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Kovalevski Exponents and Integrability Properties in Class A Homogeneous Cosmological Models

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Abstract

Qualitative approach to homogeneous anisotropic Bianchi class A models in terms of dynamical systems reveals a hierarchy of invariant manifolds. By calculating the Kovalevski Exponents according to Adler - van Moerbecke method we discuss how algebraic integrability property is distributed in this class of models. In particular we find that algebraic nonintegrability of vacuum Bianchi VII_0 model is inherited by more general Bianchi VIII and Bianchi IX vacuum types. Matter terms (cosmological constant, dust and radiation) in the Einstein equations typically generate irrational or complex Kovalevski exponents in class A homogeneous models thus introducing an element of nonintegrability even though the respective vacuum models are integrable.

1 Introduction

Einstein's theory of General Relativity provides a very elegant geometric picture of gravitational interaction. However, such a formulation becomes possible at the expense of making the field equations strongly nonlinear [1]. Even in rather symmetric case of spatially homogeneous space-times the dynamics of their general types (mixmaster models) turns out to be chaotic [2].

On the other hand, growing importance of non-linearity in fundamental as well as in phenomenological physical laws and richness of structure induced by non-linear phenomena motivated strong interest in techniques allowing one to detect and properly describe the complexity inherent to non-linear systems. One such approach aimed at investigating the integrability of non-linear systems has been initiated in works of Kovalevski [3] who in her famous contribution to the problem of rotating rigid body considered the behaviour of the solution near the essential singularities in the complex plane. Similar intuition that the position of singular points (essential singularities) depends on (arbitrary) integration constants and consequently that their mobility precludes the construction of first integrals was formulated and developed by Painlevé [5]. For the review of singularity analysis, Painlevé property and integrability of nonlinear systems see [5].

Yoshida [4] developed precise methods of investigating the restricted case of algebraic integrability of homogeneous dynamical systems. In particular he obtained a necessary condition for algebraic integrability in terms of Kovalevski exponents (i.e. the nonexistence of irrational or complex KEs). Kovalevski Exponents [3, 4] allow for reconstructing a set of essential singularities for nonintegrable systems without need of constructing the solution. The usefulness of calculating the Kovalevski exponents has also been emphasized by Adler and van Moerbecke [6]. They gave the criteria for algebraic complete integrability for Euclidean Toda-like systems. Pavlov [7] essentially extended Adler-van Moerbecke formula into the form suitable for studying indefinite Toda lattices. Thus from the existence of irrational or complex KEs we shall deduce that the system does not have additional algebraic first integrals (i.e. besides the energy integral). Of course the nonexistence of such first integrals does not imply nonintegrability (see e.g. [5]). On the other hand when the KEs are integer numbers we not only know that corresponding first integral exists but we are able to construct its form.

In this paper we calculate the Kovalevski Exponents (KEs) in the wide class of homogeneous Bianchi spacetimes represented as hamiltonian systems with indefinite kinetic energy form and exponential potential term. Our technique is a generalization of the methods previously applied in studies of Euclidean Toda systems [8]. In the present paper we applied this method for the wide class of Bianchi models with matter and cosmological constant. Our investigations are complementary to those of Demaret and Sheen [9] who performed the Painlevé perturbative test for the Bianchi Type IX model with perfect fluid, radiation and cosmological constant in order to predict some probable chaotic regimes. In all cases but the cosmological constant case, the models studied do not pass the Painlevé test, exhibit infinitely many movable logarithmic singularities and are therefore probably chaotic. In [10] (see also [11]) it was shown that the perturbation of an exact solution exhibits a movable transcendental essential singularity thus proving the nonintegrability. Further numerical evidence of the nonintegrability of the Mixmaster model by demonstrating the existence of complicated dense essential singularity patterns and infinitely-sheeted solutions with sensitive dependence on initial conditions can be found in the paper by Bountis and Drossos [12].

It is known that there exists a hierarchy in the class of homogeneous cosmological models dictated by the dimensionality of isometry groups acting on constant time hypersurfaces [15, 16]. This was in fact the strongest argument supporting the claim that mixmaster models represent a generic dynamical regime near the cosmological singularity [2, 16]. We shall explore in this paper how this structure is reflected in the property of algebraic integrability in the class of homogeneous anisotropic cosmological models.

2 Kovalevski Exponents in brief

In this section we shall recall the main facts concerning the Kovalevski Exponents. For this purpose, let us consider a system of first order ordinary differential equations:

$$\frac{dx_i}{dt} = F_i(x_1, \dots, x_n), \tag{2.1}$$

where the right hand sides F_i are rational functions of $x_1, \dots, x_n$. Moreover, assume that the functions F_i are weighted homogeneous with weights M_i i.e.

$$F_i(\alpha^{g_1} x_1, \dots, \alpha^{g_n} x_n) = \alpha^{M_i} F_i(x_1, \dots, x_n),$$

where $M_i = g_i + 1$. Then the system (2.1) is similarity invariant and there exists a particular solution

$$x_1 = c_1 t^{-g_1}, \ \dots, \ x_n = c_n t^{-g_n}, \tag{2.2}$$

where the constants c_i satisfy algebraic equations:

$$F_i(c_1, \dots, c_n) = -g_i c_i, \qquad i = 1, \dots, n. \tag{2.3}$$

When we consider variational equations of the initial system (2.1) about the reference solution (2.2) and substitute $x_i = (c_i + \zeta_i)t^{-g_i}$ we obtain the linearized system

$$t\frac{d\zeta_i}{dt} = \sum_{j=1}^{n} K_{ij}\zeta_i,$$

where

$$K_{ij} = \frac{\partial F_i}{\partial x_j}(c_1, \dots, c_n) + \delta_{ij} g_i.$$

The eigenvalues $\rho_1, \dots, \rho_n$ of the K_{ij} matrix are called the *Kovalevski Exponents.* In [4] it has been proven that a *necessary condition* for a similarity invariant system (2.1) to be algebraically integrable is that *all* Kovalevski Exponents be *rational* numbers. In other words, if at least one Kovalevski Exponent is irrational or imaginary then the system (2.1) does not have algebraic first integrals. This result is central to our subsequent investigation.

3 Bianchi class A models as indefinite Toda lattices

Einstein field equations for homogeneous Bianchi class A models as well as for diagonal class B models can be cast into a hamiltonian form [15]

$$\mathcal{H} = \frac{1}{2}\left(-p_\alpha^2 + p_{\beta_-}^2 + p_{\beta_+}^2\right) + \exp(4\alpha)V(\beta_+, \beta_-), \tag{3.1}$$

where

$$\begin{aligned} V(\beta_+, \beta_-) = {} & n_3^2 \exp(-8\beta_+) + n_1^2 \exp(4\beta_+ + 4\sqrt{3}\beta_-) + \\ & + n_2^2 \exp(4\beta_+ - 4\sqrt{3}\beta_-) - 2n_1 n_2 \exp(4\beta_+) - \\ & - 2n_1 n_3 \exp(-2\beta_+ + 2\sqrt{3}\beta_-) - 2n_2 n_3 \exp(-2\beta_+ - 2\sqrt{3}\beta_-). \end{aligned}$$

The variables α, β_+, β_- have the meaning of expansion factor and anisotropy parameters respectively [1] whereas n_1, n_2, n_3 parametrize the structure constants of isometry groups (defining the type of the spacetime).

Hamiltonian (3.1) is a *generalized indefinite Toda lattice* i.e. it is of the type

$$\mathcal{H} = \frac{1}{2}\langle p, p\rangle + \sum_{i=1}^{N} c_i \exp{(a_i, q)}, \tag{3.2}$$

where $\langle \cdot\,,\,\cdot\rangle$ is a *Lorenzian* scalar product and $(\cdot\,,\,\cdot)$ is an Euclidean scalar product, $\vec{a}$ is a real vector.

4 Kovalevski Exponents for indefinite Toda Lattices

Given generalized Toda lattice let us define new variables

$$u_i = \langle a_i, p\rangle, \qquad v_i = \exp{(a_i, q)}.$$

Transformation between (a, p) and (u, v) variables is a generalization of Flaska transformation.

Then one obtains an autonomous dynamical system with polynomial right hand sides

$$\dot{v}_i = u_i v_i, \qquad \dot{u}_i = \sum_{j=1}^{N} M_{ij} v_j, \qquad M_{ij} = -c_j \langle a_i, a_j\rangle. \tag{4.1}$$

The system (4.1) is of the type (2.1) and one can easily check that it is similarity invariant and possesses a particular solution (see (2.2)):

$$u_i = \frac{U_i}{t}, \qquad v_i = \frac{V_i}{t^2},$$

where the constants U_i, V_i fulfill the following algebraic equations (see (2.3)):

$$-2V_i = U_i V_i, \qquad -U_i = \sum_{j=1}^{n} M_{ij} V_j. \tag{4.2}$$

In order to discuss the single-valuedness of solutions one investigates behavior of solutions in the vicinity of particular solutions. Let us hence consider the first variation

$$\frac{d}{dt}\delta u_i = \sum_{j=1}^{n} M_{ij}\delta v_j, \qquad \frac{d}{dt}\delta v_i = \frac{U_i \delta v_i}{t} + \frac{V_i \delta u_i}{t^2}. \tag{4.3}$$

We seek for the particular solutions of (4.3) in the form

$$\delta u_i = \xi_i t^{\rho-1}, \qquad \delta v_i = \eta_i t^{\rho-2},$$

where ξ_i, η_i are constants and ρ are called Kovalevski Exponents and satisfy the following system of algebraic equations:

$$(\rho - 2 - U_i)\eta_i = V_i \xi_i, \qquad (\rho - 1)\xi_i = \sum_{j=1}^{n} M_{ij}\eta_j. \tag{4.4}$$

The general procedure for extracting the Kovalevski Exponents is to solve the systems (4.2) and (4.4).

Some specific cases can clearly be distinguished:

(i) $\exists\, i : V_i \neq 0;\ \forall\, i \neq j : V_j = 0$,

(ii) $\exists\, (i,j) : V_i \neq 0, V_j \neq 0;\ \forall\, k \neq i, k \neq j : V_k = 0$,

(iii) $\exists\, (i,j,k) : V_i \neq 0, V_j \neq 0, V_k \neq 0;\ \forall\, l \neq i, l \neq j, l \neq k : V_l = 0$,

and so on until the dimension N of the system is reached. Integer Kovalevski Exponents are necessary for solution to be meromorphic.

In the case (i) $\rho_1 = -1$ and $\rho_2 = 2$ are always solutions, other Kovalevski Exponents can be calculated from the generalized Adler - van Moerbecke formula:

$$\rho = 2 - \frac{M_{ij}}{M_{jj}}, \tag{4.5}$$

where $i \neq j$, $M_{jj} \neq 0$ and j is such that $V_j \neq 0$.

In the case (ii) one should solve the system:

$$\left[-M_{jj} + \frac{\rho(\rho-1)}{V_j}\right]\left[\frac{\rho(\rho-1)}{V_i} - M_{ii}\right] - M_{ij}M_{ji} = 0. \tag{4.6}$$

For this purpose it is useful to build an auxillary matrix

$$\hat{M}_{ij} = \left(\begin{array}{cc} M_{ii} & M_{ij} \\ M_{ji} & M_{jj} \end{array}\right).$$

5 Vacuum Bianchi class A models

In this class of model we have:

$$\vec{a_1} = (4,-8,0), \qquad \vec{a_2} = \left(4,4,4\sqrt{3}\right), \qquad \vec{a_3} = \left(4,4,-4\sqrt{3}\right),$$
$$\vec{a_4} = (4,4,0), \qquad \vec{a_5} = \left(4,-2,2\sqrt{3}\right), \qquad \vec{a_6} = \left(4,-2,-2\sqrt{3}\right)$$

and

$$\vec{c} = \left(n_1^2, n_2^2, n_3^2, -2n_1n_2, -2n_2n_3, -2n_1n_3\right).$$

The respective Cartan matrix reads:

$$M_{ij} = -48\left[\begin{array}{cccccc} n_1^2 & -n_2^2 & -n_3^2 & 2n_1n_2 & 0 & 0 \\ -n_1^2 & n_2^2 & -n_3^2 & 0 & 0 & 2n_1n_3 \\ -n_1^2 & -n_2^2 & n_3^2 & 0 & 2n_2n_3 & 0 \\ -n_1^2 & 0 & 0 & 0 & n_2n_3 & n_1n_3 \\ 0 & 0 & -n_3^2 & n_1n_2 & 0 & n_1n_3 \\ 0 & -n_2^2 & 0 & n_1n_2 & n_2n_3 & 0 \end{array}\right]. \tag{5.1}$$

Application of the generalized Adler - van Moerbecke formula (whenever there is only one $V_i \neq 0$) shows that all Kovalevski Exponents (corresponding to this case) are integer and equal:

$$\rho_1 = -1, \qquad \rho_2 = 2, \qquad \rho_3 = 4.$$

Let us now check if $\hat{M}_{ij}$ is degenerate. For this purpose we build an auxillary traceless symmetric matrix,

$$[\det \hat{M}_{ij}] = \begin{bmatrix} 0 & 0 & 0 & 96n_1^3n_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 96n_1n_2^2n_3 \\ 0 & 0 & 0 & 0 & 96n_2n_3^3 & 0 \\ 96n_1^3n_2 & 0 & 0 & 0 & -48n_1n_2^2n_3 & -48n_1^2n_2n_3 \\ 0 & 0 & 96n_2n_3^3 & -48n_1n_2^2n_3 & 0 & -48n_1n_2n_3^2 \\ 0 & 96n_1n_2^2n_3 & 0 & -48n_1^2n_2n_3 & -48n_1n_2n_3^2 & 0 \end{bmatrix}. \quad (5.2)$$

An inspection of (5.2) allows us to formulate the following

Conclusions

- for Bianchi I and Bianchi II models every block matrix is degenerate;
- for Bianchi VI_0 and Bianchi VII_0 models there is only one pair (V_1, V_4) leading to nondegenerate matrix $\hat{M}_{14}$; respective Kovalevski Exponents read: $\mathrm{B(VI_0)}$ $\rho_{1,2} = \frac{1}{2}(1 \pm i\sqrt{15})$, $\rho_3 = -1$, $\rho_4 = 2$ and for $\mathrm{B(VII_0)}$: $z = \rho(\rho - 1) = 1 \pm i\sqrt{7}$;
- from mixmaster models (i.e. Bianchi IX and VIII) there is six pairs of (V_i, V_j) leading to nondegenerate $\hat{M}_{ij}$; respective Kovalevski exponents are the same for Bianchi IX and VIII models and read: $z = 1 \pm i\sqrt{7}$ for pair $(1, 4)$ and $z = \pm 2$ for pairs $(3, 5)$; $(4, 5)$; $(4, 6)$; $(5; 6)$. Because $z = \rho(\rho - 1)$ then $z = 2$ corresponds to $\rho_1 = -1$, $\rho_2 = 2$ and $z = -2$ corresponds to $\rho_{1,2} = \frac{1}{2}(1 \pm i\sqrt{7})$.

Corollary 5.1. *Bianchi type* II *model is algebraically integrable.*

Corollary 5.2. *Mixmaster models inherit their algebraic nonintegrability from Bianchi* VII_0 *model.*

6 Homogeneous models with cosmological constant, dust and radiative matter

One can now consider cosmological models with nonvanishing matter terms. They are described by the hamiltonian:

$$\begin{aligned}\mathcal{H} = \frac{1}{2}\left(-p_\alpha^2 + p_{\beta_+}^2 + p_{\beta_-}^2\right) + \exp(4\alpha)V(\beta_+, \beta_-) + \\ + \Lambda\exp(6\alpha) + \mu_{\mathrm{dust}}\exp(3\alpha) + \Gamma_{\mathrm{rad}}\exp(2\alpha),\end{aligned}$$

where Λ as usually denotes the cosmological constant and the subsequent terms denote other matter sources i.e. the amount of energy-momentum carried by dust and radiative matter, respectively.

6.1 Models with cosmological constant

Let us consider the case $\Lambda \neq 0$, $\Gamma_{\mathrm{rad}} = \mu_{\mathrm{dust}} = 0$. In addition to six $\vec{a}_i$ vectors in the decomposition (3.2) there exists now one more vector:

$$\vec{a}_7 = (6, 0, 0)$$

and subsequent formulae (5.1) and (5.2) are modified correspondingly. From the formula (4.5) one can calculate the following additional Kovalevski Exponents:

$$\rho_i = 2 - 2\frac{\langle \vec{a}_i, \vec{a}_7\rangle}{\langle \vec{a}_7, \vec{a}_7\rangle} = \frac{2}{3} \qquad \text{for} \quad i = 1, \ldots, 6,$$
$$\rho_i = 2 - 2\frac{\langle \vec{a}_7, \vec{a}_i\rangle}{\langle \vec{a}_i, \vec{a}_i\rangle} = 3 \qquad \text{for} \quad i = 1, 2, 3.$$

Other Kovalevski Exponents obtained from (4.6) are:

$$z_{1,2} = -1 \pm \frac{\sqrt{7}}{2}, \qquad z_{1,2} = -\frac{3}{2} \pm \frac{\sqrt{7}}{2}.$$

Recall that $z = \rho(\rho-1)$. Hence we obtained irrational and complex values of the Kovalevski Exponents. Their existence does not depend on the value of cosmological constant. This proves that the presence of Λ introduces an element of nonintegrability in the model.

6.2 Models with dust matter

Now we choose $\mu_{\rm dust} \neq 0$ and fix $\Lambda = \Gamma_{\rm rad} = 0$. Additional seventh $\vec{a}_i$ vector in the decomposition (3.2) is:

$$\vec{a}_7 = (3, 0, 0).$$

Adler - van Moerbecke formula (4.5) gives the following Kovalevski Exponents:

$$\rho_i = 2 - 2\frac{\langle \vec{a}_i, \vec{a}_7\rangle}{\langle \vec{a}_7, \vec{a}_7\rangle} = \frac{2}{3} \qquad \text{for} \quad i = 1, \ldots, 6,$$
$$\rho_i = 2 - 2\frac{\langle \vec{a}_7, \vec{a}_i\rangle}{\langle \vec{a}_i, \vec{a}_i\rangle} = \frac{5}{2} \qquad \text{for} \quad i = 1, 2, 3.$$

Further Kovalevski Exponents obtained from (4.6) read:

$$z_{1,2} = -\frac{1}{20} \pm \frac{\sqrt{591}}{60}.$$

Therefore irrational and complex values of the Kovalevski Exponents exist irrespectively of concrete value of μ or Bianchi type.

6.3 Models with radiative matter

We take $\Gamma_{\rm rad} \neq 0$ and $\Lambda = \mu_{\rm dust} = 0$. Additional vector in the decomposition (3.2) reads now:

$$\vec{a}_7 = (2, 0, 0).$$

Adler - van Moerbecke formula (4.5) yields

$$\rho_i = 2 - 2\frac{\langle \vec{a}_i, \vec{a}_7\rangle}{\langle \vec{a}_7, \vec{a}_7\rangle} = 0 \qquad \text{for} \quad i = 1, \ldots, 6,$$
$$\rho_i = 2 - 2\frac{\langle \vec{a}_7, \vec{a}_i\rangle}{\langle \vec{a}_i, \vec{a}_i\rangle} = 18 \qquad \text{for} \quad i = 1, 2, 3.$$

Proceeding along the route of (4.6) reveals that for other Kovalevski exponents $z = \rho(\rho-1)$ parameters satisfy the following equations:

$$363z^2 - 22z - 52/3 = 0, \qquad z^2 - 4z - 1/192 = 0.$$

Both of them lead to complex Kovalevski exponents irrespectively of Γ_{rad} or concrete Bianchi type concerned.

7 Conclusions

Qualitative analysis of the dynamics of homogeneous Bianchi models revealed a remnant of the hierarchy of invariant manifolds. In the Bogoyavlenskii [15] formalism different Bianchi types lie on the components of the boundary of compact manifold on which respective dynamical system is defined. This boundary has its own boundaries thus generating a hierarchy in the class of homogeneous cosmological models which could be put into one-to-one correspondence with respective Bianchi type. The most general of homogeneous models i.e. the so called mixmaster models are nonintegrable (e.g. [10, 11, 12]). We have therefore asked the question whether there exits a respective hierarchy of nonintegrability in the full class of homogeneous cosmological models. In this paper we answered this question at the level of algebraic integrability by showing that Bianchi VI_0 and VII_0 models are algebraically nonintegrable since they have complex Kovalevski exponents. This property is propagated into more general Bianchi types (IX and VIII) according to the structure of invariant submanifolds. We have shown algebraic integrability of Bianchi type I and II models.

We have also discussed the influence of matter terms in the energy-momentum tensor on the property of algebraic integrability of homogeneous cosmological models. It turned out that inclusion of dust, cosmological constant or radiation introduces algebraic nonintegrability in the system. One can wonder whether the cosmological constant for example should appear in the Kovalevski exponents and then give integer numbers in the limit as it tended to zero. However it is well known that one cannot obtain solutions of the Einstein equations without cosmological constant from the respective ones with cosmological constant by taking the limit $\Lambda \to 0$. This property is reflected also at the level of Kovalevski exponents.

The literature concerning various aspects of homogeneous cosmological models is very big and diverse, e.g. Kramer et al. [17] gives a compendium of exact solutions of the Einstein's equations (and in large part is devoted to homogenous models), also a dynamical systems approach initiated by Bogoyavlenskii has considerably been developed by Wainwright and Ellis (for the review see [18]) and chaotic aspects of Mixmaster homogeneous models have recently been investigated (e.g. [19]) in more detailed way. However, the existence of an exact solution does not imply the existence of algebraic first integrals. Similarly, the hierarchy revealed by the dynamical systems approach concerns the generality of solutions not the property of their integrability. Also the works revealing dynamical complexity of some models (in terms of fractal basin boundaries [19]) can by no means be understood as a rigorous proof of their nonintegrability. Therefore our results presented above can be perceived as complementary to these existing in the literature and are another step toward deeper understanding of mathematical properties of homogeneous cosmological models.

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