w} - \mathrm{i}\Omega\dot{w} + 2\Omega^2 w = (\dot{w} - \mathrm{i}\Omega w)w; \tag{1.14b}$$

$$\ddot{w} - 3\mathrm{i}\Omega\dot{w} - 2\Omega^2 w = (\dot{w} - \mathrm{i}\Omega w)w; \tag{1.15}$$

$$\ddot{u} + (5/2)\Omega\dot{v} - (3/2)\Omega^2 u = a_1\left(u^2 - v^2\right) - 2a_2 uv, \tag{1.16a}$$

$$\ddot{v}-(5/2)\Omega\dot{u}-(3/2)\Omega^2 v=a_2\left(u^2-v^2\right)+2a_1uv; \tag{1.16b}$$

$$\ddot{u}+\Omega\dot{v}-2\Omega^2=\exp(u)[a_1\cos(v)-a_2\sin(v)], \tag{1.17a}$$

$$\ddot{v}-\Omega\dot{u}=\exp(u)[a_2\cos(v)+a_1\sin(v)]; \tag{1.17b}$$

$$\ddot{u}+\Omega\dot{v}+2\Omega^2 u=\dot{u}u-\dot{v}v+2\Omega uv, \tag{1.18a}$$

$$\ddot{v}-\Omega\dot{u}+2\Omega^2 v=\dot{u}v+u\dot{v}-\Omega\left(u^2-v^2\right); \tag{1.18b}$$

$$\ddot{u}+3\Omega\dot{v}-2\Omega^2 u=\dot{u}u-\dot{v}v+2\Omega uv, \tag{1.19a}$$

$$\ddot{v}-3\Omega\dot{u}-2\Omega^2 u=\dot{u}v+u\dot{v}-\Omega\left(u^2-v^2\right); \tag{1.19b}$$

$$\ddot{w}+\mathrm{i}\{(3q_1q_2+p_1q_2-p_2q_1)/[q_1(p_2-2q_2)]\}\Omega\dot{w} \\ +[(q_2/q_1)(p_1+q_1)/(p_2-2q_2)]\Omega^2 w=\alpha(\dot{w}-\mathrm{i}\Omega w)^{p_2/q_2}w^{p_1/q_1}; \tag{1.20}$$

$$\ddot{w}-3\mathrm{i}\Omega\dot{w}-2\Omega^2 w=\alpha(\dot{w}-\mathrm{i}\Omega w)^3 w^{-3}; \tag{1.21}$$

$$\ddot{w}+\mathrm{i}(nm+n-1)\Omega\dot{w}+n(m+1)\Omega^2 w=\alpha(\dot{w}-\mathrm{i}\Omega w)^{(2n+1)/n}w^m; \tag{1.22}$$

$$\ddot{w}-\mathrm{i}\ (2nm+1)\Omega\dot{w}-2nm\Omega^2 w=\alpha(\dot{w}-\mathrm{i}\Omega w)^{(2n+1)/n}w^{-(2m+1)}; \tag{1.23}$$

$$\ddot{w}-3\mathrm{i}\Omega\dot{w}-2\Omega^2 w=\alpha w(\dot{w}-\mathrm{i}\Omega w)+\beta w^3; \tag{1.24}$$

$$\ddot{w}-3\mathrm{i}\Omega\dot{w}-2\Omega w=\alpha w(\dot{w}-\mathrm{i}\Omega w)-(\alpha/3)^2 w^3; \tag{1.25}$$

$$\ddot{w}-3\mathrm{i}\Omega\dot{w}-2\Omega^2 w=\alpha w(\dot{w}-\mathrm{i}\Omega w); \tag{1.26}$$

$$\ddot{u}+3\Omega\dot{v}-2\Omega^2 u=a_1(u\dot{u}-v\dot{v}+2\Omega uv) \\ -a_2\left[\dot{u}v+u\dot{v}-\Omega\left(u^2-v^2\right)\right]+b_1u\left(u^2-3v^2\right)-b_2v\left(3u^2-v^2\right), \tag{1.27a}$$

$$\ddot{v}-3\Omega\dot{u}-2\Omega^2 v=a_2(u\dot{u}-v\dot{v}+2\Omega uv) \\ +a_1\left[\dot{u}v+u\dot{v}-\Omega\left(u^2-v^2\right)\right]+b_2u\left(u^2-3v^2\right)+b_1v\left(3u^2-v^2\right); \tag{1.27b}$$

$$\ddot{w}-\mathrm{i}\Omega\dot{w}=\dot{w}^2F(w); \tag{1.28}$$

Autonomous third order ODEs (complex and real versions)

$$\dddot{w}-10\mathrm{i}\Omega\ddot{w}-31\Omega^2\dot{w}+30\mathrm{i}\Omega^3 w=\alpha(2\dot{w}-5\mathrm{i}\Omega w)w; \tag{1.29}$$

$$\dddot{z}-10\mathrm{i}\Omega\ddot{z}-19\Omega^2\dot{z}-30\mathrm{i}\Omega^3 z=\alpha(2\dot{z}-5\mathrm{i}\Omega z)z; \tag{1.30}$$

$$\dddot{w}+31\Omega^2\dot{w}+30\mathrm{i}\Omega^3 w+5\mathrm{i}\Omega\gamma=2(\ddot{w}+5\mathrm{i}\Omega\dot{w}-\gamma)\dot{w}/w; \tag{1.31}$$

$$\dddot{w}-5\mathrm{i}\Omega\ddot{w}+\Omega^2\dot{w}-5\mathrm{i}\Omega^3 w=2w\dot{w}\left(\ddot{w}+\Omega^2 w\right)/\left(w^2+\eta\right); \tag{1.32}$$

$$\dddot{u}+10\Omega\ddot{v}-31\Omega^2\dot{u}-30\Omega^3 v \\ =2a_1(\dot{u}u-\dot{v}v+5\Omega uv)-a_2\left[2(\dot{u}v+\dot{v}u)-5\Omega\left(u^2-v^2\right)\right], \tag{1.33a}$$

$$\dddot{v}-10\Omega\ddot{u}-31\Omega^2\dot{v}+30\Omega^3 u \\ =2a_2(\dot{u}u-\dot{v}v+5\Omega uv)+a_1\left[2(\dot{u}v+\dot{v}u)-5\Omega\left(u^2-v^2\right)\right]; \tag{1.33b}$$

$$\dddot{u}+10\Omega\ddot{v}-19\Omega^2\dot{u}+30\Omega^3 v \\ =2a_1(\dot{u}u-\dot{v}v+5\Omega uv)-a_2\left[2(\dot{u}v+\dot{v}u)-5\Omega\left(u^2-v^2\right)\right], \tag{1.34a}$$

$$\dddot{v}-10\Omega\ddot{u}-19\Omega^2\dot{v}-30\Omega^3 u \\ =2a_2(\dot{u}u-\dot{v}v+5\Omega uv)-a_1\left[2(\dot{u}v+\dot{v}u)-5\Omega\left(u^2-v^2\right)\right]; \tag{1.34b}$$

First-order PDE of shock type (complex and real versions)

$$w_t-\mathrm{i}\Omega w=\alpha w_x w^{p/q}; \tag{1.35}$$

$$u_t + \Omega v = a_1(u_x u - v_x v) - a_2(u_x v + v_x u), \tag{1.36a}$$
$$v_t - \Omega u = a_2(u_x u - v_x v) + a_1(u_x v + v_x u); \tag{1.36b}$$
$$u_t + \Omega v = a_1\left[u_x\left(u^2 - v^2\right) - 2v_x uv\right] - a_2\left[2u_x uv + v_x\left(u^2 - v^2\right)\right], \tag{1.37a}$$
$$v_t - \Omega v = a_2\left[u_x\left(u^2 - v^2\right) - 2v_x uv\right] + a_1\left[2u_x uv + v_x\left(u^2 - v^2\right)\right]; \tag{1.37b}$$

Burgers-type equations (complex and real versions)

$$w_t = \left(\mathrm{i}\Omega x w + \beta w_x + \alpha w^2\right)_x; \tag{1.38}$$
$$u_t = \left[-\Omega x v + b_1 u_x - b_2 v_x + a_1\left(u^2 - v^2\right) - 2a_2 uv\right]_x, \tag{1.39a}$$
$$v_t = \left[\Omega x u + b_2 u_x - b_1 v_x + a_1\left(u^2 - v^2\right) - 2a_1 uv\right]_x; \tag{1.39b}$$

Generalized KdV-type equations

$$w_t = \mathrm{i}\Omega[2(q/p)w + x w_x] + \beta w_{xxx} + \alpha w_x w^{p/q}; \tag{1.40}$$

KdV-type equation (complex and real versions)

$$w_t = \mathrm{i}\Omega[2w + x w_x] + \beta w_{xxx} + \alpha w_x w; \tag{1.41}$$
$$u_t = -\Omega(2v + x v_x) + b_1 u_{xxx} - b_2 v_{xxx} + a_1(u_x u - v_x v) - a_2(u_x v + v_x u), \tag{1.42a}$$
$$v_t = \Omega(2u + x u_x) + b_1 v_{xxx} + b_2 u_{xxx} + a_2(u_x u - v_x v) + a_1(u_x v + v_x u); \tag{1.42b}$$

Modified KdV-type equation (complex and real version)

$$w_t = [\mathrm{i}\Omega x w + \beta w_{xx} + (\alpha/3)w^3]_x; \tag{1.43}$$
$$u_t = \left[-\Omega x v + b_1 u_{xx} - b_2 v_{xx} + (a_1/3)u\left(u^2 - 3v^2\right) - (a_2/3)v\left(3u^2 - v^2\right)\right]_x, \tag{1.44a}$$
$$v_t = \left[\Omega x u + b_2 u_{xx} + b_1 v_{xx} + (a_2/3)u\left(u^2 - 3v^2\right) + (a_1/3)v\left(3u^2 - v^2\right)\right]_x; \tag{1.44b}$$

KP-type equation (complex and real versions)

$$[w_t - \mathrm{i}\Omega w - (\mathrm{i}/2)\Omega x w_x - \mathrm{i}\Omega y w_y + \beta w_{xxx} + \alpha w_x w]_x + \gamma w_{yy} = 0; \tag{1.45}$$
$$\begin{aligned}&[u_t + \Omega v + (\Omega/2)x v_x + \Omega y v_y + b_1 u_{xxx} - b_2 v_{xxx} \\ &\quad + a_1(u_x u - v_x v) - a_2(u_x v + v_x v)]_x + c_1 u_{xx} - c_2 v_{yy} = 0,\end{aligned} \tag{1.46a}$$
$$\begin{aligned}&[v_t - \Omega u - (\Omega/2)x u_x - \Omega y u_y + b_1 v_{xxx} + b_2 x_{xxx} + b_2 u_{xxx} \\ &\quad + a_2(u_x u - v_x v) + a_1(u_x v + v_x u)]_x + c_2 u_{yy} + c_1 v_{yy} = 0.\end{aligned} \tag{1.46b}$$

These evolution equations are discussed tersely, one by one, at the end of this introductory Section 1. But let us immediately emphasize that the independent variable, t ("time"), which accounts for their evolutionary character, is of course assumed to be *real*, and also *real* is the nonvanishing constant Ω, to which the period

$$T = 2\pi/|\Omega| \tag{1.47}$$

is associated. The dependent variables are in general complex, as are other constants featured by these equations; but sometimes the evolution equations are written in the real form yielded by an explicit separation of all complex numbers into their real and imaginary parts. The main point is that *all* these evolution equations (but one, namely (1.14a) and its real version (1.17); see below) feature *many*, or in some cases *only*, solutions which are periodic with period T – or, in several cases, with periods obtained by multiplying T by a rational number. Note that the above list includes both ODEs and PDEs; in the latter case we restricted (with one exception, see (1.45), (1.46)), merely for the sake of simplicity, consideration to (few) equations in $2 = 1 + 1$ variables, namely to only one, of course *real*, "space" variable, x, in addition to the, also *real*, "time" variable t. The alert reader, after having understood how the trick which yielded these evolution equations works, will have no difficulty (rather, some fun!) in applying it more widely.

The trick is introduced in the following Section 2 to derive and discuss the three prototypical evolution equations (1.1), (1.35) and (1.38); it is then used in Section 3 to derive and discuss the other evolution equations reported in the above list (as well as some more general versions encompassing several of them).

The elementary character of the trick presumably precludes any hope to ascertain *precisely* who introduced it firstly [1, 2]. An ample use of it, in the context of classical many-body problems amenable to exact treatments, has been made in Ref. [3], to which we refer for references to previous applications in such a context; for this reason we did not include in the above list of evolution equations any (system of) ODEs describing classical many-body motions.

Let us end this introductory Section 1 by providing some details on the evolution equations listed above.

In the ODE (1.1), as well as in all subsequent ODEs in the list, dots denote differentiations with respect to the (*real*) independent variable t ("time"). In this ODE (1.1), the dependent variable $w \equiv w(t)$ is *complex*, α is a nonvanishing, but otherwise arbitrary (possibly *complex*) constant, and p/q is a *rational* number different from unity, $p \neq q$. *All nonsingular* solutions of this ODE, (1.1), are *periodic*: for a given assignment of the rational number p/q, there are generally two periods, T_1 and T_2, both integer multiples of T, see (1.47), and the initial data, $w(0)$, are divided into two complementary open sets, which yield solutions periodic with one, respectively with the other, period; these two sets of initial data are separated by (a lower dimensional set of) initial data that yield singular solutions. The solutions of this ODE, (1.1), are given, and discussed, in Section 2.

In the system of two coupled ODEs (1.2) the two dependent variables, $u \equiv u(t)$, $v \equiv v(t)$, are instead *real*, as well as the two, otherwise *arbitrary*, constants a_1, a_2; p is an *integer* larger than unity, $p > 1$; and the symbol $[[\cdots]]$ denotes the integral part of the doubly-bracketed quantity. This system, (1.2), is merely another avatar of (1.1) with $q = 1$, obtained by setting $w = u + \mathrm{i}v$, $\alpha = a_1 + \mathrm{i}a_2$.

The system (1.3) is just (1.2) with $p = 2$.

The ODE (1.4) is an avatar of (1.3) with $a_1 = 0$, $a_2 = a$, obtained by time-differentiating the first of the (1.3), by using the second of the (1.3) to eliminate $\dot{v}$, by then using the first of the (1.3) to eliminate v, and finally by setting $u(t) = [x(t)]^{-2} - \Omega/(2a)$; the fact that all (real) solutions of this real ODE are periodic with period T, see (1.47), is of course a well-known result.

In the 3 complex evolution ODEs (1.5), (1.6) and (1.7) the two constants α and γ are *arbitrary* (possibly complex), and also *arbitrary* (possibly complex) are the three constants α, γ and δ, in the complex ODE (1.8). *All (nonsingular)* solutions of these 4 (nonautonomous!) evolution ODEs are periodic with period T, see (1.47).

The 4 systems of 2 *real* coupled evolution ODEs (1.9), (1.10), (1.11) respectively (1.12) are merely the *real* avatars of (1.5), (1.6), (1.7) respectively (1.8), obtained by setting $w = u + \mathrm{i}v$, $\alpha = a_1 + \mathrm{i}a_2$, $\gamma = c_1 + \mathrm{i}c_2$, $\delta = d_1 + \mathrm{i}d_2$. Of course *all* their (*nonsingular*) solutions are periodic with period T, see (1.47).

All nonsingular solutions of (1.13), (1.14b) and (1.15) are periodic in t with period (at most) T, see (1.47). The general solutions of the evolution ODEs (1.13) and (1.14b) are displayed in Section 3, as well as conditions sufficient to guarantee their *nonsingularity* (see (3.17)). The solutions of (1.14b) coincide with the *time-derivative* of the solutions of (1.14a).

The four systems of 2 *real* coupled evolution ODEs (1.16), (1.17), (1.18), (1.19) are merely the *real* avatars of (1.13), (1.14a), (1.14b), (1.15), obtained by setting $w = u + \mathrm{i}v$, $\alpha = a_1 + \mathrm{i}a_2$; hence their (*real*) solutions can be immediately obtained from the solutions of the complex ODEs (1.13), (1.14a), (1.14b), (1.15), see immediately above.

The similarity among (1.14b) and (1.15) (and likewise among (1.18) and (1.19)) should be noted; also note that (1.15) is a special case of (1.20), which is not the case of (1.14b).

The complex evolution equation (1.20), with α an *arbitrary* (possibly complex) constant and p_1, q_1, p_2, q_2 four *integers* (but only the two rational numbers p_1/q_1, p_2/q_2 actually enter in (1.20)), is expected to possess lots of periodic solutions, as entailed by its derivation in Section 3. However, only the 3 special cases of (1.20) corresponding to (1.21), (1.22) and (1.23) are treated in any detail in Section 3 (in addition to (1.15), see above). In these 3 evolution ODEs, (1.21)–(1.23), α is again an arbitrary, possibly complex, constant; in (1.22) and (1.23) n is a *positive* integer, while m is a *nonnegative* integer in (1.22), a *positive* integer in (1.23). In all three cases, (1.21)–(1.23), *all nonsingular* solutions are periodic. In the case of (1.21), the general solution is exhibited in Section 3, and it is shown there that in this case the *nonsingular* solutions split into two sets, both however periodic with the *same* period T, see (1.47); and these two sets are separated by a lower dimensional set of initial data (characterized by (3.23) with (3.22)), to which there correspond *singular* solutions of (1.21), namely solutions such that $\dot{w}(t)$ diverges at *real* times $t = t_b$ (with t_b defined mod(T) by (3.24) with (3.22) and (3.23)).

In the other two cases, (1.22), (1.23), the situation is analogous, but richer. Depending on the values of the integers n and m, many more periodicities are possible: and again these different periodicities correspond to different sets of initial data, $w(0)$, $\dot{w}(0)$, separated by lower-dimensional sets of such data which themselves yield singular solutions (for which $\dot{w}(t)$ diverges at some *real* times $t = t_b$, generally defined mod (t_p), see (2.2)). More details in Section 3, where expressions of the general solution of these two evolution ODEs, (1.22), (1.23), are provided, albeit in somewhat implicit form.

The complex evolution ODE (1.24) features the two *arbitrary* (possibly complex) constants α, β, in addition to the *real* constant Ω; its derivation in Section 3 entails that it possesses lots of periodic solutions. Indeed the next two ODEs listed, (1.25) respectively (1.26), which are clearly two special cases of (1.24) (corresponding to $\beta = -(\alpha/3)^2$ respectively $\beta = 0$), are explicitly solved in Section 3 (see (3.38) respectively (3.41)), and it is shown there that *all* their *nonsingular* solutions are *periodic* in t with period T, see (1.47) (conditions necessary and sufficient to guarantee that these solutions be *nonsingular* for all *real* values of t are also provided there, see (3.39) respectively (3.42)).

The system of two *real* evolution ODEs (1.27) is merely the real avatar of (1.24), obtained by setting $w = u + \mathrm{i}v$, $\alpha = a_1 + \mathrm{i}a_2$, $\beta = b_1 + \mathrm{i}b_2$.

In the second-order evolution ODE (1.28) $F(w)$ is a largely arbitrary (but analytic) function; the fact that this evolution ODE, in spite of its generality (associated with the large arbitrariness of the function $F(w)$) possesses lots of periodic solutions is, however, not new [2], and we therefore forsake any further discussion of this ODE in this paper.

The *third-order* complex ODEs (1.29) respectively (1.30) are merely two avatars of the second order ODE (1.6), and the systems of two real ODEs (1.33) respectively (1.34) are merely the real versions of (1.29) respectively (1.30), obtained by setting $w(t) = u(t) + \mathrm{i}v(t)$ respectively $z(t) = u(t) + \mathrm{i}v(t)$ and $\alpha = a_1 + \mathrm{i}a_2$; likewise the *third-order* ODEs (1.31) respectively (1.32) are avatars of (1.7) respectively (1.5). These facts are proven in Section 3; they entail that *all nonsingular* solutions of these (*third-order, autonomous*) ODEs, (1.29)–(1.34), are *periodic* with period T, see (1.47).

In the PDE (1.35), as well as in all subsequent PDEs in the list, the independent variables, x and t, are supposed to be *real*; the independent variable, $w \equiv w(x,t)$, is generally *complex* (but we also exhibit in some cases the *real* avatars of these evolution PDEs: for instance (1.36) respectively (1.37) are the *real* avatars of (1.35) with $q = 1$ and $p = 1$ respectively $p = 2$, obtained by setting $w = u + \mathrm{i}v$, $\alpha = a_1 + \mathrm{i}a_2$). Of course subscripted variables, here and below, denote partial differentiations. The constant α in (1.35) is *arbitrary*, possibly complex; Ω is (as always) a *real* (nonvanishing!) constant, and q, p are two arbitrary *integers*. The initial-value problem for this evolution PDE, (1.35), is solved in Section 2, and it is proven there that there exist classes of initial data, $w(x,0) = w_0(x)$, which yield solutions periodic in t (for all values of x).

Likewise, it is shown in Section 2 that the PDE (1.38), with α and β two *arbitrary* (possibly complex) constants and of course Ω a *real* (nonvanishing!) constant, also possesses (nonsingular!) solutions which are periodic in the (*real*) independent variable t for all values of the (*real*) independent variable x. Note the explicit appearance, in this PDE, (1.38), of the variable x, which entails that this evolution PDE, (1.38), is *not* translation-invariant. Clearly (1.39) is the *real* avatar of (1.38), obtained by setting $w = u + \mathrm{i}v$, $\alpha = a_1 + \mathrm{i}a_2$, $\beta = b_1 + \mathrm{i}b_2$.

The remaining evolution PDEs of the list reported above have been derived using the trick, as described in detail in Section 3; or they are special cases, or real avatars, of previous equations in the list. It is therefore justified to expect that they all possess lots of periodic solutions; but this hunch is not substantiated, in Section 3, via the explicit exhibition of solutions, nor is it discussed in any more detail here (lest the length of this paper become excessive). Hence a more thorough investigation of these evolution PDEs, as well as of the many others that can be easily manufactured using the trick described in detail in Section 2, remains as a task for the future.

Let us end this introductory Section 1 by indicating that our presentation, as it unfolded above and as it continues below, has been deliberately adjusted to cater not only to the interest of applied mathematicians, but as well to the needs of scientists whose mathematical training is somewhat superficial, because we believe there are such potential customers who might usefully take advantage, in various applicative contexts, of the findings reported in this paper. This may have entailed occasional repetitions (for instance in the guise of presenting both the complex and the real versions of certain evolution equations), as well as the rather systematic practice of keeping, when writing evolution equations, also constants which might be eliminated by simple rescalings of (dependent and independent) variables. We hope most readers will agree that the advantages of such an approach outweigh its defects.

2 The trick

The idea is to start from an evolution equation, chosen as candidate for a modification the goal of which is to obtain a (modified) evolution equation that features (a lot of) periodic solutions; to then assume that the *independent* variable, τ, in terms of which the (original, unmodified) evolution unfolds, is *complex*; and to then introduce the *real* "time" variable, t, so that $\tau = \tau(t)$ be a *periodic* function of t, say

$$\tau \equiv \tau(t) = [\exp(\mathrm{i}\omega t) - 1]/(\mathrm{i}\omega). \tag{2.1a}$$

Hereafter ω is a *positive* number, and we set

$$t_p = 2\pi/\omega, \tag{2.2}$$

so that $\tau(t)$ is periodic in t with period t_p. Indeed as t varies over one period, say from $t = 0$ to $t = t_p$, τ travels from $\tau = 0$ back to $\tau = 0$, over a circular contour, C, in the complex τ-plane, centered at $\tau = \mathrm{i}/\omega$ and of radius $r = 1/\omega = t_p/(2\pi)$ (so that the diameter of this circle, C, has length $2/\omega$ and straddles the upper imaginary axis in the complex τ-plane, from the origin, $\tau = 0$, to $\tau = 2\mathrm{i}/\omega$). Hence any function of the complex variable τ, say $\varphi(\tau)$, which is holomorphic in τ in a region of the complex τ-plane that includes the circle C, is a periodic function of t with period t_p. But if $\varphi(\tau)$ satisfies an evolution equation, say

$$\varphi'(\tau) = F[\varphi(\tau), \tau], \qquad \varphi'(\tau) \equiv d\varphi(\tau)/d\tau, \tag{2.3}$$

with $F(\varphi, \tau)$ an *analytic* function of its two arguments, any solution $\varphi(\tau)$ of (2.3) characterized by an initial datum $\varphi(0)$ such that $F(\varphi, \tau)$ is holomorphic in the neighborhood of $\varphi = \varphi(0)$, $\tau = 0$, is itself guaranteed (by the standard existence/uniqueness/analyticity theorem for evolution equations) to be a holomorphic function of τ in a disk D centered, in the complex τ-plane, at $\tau = 0$, and having a nonvanishing radius ρ whose magnitude depends on the function $F(\varphi, \tau)$ and on the initial datum $\varphi(0)$. Clearly if the radius ρ exceeds the diameter $2/\omega$ of the circle C, $\rho > 2/\omega$, the disk D includes the circle C, hence the corresponding solution $\varphi(\tau)$, considered as a function of the *real* variable t, is then *periodic* with period t_p, see (2.2). And it is of course easy, via (2.1), to recast (2.3) as an evolution equation in terms of the (*real*) time t rather than the (*complex*) independent variable τ.

At this stage one might wonder why was it necessary to let $\tau \equiv \tau(t)$ be *complex*, see (2.1a); could not one instead choose τ to be a *real* periodic function of t, say $\tau = \omega^{-1}\sin(\omega t)$ or $\tau = \omega^{-1}\tan(\omega t)$? Of course one could. But the additional, important requirement for the basic idea to be *usefully* applicable is that the modified evolution equation obtained via this approach in terms of the (*real*) independent variable t ("time") have a *neat* look. This is essential for such an equation to be eventually identified as the appropriate evolution equation to model some natural phenomenon. To this end the change of variable (2.1a) appears particularly suitable, as the examples exhibited in this paper show; especially if it is associated with an appropriate change of *dependent* variable, typically by setting

$$w(t) = \exp(\mathrm{i}\lambda\omega t)\varphi[\tau(t)], \tag{2.4}$$

of course with $\tau(t)$ defined by (2.1a) and with λ a (*rational*) number to be chosen conveniently (see below). And a similar kind of modification can be conveniently applied to other ("space") variables in the case of PDEs, although this may also cause certain difficulties, as we shall see below.

Enough now of this introductory discussion, which was merely meant to convey the gist of this approach. Let us rather proceed and show how this technique works, by demonstrating its effectiveness in three (very simple, completely solvable) examples, one ODE and two PDEs. But before doing so, let us note that (2.1a) entails

$$\dot{\tau}(t) = \exp(\mathrm{i}\omega t), \tag{2.1b}$$

as well as

$$\tau(0) = 0, \tag{2.1c}$$
$$\dot{\tau}(0) = 1. \tag{2.1d}$$

Here of course, and throughout (in the ODE context), dots denote differentiations with respect to the (real) time t, while (in the ODE context) we always use primes to denote differentiations with respect to the (complex) independent variable τ (hence $\dot{\varphi} = \exp(\mathrm{i}\omega t)\varphi'$, see (2.1b)).

Consider the nonlinear first-order ODE

$$\varphi' = \alpha\varphi^{p/q}, \qquad \varphi \equiv \varphi(\tau). \tag{2.5}$$

Here α is an arbitrary (possibly complex) constant ($\alpha \neq 0$), and p, q are two arbitrary integers. Without loss of generality we assume q to be positive, $q > 0$, and p, q to be coprime (namely, their decompositions into products of primes contains no common factor). We also hereafter assume, for simplicity, $p \neq q$, to exclude the trivial, linear, case, see (2.5).

The initial-value problem for (2.5) is solved by the following formula:

$$\varphi(\tau) = \varphi(0)[1 - (\tau/\tau_b)]^{q/(q-p)}, \tag{2.6a}$$
$$\tau_b = [q/(p-q)]\alpha^{-1}[\varphi(0)]^{(q-p)/q}. \tag{2.6b}$$

This solution, considered as a function of τ, is holomorphic inside the disk D centered, in the complex τ-plane, at the origin, $\tau = 0$, and having radius ρ,

$$\rho = |\tau_b|; \tag{2.7}$$

indeed $\varphi(\tau)$ has (only) two (algebraic) branch points, one at $\tau = \infty$, the other at $\tau = \tau_b$, see (2.6b). Hence φ, *considered as a function of* t, is *periodic* with period t_p, see (2.2), if the initial datum, $\varphi(0)$, entails $\rho > 2/\omega$, see (2.7) and (2.6b), consistently with the above discussion. But this condition, while sufficient to guarantee periodicity with period t_p, is more stringent than necessary. A more appropriate, less stringent but nevertheless sufficient condition to insure periodicity (with period t_p) of φ as a function of t, is that the branch point τ_b, see (2.6b), fall outside the circular contour C (centered at $\tau_c = \mathrm{i}/\omega$, and of radius $r = 1/\omega$), traveled by τ when t varies over one period t_p, namely that the distance of τ_b from $\tau_c = \mathrm{i}/\omega$, $|\tau_b - \tau_c|$, exceed the radius $r = 1/\omega$:

$$|\tau_b - \mathrm{i}/\omega| > 1/\omega, \tag{2.8}$$

of course with τ_b defined by (2.6b).

If instead the initial datum, $\varphi(0)$, entails via (2.6b) that the inequality (2.8) gets reversed,

$$|\tau_b - \mathrm{i}/\omega| < 1/\omega, \tag{2.9}$$

then φ, considered as a function of t, is also periodic, but now with period

$$\tilde{t}_p = |q-p| t_p; \tag{2.10}$$

indeed in this case, (2.9), as t varies over one period t_p, τ traverses once (before recovering its original value) the branch cut that runs from $\tau = \tau_b$ to $\tau = \infty$, hence the function $\varphi(\tau)$ picks up a phase factor $\exp[2\pi \mathrm{i} q/(q-p)]$, see (2.6a); hence after $|q-p|$ such crossings the function φ recovers its original value.

The initial data, $\varphi(0)$, that satisfy via (2.6b) the *inequalities* (2.8) respectively (2.9) are separated by a (lower-dimensional) set of initial data, $\varphi(0)$, that satisfy via (2.6b) the *equality*

$$|\tau_b - \mathrm{i}/\omega| = 1/\omega. \tag{2.11}$$

This equality entails that the time t_b, defined mod (t_p) by setting (see (2.1a) and (2.6b))

$$\tau_b = [\exp(\mathrm{i}\omega t_b) - 1]/(\mathrm{i}\omega), \tag{2.12}$$

is *real*, since this formula, (2.12), via (2.11) entails

$$|\exp(\mathrm{i}\omega t_b)| = 1. \tag{2.13}$$

Hence for the set of initial data $\varphi(0)$ that satisfy (2.11) (with τ_b defined by (2.6b)) the solution φ diverges (if $p > q$) or vanishes (if $p < q$) at the real time $t = t_b \mod (t_b)$; this signifies that, at these times, the ODE (2.5) becomes *singular*.

Before proceeding further, let us note that much of the above treatment would apply even if the *rational* exponent p/q in (2.5) were replaced by an arbitrary *real*, or even *complex*, number: in the formulas above, this is most easily realized by setting $q = 1$ and letting p be an arbitrary, even complex, number. The conclusion about the *periodicity*, with period t_p, of the solution φ (considered as a function of t), would still stand, whenever the intial datum $\varphi(0)$ entails, via (2.6b), validity of the inequality (2.8); on the other hand, if the initial datum, $\varphi(0)$, entails via (2.6b) the reversed inequality (2.9), then φ, considered as a function of t, would cease altogether to be periodic (unless p is rational). The fact that the initial data which yield (2.8) respectively (2.9) are separated by a (lower dimensional) set of initial data, $\varphi(0)$, satisfying (2.11) via (2.6b), data which yield solutions that become singular at $t = t_b$, see (2.12), would also stand. However, for the results that now follow, the restriction to a *rational* exponent p/q in the right-hand-side of (2.5) plays an essential role: without this restriction no solution of the evolution equations (2.17), see below, is periodic.

Let us now obtain the ODE, with the real time t as independent variable, satisfied by the (complex) function $w(t)$, see (2.4). Clearly this definition, (2.4), entails, via (2.1b),

$$\dot{w} = \mathrm{i}\lambda\omega w + \exp[\mathrm{i}(1+\lambda)\omega t]\varphi', \tag{2.14}$$

hence, via (2.5) and (2.4),

$$\dot{w} - \mathrm{i}\lambda\omega w = \alpha \exp\{\mathrm{i}[1 + \lambda(q-p)/q]\omega t\} w^{q/p}. \tag{2.15}$$

It is now clearly convenient to set

$$\lambda = q/(p-q), \tag{2.16a}$$
$$\Omega = \lambda\omega = [q/(p-q)]\omega, \tag{2.16b}$$

so that the evolution equation satisfied by $w \equiv w(t)$ take the neater (*autonomous*!) form (1.1):

$$\dot{w} - \mathrm{i}\Omega w = \alpha w^{p/q}. \tag{2.17}$$

The solution of the initial-value problem for this equation then follows from (2.4), (2.6), (2.1a) and (2.16):

$$w(t) = w(0)\exp(\mathrm{i}\Omega t)\{1 - \alpha[w(0)]^{(p-q)/q}[\exp\{\mathrm{i}[(p-q)/q]\Omega t\} - 1]/(\mathrm{i}\Omega)\}^{q/(q-p)}. \tag{2.18}$$

And it is clear, by inspection or from the previous discussion (also keeping in mind (2.4) and (2.16)), that the set of initial data $w(0)$ such that there hold the *inequality*

$$\left|\alpha^{-1}[w(0)]^{(q-p)/q} - \mathrm{i}/\Omega\right| > 1/\Omega \tag{2.19}$$

yield solutions (2.18) which are periodic with the period T_1, equal to the minimum common integer multiple among $t_p = 2\pi/\omega = [q/(p-q)]2\pi/\Omega = |q/(p-q)|T$ and T, see (2.2) and (1.47); while the set of initial data $w(0)$ such that the reversed *inequality* hold,

$$\left|\alpha^{-1}[w(0)]^{(q-p)/q} - \mathrm{i}/\Omega\right| < 1/\Omega, \tag{2.20}$$

yield solutions (2.18) which are periodic with period $T_2 = qT$, which is indeed the minimum common integer multiple among $\tilde{t}_p = |q-p|2\pi/\omega = q2\pi/|\Omega| = qT$ and T, see (2.10) and (1.47); and these two sets of intial data are separated by (the lower dimensional set of) those data that satisfy the *equality*

$$\left|\alpha^{-1}[w(0)]^{(q-p)/q} - \mathrm{i}/\Omega\right| = 1/\Omega, \tag{2.21}$$

namely by the initial data $w(0)$ which yield solutions $w(t)$ that become singular at $t = t_b \mod (t_p)$, with t_b given by (2.12) and (2.6b) (with $\varphi(0)$ replaced by $w(0)$, since these two quantities coincide, see (2.4) and (2.1c)), and with t_p given by (2.2) (with (2.16b)).

The ODE (2.17) is the modified version of (2.5), which indeed features a lot of *periodic* solutions: in fact, as we just saw, *all* its *nonsingular* solutions are *periodic* (with periods T_1 or T_2). Let us re-emphasize that, in (2.17), the independent variable t ("time") is *real*, while the dependent variable, $w \equiv w(t)$, is *complex*, as entailed by the fact that Ω is *real*; since we assumed ω and q to be *positive*, Ω has the same sign as $p - q$, see (2.16b); but this is not a significant restriction on (2.17), indeed the assumptions that ω and q be *positive* are quite unessential (it is of course essential that neither ω nor q vanish!); these assumptions were made merely to marginally simplify our presentation. We can also assume the arbitrary (nonvanishing!) constant α to be *complex*; while the exponent p/q is of course *rational*. Other avatars of this ODE, (2.17), are displayed in the preceding Section 1, see (1.2), (1.3), (1.4).

Let us now proceed and show how the trick works for PDEs, beginning from a very simple, solvable, example.

Consider the ("shock") evolution PDE

$$\varphi_\tau = \alpha\varphi_x\varphi^{p/q}, \qquad \varphi \equiv \varphi(x,\tau). \tag{2.22}$$

Here α is again an arbitrary (possibly complex) constant and p/q is a (nonvanishing) *rational* number (q a *positive integer*, $q > 0$; p a *nonvanishing integer*, $p \neq 0$; p and q coprime). We now set (see (2.4))

$$w(x,t) = \exp(\mathrm{i}\lambda\omega t)\varphi[x,\tau(t)], \tag{2.23}$$

of course with $\tau(t)$ defined by (2.1a) and with λ a parameter to be determined (see below). This entails, via (2.1b),

$$w_t = \mathrm{i}\lambda\omega w + \exp[\mathrm{i}(1+\lambda)\omega t]\varphi_\tau, \tag{2.24a}$$

hence, via (2.22) and (2.23),

$$w_t = \mathrm{i}\lambda\omega w + \alpha\exp[\mathrm{i}(1-\lambda p/q)\omega t]w_x w^{p/q}. \tag{2.24b}$$

This suggests setting

$$\lambda = q/p, \tag{2.25a}$$

$$\Omega = \lambda\omega = (q/p)\omega, \tag{2.25b}$$

so that the evolution PDE satisfied by $w(x,t)$ take the neat form

$$w_t - \mathrm{i}\Omega w = \alpha w_x w^{p/q}. \tag{2.26}$$

This is the evolution PDE, see (1.35), we expect shall possess a lot of periodic solutions. Indeed, as can be easily verified, the solution of the initial-value (or "Cauchy") problem for (2.26) is given by the definition (2.23) with (2.1a) and with $\varphi(x,\tau)$ being the root of the (nondifferential) equation

$$\varphi = w_0\left(x + \alpha\tau\varphi^{p/q}\right), \tag{2.27}$$

where $w_0(x)$ provides the initial condition for (2.22) and (2.26) (see (2.23) and (2.1c)):

$$w(x,0) = \varphi(x,0) = w_0(x). \tag{2.28}$$

But the equation (2.27) generally has several (indeed, quite possibly, an infinity of) complex roots. Which one should be chosen? Of course, the one that develops by continuity from (2.28) at $\tau = t = 0$, as the time evolution unfolds. Indeed it is clear that, if $w_0(x)$ is an *analytic* function of its argument, x, with no singularities for *real* x, and if moreover the (*positive*) quantity

$$\Delta(x) = |\alpha|\,\rho\,|w_0(x)|^{p/q} \tag{2.29}$$

is sufficiently *small* compared to the distance, say $d(x)$, of the singularity of $w_0(z)$ in the complex z-plane closest to the real point x,

$$\Delta(x) \ll d(x), \tag{2.30}$$

then the solution φ of (2.27) that flows from (2.28) can be obtained from (2.27) by iteration, namely by setting

$$\varphi_0(x,\tau) = w_0(x), \tag{2.31a}$$

$$\varphi_n(x,\tau) = w_0\left(x + \alpha\tau[\varphi_{n-1}(x,\tau)]^{p/q}\right), \qquad n = 1, 2, 3, \ldots \tag{2.31b}$$

so that the sequence $\varphi_n(x,\tau)$ converge to $\varphi(x,\tau)$ as $n \to \infty$,

$$\varphi(x,\tau) = \lim_{n\to\infty}[\varphi_n(x,\tau)] \tag{2.32}$$

for all values of the *complex* variable τ such that $|\tau| \leq \rho$, see (2.29). This entails that, if the condition (2.30) holds for *all* (*real*) values of x (we are implicitly assuming here that one considers the PDE (2.17) for *all real* values of the space variable x), then $\varphi(x,\tau)$ is holomorphic in τ in a disk D, centered at $\tau = 0$, whose radius ρ can be made arbitrarily large by making an appropriate choice of the initial datum, see (2.28) and (2.29). This entails that there always is a set of initial data, $w_0(x)$, the modulus of which is, for all real values of x, sufficiently small (if $p/q > 0$) or sufficiently large (if $p/q < 0$) to entail that ρ exceed $2/\omega$

$$\rho > 2/\omega\,. \tag{2.33}$$

But we know from the above discussion that, whenever this happens, φ, considered as a function of t, is periodic with period t_p, see (2.2). Hence the solution $w(x,t)$ of (2.26) with such initial datum, see (2.28), is periodic in t with period

$$\tilde{T} = |p|t_p = qT. \tag{2.34}$$

see (2.23), (2.25), (2.2) and (1.47). This fulfills our expectation that the nonlinear evolution PDE (2.26) possess (a lot of) periodic solutions; note that the set of initial data that, according to the above discussion, yield *completely periodic* solutions $w(x,t)$ (namely, solutions periodic in t with period $\tilde{T}$, see (2.34), for *all real* values of x), have nonvanishing measure among all initial data, $w_0(x)$, since the fundamental restriction characterizing them is validity of the *inequality* (2.30) for all *real* values of x, with the quantity ρ in the right-hand side of the definition (2.29) of $\Delta(x)$ replaced by $2/\omega$.

While this discussion is sufficient to justify our claim that the nonlinear evolution PDE (2.26) possess a lot of completely periodic solutions, more can be said if attention is restricted to special sets of initial data. Assume for instance that the initial datum, $w_0(x)$, is a rational function:

$$w_0(x) = P(x)/Q(x), \tag{2.35}$$

where $P(x)$, $Q(x)$ are two polynomials (and of course $Q(x)$ has no real zeros, $Q(x) \neq 0$ for real x, so that $w_0(x)$, see (2.35), is *nonsingular* for real x). Then (2.27) becomes

$$\varphi\, Q\left(x + \alpha\tau\varphi^{p/q}\right) = P\left(x + \alpha\tau\varphi^{p/q}\right), \tag{2.36}$$

which is a polynomial equation for the unknown quantity

$$y = \varphi^{\mathrm{sign}(p)/q}. \tag{2.37}$$

The degree N of this polynomial equation in y is clearly the largest of the two positive integers $q + N^{(Q)}|p|$, $N^{(P)}|p|$ if p is *positive*, and instead the largest of the two positive integers $N^{(Q)}|p|$, $q + N^{(P)}|p|$ if p is negative, with $N^{(P)}$ respectively $N^{(Q)}$ the degree of the polynomial $P(x)$ respectively $Q(x)$ (of course in order that $w_0(x)$ be localized in space, namely that it vanish as $x \to \pm\infty$, the degree of $Q(x)$ should *exceed* the degree of $P(x)$, $N^{(Q)} > N^{(P)}$). The coefficients of the polynomial equation (2.36) in y, see (2.37), are of course *periodic* in t with period t_p, see (2.1a) and (2.2). Hence all its zeros will also be *periodic*, with the same period t_p, or a period which is a (finite integer) multiple of t_p if during the time evolution the zeros get reshuffled. Hence we can conclude that *all nonsingular* solutions of the nonlinear evolution PDE (2.26) evolving from a *rational* initial datum $w_0(x)$ will be *completely periodic* (note that in this rational case there is no restriction on the overall size of the initial datum $w(x)$).

The third, and last, example we discuss (in this Section 2 meant to illustrate how the trick works) takes as starting point the ("Burgers") evolution PDE

$$\varphi_\tau - \beta\varphi_{\xi\xi} = 2\alpha\varphi_\xi\varphi, \qquad \varphi \equiv \varphi(\xi,\tau). \tag{2.38}$$

The arbitrary (possibly complex, but nonvanishing) constants α, β could be rescaled away, but we prefer to keep them (as well as the standard factor 2 in the right-hand side). Here of course subscripted variables denote partial differentiations.

Note that we have introduced a new "space" variable, ξ. The motivation for doing so is that, in addition to the change of variable (2.1a) (from the *complex* independent variable τ to the *real* variable t, "time"), and to the change of dependent variable analogous to (2.23) and (2.4),

$$w(x,t) = \exp(\mathrm{i}\lambda\omega t)\varphi[\xi(t),\tau(t)], \tag{2.39}$$

it is now convenient to introduce as well a (time-dependent) rescaling of the "space" independent variable, by setting

$$\xi = x\exp(\mathrm{i}\mu\omega t). \tag{2.40}$$

As we will immediately see this has the advantage to yield, via a convenient choice of the two numbers λ and μ, see (2.39) and (2.40), quite a neat form for the nonlinear evolution PDE satisfied by $w(x,t)$ (but there is also a drawback, see below). Note that (2.39) and (2.40) entail, via (2.1c),

$$\varphi(x,0) = w(x,0). \tag{2.41}$$

Clearly (2.39) also entails, via (2.1b) and (2.40),

$$w_t = \mathrm{i}\lambda\,\omega w + \mathrm{i}\mu\omega x\exp[\mathrm{i}(\mu+\lambda)\omega t]\varphi_\xi + \exp[\mathrm{i}(1+\lambda)\omega t]\varphi_\tau, \tag{2.42a}$$

while (2.39) with (2.40) entail

$$w_x = \exp[\mathrm{i}(\mu+\lambda)\omega t]\varphi_\xi, \tag{2.42b}$$

$$w_{xx} = \exp[\mathrm{i}(2\mu+\lambda)\omega t]\varphi_{\xi\xi}. \tag{2.42c}$$

Hence the evolution PDE (2.38) satisfied by $\varphi(\xi, \tau)$ translates into the following evolution PDE satisfied by $w(x, t)$:

$$w_t - \mathrm{i}\lambda\omega w - \mathrm{i}\mu\omega x w_x - \beta \exp[\mathrm{i}(1-2\mu)\omega t] w_{xx} = 2\alpha \exp[\mathrm{i}(1-\mu-\lambda)\omega t] w_x w. \quad (2.43)$$

It is therefore natural to set

$$\lambda = \mu = 1/2, \quad (2.44\text{a})$$

$$\omega = 2\Omega, \quad (2.44\text{b})$$

so that (2.43) take the following neat form (see (1.38)):

$$w_t - \mathrm{i}\Omega(w + x w_x) - \beta w_{xx} = 2\alpha w_x w, \quad (2.45\text{a})$$

or equivalently

$$w_t = \left(\beta w_x + \mathrm{i}\Omega x w + \alpha w^2\right)_x . \quad (2.45\text{b})$$

This is the evolution PDE we expect shall possess a lot of solutions, $w \equiv w(x, t)$, *completely periodic* in the *real* "time" t. This hunch is entailed by the above derivation of this evolution PDE (from the evolution PDE (2.38), via the changes of dependent and independent variables (2.39), (2.1a) and (2.40), with (2.44)), and it is indeed demonstrated by the solutions we exhibit below. Note however that this evolution PDE, (2.45), features an explicit dependence on the space coordinate, x; hence it lacks translation invariance. This explicit dependence originates from the change of (independent, space) variable (2.40), which is on the other hand instrumental to eliminate, via the assignment (2.44a), any explicit time dependence from (2.45): see (2.43), and note that forsaking the change of (space, independent) variable (2.40) from ξ to x amounts to setting $\mu = 0$. But this change of variables, (2.40) with (2.44), has another, possibly unpleasant, consequence: it may cause the solutions $w \equiv w(x, t)$ of (2.45) (which are generally obtained via (2.39) from the solutions of the solvable Burgers equation (2.38)) to lose the property of (space) localization (namely the property to vanish when the *real* variable diverges, $x \to \pm\infty$), and it may also cause the solutions, $w \equiv w(x, t)$, to become *singular* for some finite *real* values of the variables x and t: note that, as x varies over the real axis, $-\infty < x < \infty$, and t varies over one period, say $0 \leq t < t_p$, see (2.2), the variable ξ, see (2.40), sweeps (in fact, twice) the *entire complex* ξ-plane; while of course $\varphi(\xi, \tau)$ cannot be singularity-free in the *entire* complex ξ-plane, if it does depend at all upon the variable ξ; hence the singularities of $\varphi(\xi, \tau)$ in the complex ξ-plane are likely to show up, via (2.39), as singularites of $w(x, t)$ for *real* x and t. This phenomenon, which might severely reduce the applicability of (2.45) (and of all the other evolution PDEs exhibited in this paper for the derivation of which the change of variable (2.40) was instrumental; but note that the evolution PDE discussed above, see (2.26), does *not* belong to this class), is indeed exhibited by many of the solutions of (2.45) displayed below; however, not by *all* these solutions, as we now show.

The solutions $w \equiv w(x, t)$ of (2.45) can be obtained via (2.39) from the solutions $\varphi \equiv \varphi(\xi, \tau)$ of the Burgers equation (2.38). These solutions, $\varphi \equiv \varphi(\xi, \tau)$, can themselves be obtained via the Cole–Hopf transformation,

$$\varphi(\xi, \tau) = \alpha^{-1} \psi_\xi(\xi, \tau) / \psi(\xi, \tau), \quad (2.46)$$

from the solutions of the *linear* ("heat", or "Schrödinger") PDE

$$\psi_\tau = \beta\psi_{\xi\xi}, \psi \equiv \psi(\xi, \tau). \tag{2.47}$$

In this manner one can easily manufacture large classes of solutions $\varphi(\xi, \tau)$ of the Burgers equation (2.38), hence, via (2.39), large classes of solutions $w(x, t)$ of the evolution PDE (2.45); and one can also generally solve, in this manner, the initial-value problem for (2.45). For the reasons explained above, one then expects many of these solutions $w(x, t)$ to depend *periodically* on the real time t, but also to become singular for some *real* values of the "space" and "time" variables x and t. But there also exist solutions of (2.45) which are *nonsingular* for *all real* values of x and t, and which are moreover *localized* in x (namely, such that $w(\pm\infty, t) = 0$) and of course *periodic* in t.

These statements are validated by the following two explicit examples.

The solution $w(x, t)$ of (2.45) that corresponds to the solution

$$\psi(\xi, \tau) = A - \exp\left(k\xi + \beta k^2\tau\right) \tag{2.48}$$

of the *linear* PDE (2.47), reads

$$w(x, t) = -(kB/\alpha)\exp(\mathrm{i}\Omega t) \times \left\{B - \exp\left[-kx\exp(\mathrm{i}\Omega t) + \left[\mathrm{i}\beta k^2/(2\Omega)\right]\exp(2\mathrm{i}\Omega t)\right]\right\}^{-1}, \tag{2.49a}$$

$$B = A^{-1}\exp\left[-\mathrm{i}\beta k^2/(2\Omega)\right]. \tag{2.49b}$$

Here k and B (or, equivalently, k and A, see (2.49b)) are two *arbitrary* (possibly complex) constants. This solution is obviously *periodic* in t with period T, see (1.47); but (as it can be easily shown, see the Appendix) it *always* becomes singular at some *real* value of x and t.

Another interesting class of solutions of (2.45) are characterized by the property to be, for all time, *rational* in x. They are easily obtained, via (2.39), (2.40) and (2.1a), from (2.46) with $\psi(\xi, \tau)$ a solution of the linear PDE (2.47) which is, for all time, polynomial in ξ (say, of degree N):

$$\psi(\xi, \tau) = \sum_{m=0}^{N} \gamma_m(\tau)\xi^{N-m}. \tag{2.50}$$

It is easily seen that such polynomial solutions of (2.47) exist and that their coefficients are characterized by the ODEs

$$\gamma_0' = \gamma_1' = 0, \qquad \gamma_m' = \beta(N - m + 2)(N - m + 1)\gamma_{m-2}, \qquad m = 2, \ldots, N, \tag{2.51}$$

which could be easily solved in explicit form.

These rational solutions $w(x, t)$ of (2.45) are clearly *periodic* in t with period T (see (2.39), (2.40), (2.1a), (2.44) and (1.47)), and they are, for all time, *spatially localized*, namely they vanish as $x \to \pm\infty$, $w(\pm\infty, t) = 0$ (albeit the vanishing is slow, being generally proportional to $|x|^{-1}$ as $x \to \pm\infty$). But is there any such solution which remains *nonsingular* for all *real* values of x and t?

To gain some insight about this question, we note that *all* these rational solutions $w(x,t)$ also admit the following representation:

$$w(x,t) = (\beta/\alpha)\sum_{n=1}^{N}[x - z_n(t)]^{-1}, \tag{2.52}$$

with the N *complex* quantities $z_n(t)$ characterized by arbitrary "initial conditions", $z_n(0) = z_n^{(0)}$, and by the following system of first-order ODEs:

$$\dot{z}_n + \mathrm{i}\Omega z_n = -2\beta \sum_{m=1, m\neq n}^{N} (z_n - z_m)^{-1}. \tag{2.53}$$

As it is well known [3], and as it is indeed implied by the above treatment, these N quantities $z_n(t)$ are simply related to the N zeros of the time-dependent polynomial (2.50); they are of course all periodic functions of time, with period at most $\tilde{T} = T \cdot N!$.

The issue about the existence of *nonsingular* solutions $w(x,t)$ (rational in x) of (2.45) hinges now on the question whether, for given β and (*real!*) Ω, and for some given N, there exist initial (of course *complex*) values $z_n(0)$ such that *none* of the quantities $z_n(t)$ is *real* for any real time. The structure of (2.52) might seem to suggest a negative reply; note for instance that the center-of-mass of the quantities $z_n(t)$,

$$Z(t) = N^{-1}\sum_{n=1}^{N} z_n(t), \tag{2.54}$$

evolves in time according to the equation

$$\dot{Z} + \mathrm{i}\Omega Z = 0, \tag{2.55a}$$

hence according to the law

$$Z(t) = Z(0)\exp(-\mathrm{i}\Omega t), \tag{2.55b}$$

which clearly entails that $Z(t)$, whatever the choice of the initial datum $Z(0)$, at some (*real*) time does indeed become *real*.

But the pessimism about the existence of *nonsingular* solutions $w(x,t)$ (rational in x) of (2.45) turns out to be unwarranted: there do exist such *nonsingular* solutions, as shown by the following example (for $N = 2$). Indeed such a solution $w(x,t)$ of (2.45) reads, as can be easily verified,

$$w(x,t) = 2[x - a\exp(-\mathrm{i}\Omega t)]\left\{[x - a\exp(-\mathrm{i}\Omega t)]^2 - \mathrm{i}(\beta/\Omega) - b\exp(-2\mathrm{i}\Omega t)\right\}^{-1}, \tag{2.55}$$

with a and b two arbitrary (complex) constants; and we show in the Appendix that a condition *sufficient* to guarantee that this solutions, (2.56), be *nonsingular* for *all real* values of x and t, is validity of the *inequality*

$$|\mathrm{Re}\,(\beta)/\Omega| > |b| + 4|a|^2. \tag{2.56}$$

Note that, if $a = 0$, this solution $w(x,t)$, see (2.56), of the evolution PDE (2.45) is an *odd* function of x, and it is *completely periodic* in t with period $T/2$, see (1.47); if $a \neq 0$, this solution $w(x,t)$, see (2.56), has no definite parity as a function of x, and it is *completely periodic* in t with period T, see (1.47).

3 Derivation of evolution equations featuring periodic solutions

In the preceding Section 2 we have discussed in some detail, using three representative examples, the trick whereby evolution equations can be modified so that the modified versions (have perhaps a neat form and) generally feature a lot of, and in some cases only, *periodic* solutions. In this Section 3 we apply this trick to other evolution equations, and we thereby obtain all the evolution equations exhibited in the introductory Section 1. Our treatment below is however somewhat terser than in the preceding Section 2.

We treat firstly ODEs, then PDEs in $1 + 1 = 2$ variables (space and time), and finally one example in $1 + 2 = 3$ variables.

Derivation of (1.5), (1.6) and (1.7.). *All* solutions of the first Painlevé ODE

$$\varphi'' = \alpha\varphi^2 + \beta\tau, \qquad \varphi \equiv \varphi(\tau), \tag{3.1}$$

are *meromorphic* functions of the complex independent variable τ (see, for instance, [4]). The two arbitrary (complex) constants α, β could be eliminated by rescaling the dependent and independent variables, but we prefer to keep them. Of course here and below primes denote differentiations with respect to τ.

We now introduce the new independent variable t ("time") by setting

$$\tau \equiv \tau(t) = -(\mathrm{i}/\omega)\exp(\mathrm{i}\omega t). \tag{3.2}$$

Here and throughout ω is a *positive* number, $\omega > 0$, to which we associate the period t_p, see (2.2). Note the analogy, but also the difference, among (3.2) and (2.1a).

We moreover introduce the new (complex) dependent variable $w \equiv w(t)$ via a relation closely analogous to (2.4):

$$w(t) = \exp(\mathrm{i}\lambda\omega t)\,\varphi[\tau(t)], \tag{3.3}$$

where λ is a number to be chosen conveniently, see below. Clearly these formulas entail that, if λ is a *rational* number, the property of $\varphi(\tau)$ to be *meromorphic* entails that $w(t)$ is *periodic* in t with a period which is an *integer* multiple of t_p, see (2.2).

On the other hand, via (3.2) and (3.3), the evolution ODE (3.1) yields

$$\ddot{w} = \mathrm{i}(2\lambda + 1)\omega\dot{w} + \lambda(\lambda + 1)\omega^2 w + \alpha\exp[\mathrm{i}(2 - \lambda)\omega t]w^2 + \gamma\exp[\mathrm{i}(\lambda + 3)\omega t], \tag{3.4a}$$

where we set

$$\beta = \mathrm{i}\omega\gamma. \tag{3.4b}$$

Of course here and below dots denote differentiations with respect to the (*real*) time t.

Three choices of λ appear of special interest: $\lambda = -1/2$, $\lambda = 2$, respectively $\lambda = -3$. The corresponding evolution ODEs have been reported in Section 1, see (1.5) (where we set $\omega = 2\Omega$), (1.6) (where we set $\omega = \Omega$) respectively (1.7) (where we also set $\omega = \Omega$). The (extremely simple!) periodicity property of their solutions, as implied by (3.2) and (3.3), are mentioned there. Note the difference among the transformations (2.1a) and (3.2), including the fact that, for $\omega \to 0$, (2.1a) becomes the identity $\tau = t$ while (3.2) has no limit. This accounts for the fact that, for $\Omega = 0$, (1.5) does *not* reproduce the Painlevé equation (3.1). Analogous remarks apply to the analogous cases treated below.

Derivation of (1.8). The treatment is closely analogous to that described immediately above, except that one now starts from the second Painlevé ODE [4],

$$\varphi'' = \alpha\varphi^3 + \beta\tau\varphi + \delta, \qquad \varphi \equiv \varphi(\tau), \tag{3.5}$$

rather than from the first, see (3.1). Here again we keep some constants that could be removed by appropriate rescalings. One then uses (3.2) and (3.3), and obtains thereby

$$\ddot{w} = \mathrm{i}(2\lambda+1)\omega\dot{w} + \lambda(\lambda+1)\omega^2 w \\ + \alpha\exp[2\mathrm{i}(1-\lambda)\omega t]w^3 + \gamma\exp(3\mathrm{i}\omega t)w + \delta\exp[\mathrm{i}(2+\lambda)\omega t] \tag{3.6a}$$

with

$$\gamma = -\mathrm{i}\beta/\omega. \tag{3.6b}$$

This suggests setting $\lambda = 1$, $\omega = \Omega$, and one thereby gets (1.8).

Of course the fact [4] that all solutions of the second Painlevé ODE, (3.5), are meromorphic functions of τ entails, via (3.2) and (3.3) (with $\lambda = 1$, $\omega = \Omega$) that *all nonsingular* solutions of (1.8) are periodic functions of the (*real*) time t, with period T, see (1.47).

Derivation of (1.13). One starts from

$$\varphi'' = \alpha\varphi^2 \tag{3.7}$$

and sets

$$w(t) = \exp(\mathrm{i}\Omega t)\varphi(\tau) \tag{3.8a}$$

with τ given by (2.1a) and

$$\Omega = 2\omega \tag{3.8b}$$

(this corresponds to (3.3) with $\lambda = 2$). This entails that the *general* solution of (1.13) reads

$$w(t) = (6/\alpha)\exp(\mathrm{i}\Omega t)\mathcal{P}[-(2\mathrm{i}/\Omega)\exp(\mathrm{i}\Omega t/2) + \beta;\ 0, g_3], \tag{3.9}$$

with β and g_3 arbitrary (complex) constants, and

$$\mathcal{P}(z; g_2, g_3) \equiv \mathcal{P}(z|\omega, \omega') \tag{3.10}$$

the Weierstrass doubly-periodic elliptic function which satisfies the ODE

$$\mathcal{P}''(z) = 6\mathcal{P}^2(z) - g_2/2. \tag{3.11}$$

Note the (standard) notation (3.10), which entails that ω and ω' are the two semiperiods of the Weierstrass function (note that the quantity ω in the right-hand side of (3.10) has nothing to do with the (*real*) quantity ω in (2.1a) and (3.8b)). Also note that a necessary and sufficient condition to guarantee that the solution (3.9) be *nonsingular* for all (*real*) values of the time t is the inequality

$$|\beta| \neq |2/\Omega|, \tag{3.12}$$

and that *all* nonsingular solutions (3.9) are *periodic* in t with period $2T = 4\pi/\Omega$, see (1.47) (only in exceptional cases, namely for special values of β and g_3, they might be *periodic* with period $T = 2\pi/\Omega$, due to the *periodicity* of the Weierstrass function).

Derivation of (1.14a), (1.14b). One starts from the equation

$$\varphi'' = \alpha \exp(\varphi), \tag{3.13}$$

and sets

$$w(t) = 2\mathrm{i}\omega t + \varphi[\tau(t)] \tag{3.14}$$

with $\tau(t)$ given by (2.1a). This yields (1.14a) (with $\omega = \Omega$). The position (3.14) suggests that the solutions $w \equiv w(t)$ of (3.13) are not periodic in t, but that their time-derivatives, $\dot{w}(t)$, are indeed periodic, with period T, see (1.47). This expectation is confirmed by the following expression of the *general* solution of (1.14a),

$$\begin{aligned} w(t) &= a\exp(\mathrm{i}\Omega t) + b + 2\mathrm{i}\Omega t \\ &\quad - 2\log\left\{(\alpha/\Omega^2)\left(2a^2\right)^{-1}\exp(b) + \exp[a\exp(\mathrm{i}\Omega t)]\right\}, \end{aligned} \tag{3.15}$$

which features the two *arbitrary* constants a, b, and which entails

$$\begin{aligned} \dot{w}(t) &= \mathrm{i}\Omega a\exp(\mathrm{i}\Omega t) + 2\mathrm{i}\Omega\Big[1 - a\exp(\mathrm{i}\Omega t)\exp[a\exp(\mathrm{i}\Omega t)] \\ &\quad \times \left\{\left(\alpha/\Omega^2\right)\left(2a^2\right)^{-1}\exp(b) + \exp[a\exp(\mathrm{i}\Omega t)]\right\}^{-1}\Big]. \end{aligned} \tag{3.16}$$

Clearly all nonsingular functions $\dot{w}(t)$, see (3.16), are periodic in t with period T, see (1.47), and it is also easily seen that a condition sufficient to guarantee that the solution $w(t)$, see (3.15), as well of course as its time-derivative, see (3.16), be *nonsingular* for *all* (*real*) times, is validity of the inequality

$$|a| \neq \left|b - \log\left(-2a^2\Omega^2/\alpha\right)\right|. \tag{3.17}$$

To obtain (1.14b), time-differentiate (1.14a), eliminate $\alpha\exp(w)$ using (1.14a), and then replace formally (as a notational change) $\dot{w}(t)$ with $w(t)$. This derivation entails of course that the *general* solution, $w = w(t)$, of (1.14b) is provided by the right-hand side of (3.16); it is clearly *nonsingular* if the inequality (3.17) holds, and periodic in t with period T, see (1.47).

Derivation of (1.20), (1.21), (1.22) and (1.23). Here we take as starting point the ODE

$$\varphi'' = \alpha(\varphi')^{p_2/q_2}\varphi^{p_1/q_1}, \qquad \varphi \equiv \varphi(\tau), \tag{3.18}$$

where, as usual, we keep the *arbitrary* (nonvanishing) constant α in spite of the fact that it could be easily rescaled away, and we introduce the 4 *integers* p_1, q_1, p_2, q_2 to write the two *rational* exponents in the right-hand side of this ODE, (3.18). We do not dwell on the restrictions to be imposed on these integers, which we trust are self-evident from (1.20) as well as from the equations written below.

We now use the change of independent variable (2.1a), as well as the change of dependent variable (3.3), the latter with

$$\lambda = -q_1(2q_2 - p_2)/(q_1q_2 - p_1q_2 - p_2q_1), \tag{3.19a}$$

$$\lambda\omega = \Omega, \tag{3.19b}$$

and we thereby get for $w \equiv w(t)$ the (*autonomous*!) ODE (1.20). Hence the solutions of this ODE, (1.20) , can be obtained, via (2.1) and (3.3) with (3.19), from the general solution of (3.18), which is yielded by the quadrature formula

$$\int^{\varphi} dx \left[x^{(p_1+q_1)/q_1} + a^2\right]^{q_2/(p_2-2q_2)}$$
$$= b + \{-\alpha q_1(p_2 - 2q_2)/[(p_1 + q_1)q_2]\}^{-q_2/(p_2-2q_2)}\tau, \tag{3.20}$$

where a^2 and b are two *arbitrary* constants.

We forsake here a discussion of (3.20) for an arbitrary choice of the two rational numbers p_1/q_1 and p_2/q_2, and we limit our consideration to three examples.

The first obtains by setting $p_1/q_1 = -3$, $p_2/q_2 = 3$, entailing $\lambda = 1$ (see 3.19a)); thereby (1.20) becomes (1.21). In this case (3.20) reads

$$-\varphi^{-1} + a^2\varphi = b + (2/\alpha)\tau, \tag{3.21a}$$

entailing, say,

$$\varphi = \left[b + (2/\alpha)\tau - \left\{[b + (2/\alpha)\tau]^2 + 4a^2\right\}^{1/2}\right] / \left(2a^2\right), \tag{3.21b}$$

so that (see (2.1) and (3.3), (3.19))

$$\varphi(0) = w(0) = \left[b - \left(b^2 + 4a^2\right)^{1/2}\right] / \left(2a^2\right), \tag{3.22a}$$

$$\varphi'(0) = \dot{w}(0) - \mathrm{i}\Omega w(0) = \left[1 - b\left(b^2 + 4a^2\right)^{-1/2}\right] / \left(\alpha a^2\right); \tag{3.22b}$$

and it is easily seen that the initial data, $w(0)$ and $\dot{w}(0)$, of (1.21) are split into two sets, all of which however yield nonsingular solutions periodic with period T, see (1.47). These two sets are separated by a (topologically nontrivial, lower dimensional) set of initial data, characterized, see (3.21) (with (2.1a), (3.19b) and $\lambda = 1$), by the *equalities*

$$|1 + \Omega\alpha(\pm a - \mathrm{i}b/2)| = 1; \tag{3.23}$$

and clearly these special initial data yield solutions which become singular (namely, which are such that $\dot{w}$ diverges) at the *real* times t_b, characterized by the relation

$$\exp(\mathrm{i}\Omega t_b) = 1 + \Omega\alpha(\pm a - \mathrm{i}b/2) \tag{3.24}$$

(the fact that the values of t_b, as defined mod (T) by this equation, (3.24), are *real* is of course implied by (3.23)).

The second example obtains by setting

$$p_1/q_1 = m, \qquad p_2/q_2 = (2n+1)/n, \tag{3.25a}$$

entailing (see (3.19a))

$$\lambda = -(nm+n+1)^{-1}, \tag{3.25b}$$

with m a *nonnegative* integer and n a *positive* integer; thereby (1.20) becomes (1.22). In this case (3.20) reads

$$P_{nm+n+1}(\varphi) = b + \left\{-[n(m+1)]^{-1}\alpha\right\}^{-n}\tau, \tag{3.26}$$

where $P_{nm+n+1}(\varphi)$ is a polynomial of degree $nm+n+1$ in φ, whose coefficients are time-independent (for given n and m, they only depend on the constant a). Since τ, see (2.1), is a periodic function of t with period t_p, see (2.2), clearly the set of the $nm+n+1$ roots of (3.26) is also periodic with the same period; but each root, if followed continuously as function of t, need only be periodic with period $t_p \cdot (nm+n+1)!$. Hence we may conclude that *all nonsingular* solutions $w(t)$ of (1.22) are *periodic* with a period $\tilde{T}$ which is at least the minimum common integer multiple among T, see (1.47), and

$$t_p = T/(nm-n+1), \tag{3.27a}$$

and at most the minimum common integer multiple among T, see (1.47), and

$$\tilde{t}_p = t_p(nm+n+1)! = T(nm+n)! \tag{3.27b}$$

(see (3.19), (3.25) and (1.47)). The singular solutions correspond to lower dimensional sets of initial data $w(0)$, $\dot{w}(0)$, such that, for some *real* time, (at least) 2 roots of (3.26) coincide, at which time $\dot{w}(t)$ diverges.

The third example obtains by setting

$$p_1/q_1 = -(2m+1), \qquad p_2/q_2 = (2n+1)/n, \tag{3.28a}$$

entailing (see (3.19a))

$$\lambda = (2nm-1)^{-1}, \tag{3.28b}$$

with m and n *positive* integers; thereby (1.20) becomes (1.23). In this case (3.20) reads

$$P_{2mn-1}(\varphi) = \left[b + (2mn)^n\alpha^{-n}\tau\right]\varphi^{2mn-1}. \tag{3.29}$$

Hence, by an analysis closely analogous to that given above, in this case we also conclude that *all nonsingular* solutions $w(t)$ of (1.23) are *periodic* with a period $\tilde{T}$ which is at least the minimum common integer multiple among T, see (1.47), and

$$t_p = T/(2nm-1) \tag{3.30a}$$

and at most the minimum common integer multiple among T, see (1.47), and

$$\tilde{t}_p = t_p(2nm-1)! = T(2nm-2)! \tag{3.30b}$$

(see (3.19), (3.28) and (1.47)).

Derivation of (1.24), (1.25) and (1.26). We take as starting point the evolution ODE

$$\varphi'' = \alpha\varphi'\varphi + \beta\varphi^3, \qquad \varphi \equiv \varphi(\tau), \tag{3.31}$$

and make the (by now standard) changes of dependent and independent variables (3.3) and (2.1). This yields

$$\ddot{w} = \mathrm{i}(2\lambda+1)\omega\dot{w} + \lambda(\lambda+1)\omega^2 w$$
$$+ \alpha\exp[\mathrm{i}(1-\lambda)\omega t]w(\dot{w} - \mathrm{i}\lambda\omega w) + \beta\exp[2\mathrm{i}(1-\lambda)\omega t]w^3, \tag{3.32}$$

and this yields (1.24) by setting

$$\lambda = 1, \tag{3.33a}$$
$$\omega = \Omega. \tag{3.33b}$$

The other two ODEs, (1.25) respectively (1.26), are clearly the two special cases of (1.24) corresponding to $\beta = -\alpha^2/9$ respectively $\beta = 0$. The motivation for singling out these two cases is because, by setting

$$\varphi(\tau) = \gamma\psi'(\tau)/\psi(\tau), \tag{3.34a}$$
$$\gamma = \left[\alpha - \left(\alpha^2 + 8\beta\right)^{1/2}\right]/(2\beta), \tag{3.34b}$$

one transforms (3.31) into

$$\psi'''\psi = \eta\psi'\psi'', \tag{3.35a}$$
$$\eta = \alpha^2\left\{1 + 6\left(\beta/\alpha^2\right) - \left[1 + 8\left(\beta/\alpha^2\right)\right]^{1/2}\right\}/(2\beta), \tag{3.35b}$$

and for $\beta = -\alpha^2/9$ respectively $\beta = 0$ one gets from (3.35b) $\eta = 0$ respectively $\eta = 1$, two values for which (3.35a) is particularly easy to integrate.

Indeed in the first case, $\beta = -\alpha^2/9$, $\eta = 0$ (which entails $\gamma = -3/\alpha$, see (3.34b)) one gets from (3.35a)

$$\psi(\tau) = a + b\tau + \tau^2 \tag{3.36}$$

hence (see (3.34a))

$$\varphi(\tau) = -(3/\alpha)(b + 2\tau)/\left(a + b\tau + \tau^2\right) \tag{3.37}$$

with a and b two *arbitrary* constants. Via (3.3) (with (3.33)) and (2.1) this entails

$$w(t) = 3\mathrm{i}(\Omega/\alpha)[(2-B)\exp(\mathrm{i}\Omega t) - 2\exp(2\mathrm{i}\Omega t)]$$
$$\times [1 - A - B + (B-2)\exp(\mathrm{i}\Omega t) + \exp(2\mathrm{i}\Omega t)]^{-1} \tag{3.38a}$$

with A and B,

$$A = a\omega^2, \qquad B = \mathrm{i}\omega b, \tag{3.38b}$$

two arbitrary constants. This is the general solution of (1.25); it is clearly *periodic* in t with period T, see (1.47), and it is *nonsingular* for *all (real)* values of t provided the following *inequality* holds:

$$\left|1 - B/2 \pm \left[(B/2)^2 + A\right]^{1/2}\right| \neq 1. \tag{3.39}$$

In the second case, $\beta = 0$, the ODE (3.28) is easily integrated, to yield

$$\varphi(\tau) = (a/\alpha)[b + \exp(a\tau)]/[b - \exp(a\tau)], \tag{3.40}$$

with a and b arbitrary constants. Via (3.3) and (2.1) this entails

$$w(t) = \mathrm{i}\Omega(A/\alpha)\exp(\mathrm{i}\Omega t)\{B + \exp[A\exp(\mathrm{i}\Omega t)]\}/\{B - \exp[A\exp(\mathrm{i}\Omega t)]\} \tag{3.41a}$$

with A and B,

$$A = -\mathrm{i}a/\Omega, \qquad B = b\exp(A), \tag{3.41b}$$

two *arbitrary* constants. This is the general solution of (1.26); it is clearly *periodic* in t with period T, see (1.47), and it is *nonsingular* provided the following *inequality* holds:

$$|A| \neq |\log(B)| \tag{3.42}$$

(see (3.41)).

Derivation of (1.29), (1.30), (1.31) and (1.32). Time-differentiation of (1.6) yields

$$\dddot{w} - 5\mathrm{i}\Omega\ddot{w} - 6\Omega^2\dot{w} = 2\alpha w\dot{w} + 5\mathrm{i}\Omega\gamma\exp(5\mathrm{i}\Omega t), \tag{3.43}$$

and using again (1.6) to eliminate the last term in the right-hand side of this equation, (3.43), one gets (1.29). Note that the constant γ, see (1.6), does not appear in (1.29). Hence *all* solutions of (1.29) also satisfy (1.6) (for some appropriate value of γ); or, equivalently, the solution of the initial-value problem for (1.29) (namely, of the problem to evaluate the solution $w(t)$ of (1.29) which corresponds to given initial data $w(0)$, $\dot{w}(0)$, $\ddot{w}(0)$) is provided by the solution of the initial-value problem for (1.6) (with the same data $w(0)$, $\dot{w}(0)$, and with $\gamma = \ddot{w}(0) - 5\mathrm{i}\Omega\dot{w}(0) - 6\Omega^2 w(0)$). Hence the solutions of (1.29) have the same periodicity properties as the solutions of (1.6).

As for (1.30), it is merely another avatar of (1.29). Indeed if, in this ODE, (1.29), we set $w(t) = z(t) + c$, $\dot{c} = 0$, we get

$$\begin{aligned}\dddot{z} - 10\mathrm{i}\Omega\ddot{z} - \left(31\Omega^2 + 2\alpha c\right)\dot{z} \\ + 10\mathrm{i}\left(3\Omega^2 + \alpha c\right)\Omega z + 5\mathrm{i}\left(6\Omega^2 + \alpha c\right)\Omega c = \alpha(2\dot{z} - 5\mathrm{i}\Omega z)z,\end{aligned} \tag{3.44}$$

which is a more general avatar of (1.29) than (1.30), to which it reduces for $\alpha c = -6\Omega^2$.

The derivation of (1.31) from (1.7) is completely analogous to the derivation given above of (1.29) from (1.6); and the comments given above on the possibility to obtain *all* the solutions of (1.29) from those of (1.6) are as well applicable now, except that the role previously played by the constant γ (which appears in (1.6) but not in (1.29)) is now played by the constant α (which appears in (1.7) but not in (1.31)).

The starting point to obtain (1.32) is (1.5), which we now write as follows (by setting $\gamma = \alpha\eta$):

$$\ddot{w} + \Omega^2 w = \alpha \left(w^2 + \eta\right) \exp(5\mathrm{i}\Omega t). \tag{3.45}$$

We now time-differentiate this ODE, (3.45), and then eliminate the explicitly time-dependent term using again (3.45). This yields (1.32). Again we note that the constant α, which appears in (3.45), has dropped out of (1.32); hence we may again conclude that *all* solutions of (1.32) also satisfy (3.45) (with an appropriate value of α, including $\alpha = 0$), and that the solution of the initial-value problem for (1.32) can be obtained from the solution of the initial-value problem for (3.45) (or, equivalently, (1.5)).

Derivation of (1.40), (1.41), (1.42), (1.43) and (1.44). Here we take as starting point the evolution PDE

$$\varphi_\tau = \beta\varphi_{\xi\xi\xi} + \alpha\varphi_\xi\varphi^{p/q}, \qquad \varphi \equiv \varphi(\xi, \tau), \tag{3.46}$$

with α and β two *arbitrary* (possibly complex) constants and p, q two arbitrary *integers* ($q \neq 0$). We then use the transformation formulas (2.1), (2.39) and (2.40). We thereby get, for the new dependent variable $w \equiv w(x, t)$,

$$\begin{aligned} w_t = {} & \mathrm{i}\lambda\omega w + \mathrm{i}\mu\omega x w_x \\ & + \beta \exp[\mathrm{i}(1 - 3\mu)\omega t] w_{xxx} + \alpha \exp\{\mathrm{i}[1 - \mu - \lambda(p/q)]\omega t\} w_x w^{p/q}. \end{aligned} \tag{3.47}$$

Hence we set

$$\mu = 1/3, \qquad \lambda = 2p/(3q), \qquad \omega = 3\Omega, \tag{3.48}$$

and we thereby get (1.40).

The evolution PDE (1.41) is the special case of (1.40) with $q = p = 1$; for $\Omega = 0$ it reduces to the well-known (*integrable*!) "Korteweg-de Vries" (KdV) equation.

Likewise, the evolution PDE (1.43) is the special case of (1.40) with $q = 1$, $p = 2$; for $\Omega = 0$ it reduces to the well-known (*integrable*!) "modified Korteweg-de Vries" equation.

As for the two systems of two *real* PDEs (1.42) respectively (1.44), they are clearly the *real* avatars of (1.41) respectively (1.43), obtained by setting $w = u + \mathrm{i}v$, $\alpha = a_1 + \mathrm{i}a_2$, $\beta = b_1 + \mathrm{i}b_2$.

Finally, to obtain (1.45), as well as (1.46) which is just the *real* version of the *complex* equation (1.45) (via $w = u + \mathrm{i}v$, $\alpha = a_1 + \mathrm{i}a_2$, $\beta = b_1 + \mathrm{i}b_2$, $\gamma = c_1 + \mathrm{i}c_2$), we start from the Kadomtsev–Petviashvili (KP) equation

$$(\varphi_\tau + \beta\varphi_{\xi\xi\xi} + \alpha\varphi_\xi\varphi)_\xi + \gamma\varphi_{\eta\eta} = 0 \tag{3.49}$$

and we then set

$$w(x, y, t) = \exp(\mathrm{i}\Omega t)\varphi(\xi, \eta, \tau) \tag{3.50a}$$

with

$$\xi = x \exp(\mathrm{i}\Omega t/2), \tag{3.50b}$$

$$\eta = y \exp(\mathrm{i}\Omega t), \tag{3.50c}$$

and $\tau \equiv \tau(t)$ given by (2.1a) with

$$\Omega = 2\omega, \qquad \omega = \Omega/2. \tag{3.50d}$$

Appendix

In this Appendix we prove some results whose detailed treatment in the context of the paper would have been too distracting.

The first result we prove is that the solution (2.49) of the evolution PDE (2.45) is singular for some *real* value of x and t, namely that, for *any* choice of the *complex* numbers α, β, k and B, and of the *real* number Ω, there exist some *real* value of x and t (the latter, of course, defined mod (T), see (1.47)) such that

$$B = \exp\left\{-kx \exp(\mathrm{i}\Omega t) + \left[\mathrm{i}\beta k^2/(2\Omega)\right] \exp(2\mathrm{i}\Omega t)\right\}. \tag{A.1a}$$

Indeed this equation entails

$$\log(B) = -kx \exp(\mathrm{i}\Omega t) + \left[\mathrm{i}\beta k^2/(2\Omega)\right] \exp(2\mathrm{i}\Omega t) \tag{A.1b}$$

namely

$$x = a \exp(-\mathrm{i}\Omega t) + b \exp(\mathrm{i}\Omega t), \tag{A.1c}$$

with

$$a = -k^{-1} \log(B), \qquad b = \mathrm{i}\beta k/(2\Omega). \tag{A.1d}$$

We now show that, for any arbitrary choice of the two complex numbers a, b in (A.1c), and for an appropriate choice of the *real* quantity Ωt, the solution (A.1c) of (A.1a) is *real*. Indeed (A.1c) entails

$$\mathrm{Re}\,(x) = [\mathrm{Re}\,(a) + \mathrm{Re}\,(b)] \cos(\Omega t) + [\mathrm{Im}\,(a) - \mathrm{Im}\,(b)] \sin(\Omega t), \tag{A.1e}$$

$$\mathrm{Im}\,(x) = [-\mathrm{Re}\,(a) + \mathrm{Re}\,(b)] \sin(\Omega t) + [\mathrm{Im}\,(a) + \mathrm{Im}\,(b)] \cos(\Omega t). \tag{A.1f}$$

Now choose t (mod (T), see (1.47)) so that

$$\tan(\Omega t) = [\mathrm{Im}\,(a) + \mathrm{Im}\,(b)]/[\mathrm{Re}\,(a) - \mathrm{Re}\,(b)], \tag{A.2a}$$

entailing

$$\sin(\Omega t) = [\mathrm{Im}\,(a) + \mathrm{Im}\,(b)]/D, \tag{A.2b}$$

$$\cos(\Omega t) = [\mathrm{Re}\,(a) - \mathrm{Re}\,(b)]/D, \tag{A.2c}$$

$$D = \left\{[\mathrm{Im}\,(a) + \mathrm{Im}\,(b)]^2 + [\mathrm{Re}\,(a) - \mathrm{Re}\,(b)]^2\right\}^{1/2}. \tag{A.2d}$$

Note that a *real* choice of t such that (A.2a) hold can *always* be made; and, via (A.1f), this choice entails that $\mathrm{Im}\,(x)$ vanishes, namely that x is *real*; indeed x is then given, via (A.2b), (A.2c), (A.2d), by the explicit expression

$$x = \left(|a|^2 - |b|^2\right)/D. \tag{A.3}$$

This completes our proof.

Next we prove that the inequality (2.57) is (of course for Ω *real*) sufficient to guarantee that the solution $w(x,t)$, see (2.56), is *nonsingular* for all *real* values of x and t, namely that the two roots

$$x_s(t) = a\exp(-\mathrm{i}\Omega t) + s[\mathrm{i}(\beta/\Omega) + b\exp(-2\mathrm{i}\Omega t)]^{1/2}, \qquad s = \pm, \tag{A.4}$$

of the polynomial in x in the denominator in the right-hand side of (2.56) have nonvanishing imaginary parts for *all* (*real*) values of the time t, if the inequality (2.57) holds.

To prove this, we note first of all that the reality of Ω and t obviously entails

$$|\mathrm{Im}\,[\mathrm{i}(\beta/\Omega) + b\exp(-2\mathrm{i}\Omega t)]| \geq |\mathrm{Re}\,(\beta)/\Omega| - |b|, \tag{A.5a}$$

hence (if $|\mathrm{Re}\,(\beta)/\Omega| > |b|$, as indeed implied by (2.57))

$$\left|\mathrm{Im}\,[\mathrm{i}(\beta/\Omega) + b\exp(-2\mathrm{i}\Omega t)]^{1/2}\right| > (|\mathrm{Re}\,(\beta)/\Omega| - |b|)^{1/2}/2, \tag{A.5b}$$

hence

$$\left|\mathrm{Im}\,\left\{[a\exp(-\mathrm{i}\Omega t)] \pm [\mathrm{i}(\beta/\Omega) + b\exp(-2\mathrm{i}\Omega t)]^{1/2}\right\}\right| \\ > (|\mathrm{Re}\,(\beta)/\Omega| - |b|)^{1/2}/2 - |a|. \tag{A.5c}$$

This entails, via (A.4),

$$|\mathrm{Im}\,[x_s(t)]| > (|\mathrm{Re}\,(\beta)/\Omega| - |b|)^{1/2}/2 - |a|, \qquad s = \pm, \tag{A.5d}$$

hence, if (2.57) holds,

$$|\mathrm{Im}\,[x_s(t)]| > 0, \qquad s = \pm, \tag{A.6}$$

QED.

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