

# Distinguishing Three-Dimensional Lens Spaces $L(7, 1)$ and $L(7, 2)$ by Means of Classical Pentagon Equation

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## Abstract

We construct new topological invariants of three-dimensional manifolds which can, in particular, distinguish homotopy equivalent lens spaces  $L(7, 1)$  and  $L(7, 2)$ . The invariants are built on the base of a classical (not quantum) solution of pentagon equation, i.e. algebraic relation corresponding to a “2 tetrahedra  $\rightarrow$  3 tetrahedra” local re-building of a manifold triangulation. This solution, found earlier by one of the authors, is expressed in terms of metric characteristics of Euclidean tetrahedra.

## 1 Introduction

The present work is a direct continuation of paper [1]. We recall that a new construction was proposed in it of an invariant of 3-dimensional closed oriented piecewise-linear manifolds. The invariant was built on the base of a classical (not quantum) solution of pentagon equation, i.e. algebraic relation corresponding to a “2 tetrahedra  $\rightarrow$  3 tetrahedra” local re-building of a manifold triangulation (formula (5) in [1]. Below, we are using notations like [1, (5)] for references of such kind).

Only the simplest realization of these ideas was done in paper [1]. An invariant  $I$  was constructed that was expressed in terms of Euclidean metric values assigned to every tetrahedron. Then  $I$  was calculated for the sphere  $S^3$ , the projective space  $\mathbb{R}P^3$ , and also for some lens spaces  $L(p, q)$  (it was not described in paper [1] how the calculations were done for  $L(p, q)$ . It will suffice for us now if we say that they were a direct generalization of calculations for  $\mathbb{R}P^3$ ).

For all tested examples of lens spaces we got

$$I(L(p, q)) = \frac{1}{p^3}. \quad ([1, (39)])$$

This showed that our invariant  $I$  in such form as presented in [1] was not very interesting for topology because it could not distinguish the lens spaces with different  $q$ . At the same time, in the concluding Section 7 of paper [1] an idea was proposed how to modify  $I$  in the way that could hopefully lead to a more refined invariant. The idea consists, briefly,

in using some mappings of the manifold's *universal covering* in the Euclidean space  $\mathbb{R}^3$  in order to put metric values in correspondence to tetrahedra.

In the present work we make this idea concrete and show that in such way one can indeed construct more refined invariants. Our key example will be lens spaces  $L(7, 1)$  and  $L(7, 2)$ . As is well-known, they are homotopy equivalent but not homeomorphic, and can serve as a natural test of nontriviality for topological invariants. We would like to say at once that our new invariants pass this test and thus can, most likely, compete successfully with the quantum invariants of 3-dimensional manifolds. A remark on possible advantages of classical invariants is made in the concluding Section 4.

We consider triangulations or “pre-triangulations” of 3-dimensional closed oriented manifolds. By pre-triangulation we understand a manifold represented as a “pre-simplicial complex” in the sense of textbook [2]: the manifold is decomposed in simplices of dimensions 3, 2, 1 and 0 in such way that the boundary of a given simplex may consist of *coinciding* simplices of lower dimensions (in particular, in Section 3 we decompose lens spaces in tetrahedra in such way that only two vertices of each tetrahedron are different and, moreover, there are sometimes coinciding ones between the six edges of a tetrahedron. Moreover, the simplices of dimensions  $> 0$  may not be determined by the set of their vertices).

We are going to put metric values (edge lengths, dihedral angles, volumes) in correspondence to all tetrahedra in the pre-triangulation, as it was done in paper [1]. To be exact, we will do the following. Choose a consistent orientation of all tetrahedra in the pre-triangulation (using the manifold's orientability). Assign to all edges (one-dimensional cells of the complex) positive real numbers — lengths — in such way that every individual tetrahedron could be placed in a 3-dimensional Euclidean space. This means that one can calculate in a usual way dihedral angles at each edge in every tetrahedron, and one can assume that those angles take values from 0 to  $\pi$ . Next, we assign to each tetrahedron a “+” or “-” sign and impose the following condition: the algebraic sum of dihedral angles around each edge must equal 0 modulo  $2\pi$ , where “algebraic sum” means that each dihedral angle, which is calculated, as we have mentioned, from the six edge lengths of some tetrahedron, is taken with the sign “+” or “-” that is assigned to that tetrahedron. This is exactly the condition that is implied in formulas of paper [1], starting from [1, (5)].

Such assignment (of lengths to edges and signs to tetrahedra) was realized in paper [1] in the following way. All *vertices* of the complex were mapped into the 3-dimensional Euclidean space  $\mathbb{R}^3$ . For every tetrahedron in the complex we took the images of its four vertices in  $\mathbb{R}^3$  and demanded that the tetrahedron spanned by them should be nondegenerate (of course, this is possible only if all the four vertices are different). Assume also that we have fixed the orientation of  $\mathbb{R}^3$ . Then a tetrahedron in the pre-triangulation either conserves its orientation under the mapping to  $\mathbb{R}^3$  or changes it (recall that we have fixed a consistent orientation for the tetrahedra). In the first case, we assign the “+” sign to it, in the second — the “-” sign. As for the edge lengths, they are, of course, the distances between the relevant vertices in  $\mathbb{R}^3$ .

In the present paper, we generalize this construction according to the remark in Section 7 of paper [1] (and it is exactly this generalization that leads us to our new results). We take the *universal covering* of the pre-triangulation of manifold  $M$  considered as a cell complex. Its vertices are divided in classes — the inverse images of each given vertex with respect to the covering map. Choose a representative in each class and map it into some

point in the space  $\mathbb{R}^3$ . Next, fix a homomorphism  $\varphi: \pi_1(M) \rightarrow E_3$  of the fundamental group of manifold  $M$  in the group of motions of 3-dimensional Euclidean space. Suppose that some vertices  $A_1$  and  $A_2$  in the universal covering belong to the same class, then  $A_2 = g(A_1)$ , where  $g \in \pi_1(M)$ . Denote the images of vertices in  $\mathbb{R}^3$  by the same letters with the tilde. We demand that for all such vertices  $\tilde{A}_2 = \varphi(g)(\tilde{A}_1)$ .

This construction yields (in the general position) a one-to-one correspondence between the vertices in the universal covering of our complex and their images in  $\mathbb{R}^3$ . As for simplices of dimension  $> 0$ , each of them yields thus a simplex in  $\mathbb{R}^3$  — the convex hull of corresponding vertices (we emphasize that we do not care about possible intersections of such simplices in  $\mathbb{R}^3$  but we demand that all the vertices of each tetrahedron in the *universal covering* should be different). Hence we obtain lengths for the edges of the universal covering and signs — for its 3-dimensional cells in the same way as it was done two paragraphs before. Then, it is clear that if two edges are mapped in the same one by the covering map then their lengths obtained from our construction are equal (because their images in  $\mathbb{R}^3$  are taken one into another by elements  $\varphi(g) \in E_3$ ). The same holds for the tetrahedron signs. Thus, we have assigned correctly the lengths to the edges and the signs to the tetrahedra of the manifold  $M$ 's pre-triangulation itself and not only to its universal covering.

In Section 3 we present a concrete example of this construction for the case of lens spaces.

It can be shown that our construction with the universal covering yields all possible ways of assigning lengths to the edges and signs to the tetrahedra with the condition that the algebraic sum of dihedral angles around each edge must be 0 modulo  $2\pi$ . We will not, however, use this fact in the present paper.

**Important remark:** A homomorphism  $\varphi: \pi_1(M) \rightarrow E_3$  takes part in our construction. This will lead to the fact that our invariant corresponds in reality to a *pair*  $(M, \varphi)$ . In particular, for  $M = L(7, q)$  there exist, as we will see, three non-equivalent  $\varphi$ , excluding the trivial homomorphism whose image is unity (in [1] we were dealing, in essence, with exactly this “trivial” case).

Thus, we have assigned metric values to elements of pre-triangulation in a more general way than in paper [1]. Most of the other constructions in [1] do not need any generalization: the main thing that we are going to do is calculate our key differential form [1, (30)] and extract a numeric invariant from it.

The contents of the remaining sections is as follows. In Section 2 we write down general formulas for the partial derivatives of “defect angles” (we recall *en passant* their definition) with respect to the edge lengths. The matrix of such derivatives is needed for constructing our invariants. In Section 3 we are concerned with calculations as such for lens spaces  $L(7, 1)$  and  $L(7, 2)$ . In the concluding Section 4 we put forward a conjecture concerning the value of our invariants for *any* lens spaces, remark on the possible advantages of classical invariants, and discuss some intriguing unsolved problems.

## 2 Derivatives of defect angles w.r.t. edge lengths

In the Introduction we mapped the universal covering of a pre-triangulated manifold  $M$  in Euclidean space  $\mathbb{R}^3$ . The pull-back of one cell (simplex) from  $M$  with respect to the

covering map was a set of  $N = \text{card}(\pi_1(M))$  cells which were mapped, according to our construction, in isometric simplices in  $\mathbb{R}^3$ .

Suppose that Euclidean tetrahedra  $\tilde{A}_1\tilde{B}_1\tilde{C}_1\tilde{D}_1, \dots, \tilde{A}_N\tilde{B}_N\tilde{C}_N\tilde{D}_N$  correspond in such way to a tetrahedron  $ABCD$  lying in  $M$ . Now we, first, will no longer write the tildes (that stayed for points in  $\mathbb{R}^3$ , see Introduction. We thus identify the vertices in the universal covering and their images in  $\mathbb{R}^3$ ). Second, we define the oriented volume of tetrahedron  $ABCD$  as  $1/6$  of the following triple scalar product:

$$6V_{ABCD} = \overrightarrow{A_1B_1} \overrightarrow{A_1C_1} \overrightarrow{A_1D_1} \left( = \dots = \overrightarrow{A_NB_N} \overrightarrow{A_NC_N} \overrightarrow{A_ND_N} \right). \quad (2.1)$$

Our next task is to make some formulas of paper [1] more accurate. Namely, we are going to take carefully into account the orientation of angles and volumes and thus remove the absolute value signs. Introduce (small) defect angles around edges like in paper [1]: slightly (and otherwise arbitrarily) change the edge lengths; after that the algebraic sums of dihedral angles around edges cease to be zero. By definition, such sum around a given edge  $a$  is  $-\omega_a$ , where  $\omega_a$  is called defect angle (defined modulo  $2\pi$ ). Note that the sign of expression (2.1) is nothing but the sign corresponding to the tetrahedron, i.e. the sign with which the dihedral angles of the tetrahedron enter in each algebraic sum ( $-\omega_a$ ).

We will need formulas for the partial derivatives  $\partial\omega_a/\partial l_b$  of defect angles with respect to edge lengths taken when all  $\omega_a = 0$ . We are going to express these derivatives in terms of those lengths and oriented tetrahedron volumes. Nonzero derivatives are obtained in the following cases.

### 1st case

Edges  $a = DE$  and  $b = AB$  are skew edges of tetrahedron  $ABDE$ , and there is no more tetrahedron in the pre-triangulation that would have as its edges both  $a$  and  $b$ :

$$\frac{\partial\omega_{DE}}{\partial l_{AB}} = -\frac{l_{AB}l_{DE}}{6} \frac{1}{V_{ABED}}. \quad (2.2)$$

This formula coincides, in essence, with formula [1, (3)]. The only difference is that we now take into account the signs of angles and volumes.

### 2nd case

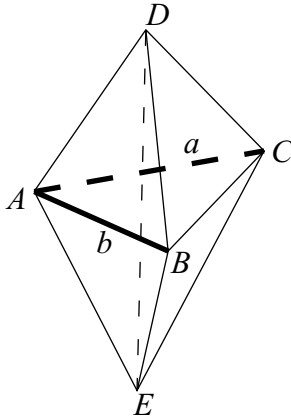
Edges  $a = AC$  and  $b = AB$  belong to the common face of two neighboring tetrahedra and, again, there is no more tetrahedron that would have both  $a$  and  $b$  as its edges (Fig. 1):

$$\frac{\partial\omega_{AC}}{\partial l_{AB}} = \frac{l_{AB}l_{AC}}{6} \frac{V_{BCED}}{V_{ABCD}V_{EABC}}. \quad (2.3)$$

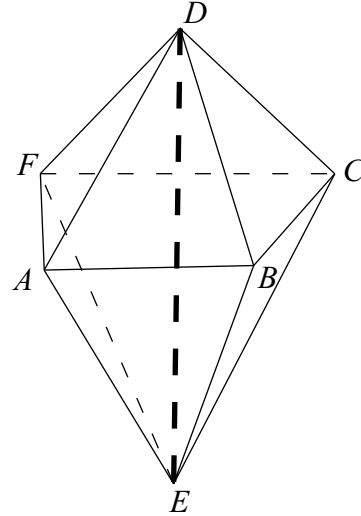
To prove this formula, we draw edge  $DE$  in Fig. 1. Suppose that lengths  $l_a$  and  $l_{DE}$  in Fig. 1 are free to change, while the lengths of remaining eight edges are fixed. We get:

$$\frac{\partial l_{DE}}{\partial l_b} = -\frac{l_b}{l_{DE}} \frac{V_{CEDA}V_{BCED}}{V_{ABCD}V_{EABC}} \quad (2.4)$$

(this is, in essence, formula [1, (1)] but now with the proper notice to volume signs).



**Figure 1.** Illustration to formula (2.3).



**Figure 2.** Illustration to formula (2.7).

Denote  $\varphi_{AC}$  the dihedral angle at edge  $AC$  in tetrahedron  $CEDA$ . Then

$$\frac{\partial \varphi_{AC}}{\partial l_{DE}} = \frac{l_{AC} l_{DE}}{6V_{CEDA}}. \quad (2.5)$$

Multiplying (2.4) by (2.5) and using the fact that  $\partial \omega_{AC} / \partial l_b = -\partial \varphi_{AC} / \partial l_b$ , we get formula (2.3).

### 3rd case

Edge  $a = b = DE$  is common for exactly three tetrahedra  $ABED$ ,  $BCED$  and  $CAED$ , and no one of those tetrahedra contains this edge more than one time:

$$\frac{\partial \omega_{DE}}{\partial l_{DE}} = -\frac{l_{DE}^2}{6} \frac{V_{ABCD} V_{EABC}}{V_{ABED} V_{BCED} V_{CAED}}. \quad (2.6)$$

This is simply formula [1, (5)] but with all signs taken into account.

### 4th case

Edge  $a = b = DE$  is common for  $> 3$  tetrahedra, and again no one of those tetrahedra contains this edge more than one time.

We write out this formula only for the case of four tetrahedra (in the case of their greater number, the generalization is obvious) depicted in Fig. 2:

$$\frac{\partial \omega_{DE}}{\partial l_{DE}} = -\frac{l_{DE}^2}{6} \left( \frac{V_{ABCD} V_{EABC}}{V_{ABED} V_{BCED} V_{CAED}} + \frac{V_{ACFD} V_{EACF}}{V_{ACED} V_{CFED} V_{FAED}} \right). \quad (2.7)$$

Here in the right-hand side we wrote the right-hand side of (2.6) plus the similar term obtained from it by replacing  $C \rightarrow F$ ,  $B \rightarrow C$ . Formula (2.7) comes out if we draw the diagonal  $AC$  in Fig. 2 and apply formula (2.6) to each of figures  $ABCDE$  and  $ACFDE$ .

Adding up the defect angles around  $DE$  in each of those figures we get nothing else than  $\omega_{DE}$  for the whole figure  $ABCFDE$ , because the “redundant” dihedral angle, namely the angle at  $DE$  in tetrahedron  $AECD$ , enters two times with different signs.

### 5th case

In reality, our definition of pre-triangulation admits such glueings between faces and, consequently, edges of tetrahedra where different *combinations* of above mentioned cases appear. To be exact, the edge  $b$  may

- lie opposite to edge  $a$  (1st case) in one or more than one tetrahedron;
- belong to the same two-dimensional face as  $a$  (2nd case), and again there may be more than one such faces;
- coincide with  $a$  (as in the 3rd or 4th case),

and these possibilities (as we will see in Section 3) do not exclude one another. It is also worth mentioning that if a tetrahedron contains a given edge  $a$  several times then  $(-\omega_a)$  includes, of course, the *sum of all* corresponding dihedral angles of that tetrahedron multiplied by the sign of its oriented volume (2.1).

Thus, there may be several “ways of influence” of differential  $dl_b$  on differential  $d\omega_a$ . Clearly, the terms corresponding to those ways sum together. Formulas (3.10) and (3.12) below are good examples of how the expression for  $\partial\omega_a/\partial l_b$  (in them, it is divided by  $l_a l_b$ ) can look like.

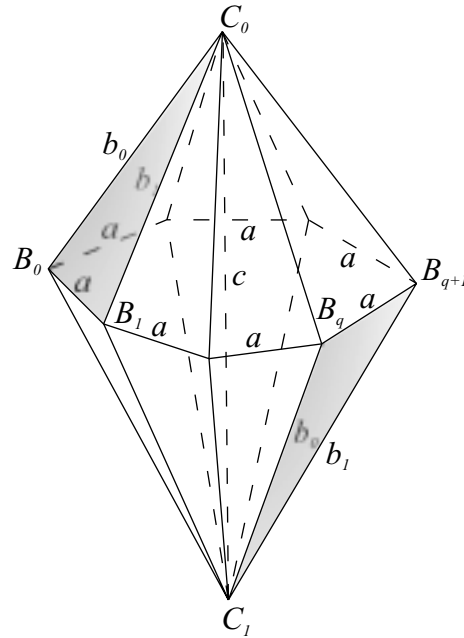
The partial derivatives that we were dealing with in this section will be used to calculate the differential form [1, formula (30)] from which we will extract manifold invariants. The formulas and arguments of paper [1] leading to that formula remain valid if we only modify the notion of “permitted length configuration”, extending the word “permitted” to any set of edge lengths obtained according to the construction with universal covering presented in the Introduction to this work.

The remarks in the end of Section 4 of [1] (after Theorem 4) on the behavior of form [1, (30)] under the adding of a new vertex to the (pre-)triangulation remain valid as well. In formulas [1, (31) and (32)], one can take as point  $E$  any one of “copies”  $E_1, \dots, E_N$  — images in  $\mathbb{R}^3$  of its pull-back with respect to the covering map. On the other hand, we will see in the next section that we must change the “standard” differential form [1, (33)] in order to calculate our new invariants for  $L(7, 1)$  and  $L(7, 2)$ .

## 3 Calculation of invariants for $L(7, 1)$ and $L(7, 2)$

We recall some facts about lens spaces. Let  $p > q > 0$ ,  $p \geq 3$ , be a pair of relatively prime integers. Consider a  $p$ -gonal bipyramid, i.e. the union of two cones over a regular  $p$ -gon. Denote as  $B_0, B_1, \dots, B_{p-1}$  the vertices of the  $p$ -gon, and by  $C_0$  and  $C_1$  — the cone vertices. For every  $i$ , we glue face  $B_i C_0 B_{i+1}$  to face  $B_{i+q} C_1 B_{i+q+1}$  (the subscripts are taken modulo  $p$ ; the vertices are glued in the order in which they are written). What we thus obtain is exactly the lens space  $L(p, q)$  (Fig. 3).

$L(p, q)$  is a 3-dimensional closed oriented manifold. Its fundamental group  $\pi_1(L(p, q))$  is isomorphic to  $\mathbb{Z}_p$ . As a generator of  $\pi_1(L(p, q))$  one can take a path going along the edge



**Figure 3.** Lens space as a bipyramid with identified faces.

$a = B_i B_{i+1}$  (note that points  $B_i$  as well as edges  $B_i B_{i+1}$  are identified among themselves for all  $i$ ), or along the edge  $c = C_0 C_1$ .

The spaces  $L(7, 1)$  and  $L(7, 2)$  that we are going to consider are interesting because they are homotopy equivalent [2] but not homeomorphic [3]. We will obtain one possible proof of their non-homeomorphism when we find out that our invariants for those spaces do not coincide.

A homomorphism of the fundamental group of a lens space in the group  $E_3$  of motions of Euclidean space can be characterized by an integer  $k$  if we demand that  $a \in \pi_1(L(p, q))$  be mapped in the rotation through angle  $2\pi k/p$  around some axis. That axis can be taken by definition as the coordinate axis  $z$ . There exist three essentially different nonzero  $k$  for  $p = 7$  ( $k = 0$  leads to the case of paper [1]):  $k = 1, 2$  and  $3$ .

The image of the universal covering of space  $L(p, q)$  (if  $k \neq 0$ ) in  $\mathbb{R}^3$  looks as follows. Vertices  $B_j$  (“copies” of one and the same vertex  $B$ ) are situated in vertices of a regular  $p$ -gon for which axis  $z$  is the symmetry axis. The same applies to vertices  $C_j$  (“copies” of vertex  $C$ ). Thus, our bipyramid has only two different [not identified with each other] vertices:  $B$  and  $C$ ). We can assume that the cylindrical coordinates of points  $B_j$  and  $C_j$  are:

$$B_j \left( \rho, \frac{2\pi j k}{p}, 0 \right), \quad C_j \left( \sigma, \alpha + \frac{2\pi q j k}{p}, s \right), \quad (3.1)$$

where  $\rho > 0$ ,  $\sigma > 0$ , while  $\alpha$  and  $s$  are any real numbers.

Return to Fig. 3. Join points  $C_0$  and  $C_1$  with an edge. Then the bipyramid finds itself decomposed in  $p$  tetrahedra grouped around edge  $C_0 C_1$ . Assume the following notations

for edges (notation “ $a$ ” and “ $c$ ” have been already introduced but we repeat them):

$$a = B_0 B_1 = \cdots = B_{p-1} B_0, \quad (3.2)$$

$$b_j = C_0 B_j = C_1 B_{j+q \pmod{p}}, \quad j = 0, \dots, p-1, \quad (3.3)$$

$$c = C_0 C_1. \quad (3.4)$$

On the whole, there are  $p + 2$  edges, and this will be the size of square matrix  $A = (\partial\omega_i/\partial l_j)$  entering in formula [1, (30)]. Both rows and columns of this matrix correspond to edges  $a, b_0, \dots, b_{p-1}, c$ . Decompose this set of edges in subsets  $\mathcal{C} = \{b_1, \dots, b_{p-2}\}$  and  $\bar{\mathcal{C}} = \{a, b_0, b_{p-1}, c\}$ . Recall our description of the image of universal covering of a lens space in  $\mathbb{R}^3$ . It is obvious from geometric considerations that this image is determined by four parameters, say,  $\rho, \sigma, s$  and  $\alpha$  in formulas (3.1), or otherwise by the lengths of four edges which we have included in  $\bar{\mathcal{C}}$ . The fact important for us is that these four lengths (in the general position) can take arbitrary infinitesimal increments from which the other length differentials are determined unambiguously, and one can assume that they are determined from conditions  $d\omega_i = 0$ , where the letter  $i$  runs over all edges. According to paper [1], this means that our decomposition of the set of edges in  $\mathcal{C}$  and  $\bar{\mathcal{C}}$  is suitable for use in formula [1, (30)].

We will slightly modify that formula by introducing a matrix

$$F \stackrel{\text{def}}{=} \begin{pmatrix} 1 & \partial\omega_i \\ l_i l_j & \partial l_j \end{pmatrix} \quad (3.5)$$

instead of  $A$ . Then the form [1, (30)] is written as

$$\left| \frac{\bigwedge_{\bar{\mathcal{C}}} l dl}{\sqrt{|\det(F|_{\mathcal{C}}) \prod 6V|}} \right|, \quad (3.6)$$

where, of course,  $F|_{\mathcal{C}}$  is the diagonal submatrix of matrix  $F$  corresponding to subset  $\mathcal{C}$ .

Our next task is to calculate explicitly the matrix elements of  $F|_{\mathcal{C}}$  for  $L(7, 1)$  and  $L(7, 2)$ . We will explain how the calculations are done on the example of two types of matrix elements of  $L(7, 1)$ , while we will simply write out the result for all other matrix elements.

At the moment, we are considering the case of arbitrary  $p$  and  $q$ .

Introduce a brief notation

$$V_i = V_{C_0 C_1 B_{i+1} B_i}. \quad (3.7)$$

An easy calculation shows that

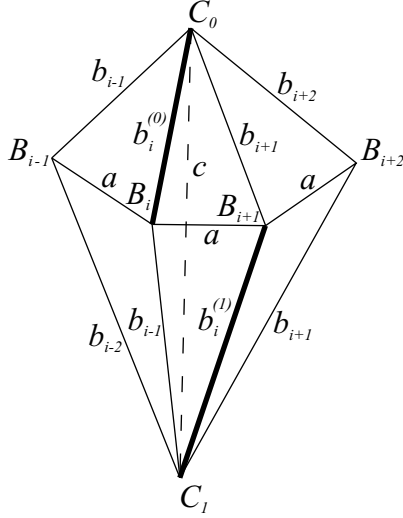
$$V_i = \frac{1}{6} R \sin \left( \alpha + \frac{\pi k(q-1-2i)}{p} \right), \quad (3.8)$$

where we have denoted

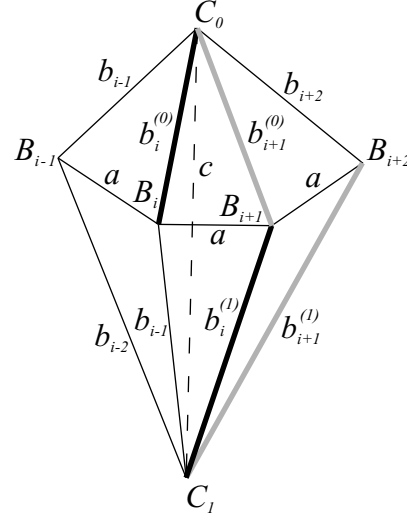
$$R = 4\rho\sigma s \sin \frac{\pi k}{p} \sin \frac{\pi q k}{p}. \quad (3.9)$$

Now set  $p = 7, q = 1$ .





**Figure 4.** A fragment of bipyramid — illustration to formula (3.10). Two copies of edge  $b_i$  are drawn in thick lines.



**Figure 5.** A fragment of bipyramid — illustration to formula (3.12). Different types of thick lines show edges  $b_i$  and  $b_{i+1}$  (each in two copies glued together).

### Diagonal matrix elements

$$(F|c)_{i,i} = \frac{1}{l_{b_i}^2} \frac{\partial \omega_{b_i}}{\partial l_{b_i}}, \quad i = 1, \dots, 5.$$

In Fig. 4 a fragment of bipyramid is presented containing both copies of edge  $b_i$  that are glued together. We denote them  $b_i^{(0)}$  and  $b_i^{(1)}$ . Each of them belongs to two tetrahedra, thus, the total number of tetrahedra around edge  $b_i$  is four. This would be the 4th case from Section 2 be it not combined with the 1st case (thus giving the 5th case as a result), because edge  $b_i$  turns out also to lie opposite itself in tetrahedron  $C_0C_1B_{i+1}B_i$ . This leads to the adding of two more identical terms of type (2.2) (the fact that there are two such terms can be explained as follows:  $b_i^{(0)}$  influences the dihedral angle at  $b_i^{(1)}$ , while  $b_i^{(1)}$  influences the dihedral angle at  $b_i^{(0)}$ ).

The result is the following combination of formulas (2.6) and (2.2):

$$(F|c)_{i,i} = -\frac{1}{6} \left( \frac{V_{C_0C_1B_{i+2}B_i} V_{C_0B_iB_{i+1}B_{i+2}}}{V_{C_0C_1B_{i+2}B_{i+1}} V_{C_0C_1B_{i+1}B_i} V_{C_1B_iB_{i+1}B_{i+2}}} + \frac{V_{C_0C_1B_{i+1}B_{i-1}} V_{C_1B_{i-1}B_iB_{i+1}}}{V_{C_0C_1B_iB_{i-1}} V_{C_0C_1B_{i+1}B_i} V_{C_0B_{i-1}B_iB_{i+1}}} + \frac{2}{V_{C_0C_1B_{i+1}B_i}} \right). \quad (3.10)$$

Formula (3.10) can be simplified considerably if we take into account that points  $B_j$  and  $C_j$  are located in two parallel planes. This gives, firstly:

$$V_{C_0B_iB_{i+1}B_{i+2}} = V_{C_1B_iB_{i+1}B_{i+2}}, \quad V_{C_0B_{i-1}B_iB_{i+1}} = V_{C_1B_{i-1}B_iB_{i+1}},$$

which leads at once to partial cancellation of numerators and denominators in two first terms in parentheses in the right-hand side of (3.10). Further simplification of those very

terms is achieved by noting that the volumes that remain in their numerators are expressed through the volumes in their denominators as follows:

$$\begin{aligned} V_{C_0C_1B_{i+2}B_i} &= V_{C_0C_1B_{i+1}B_i} + V_{C_0C_1B_{i+2}B_{i+1}}, \\ V_{C_0C_1B_{i+1}B_{i-1}} &= V_{C_0C_1B_iB_{i-1}} + V_{C_0C_1B_{i+1}B_i}. \end{aligned}$$

Finally we get

$$(F|c)_{i,i} = -\frac{4}{6V_i} - \frac{1}{6V_{i-1}} - \frac{1}{6V_{i+1}}. \quad (3.11)$$

### Next-to-diagonal matrix elements

$$(F|c)_{i,i+1} = \frac{1}{l_{b_i}l_{b_{i+1}}} \frac{\partial \omega_{b_{i+1}}}{\partial l_{b_i}}, \quad i = 1, \dots, 4.$$

In Fig. 5 a fragment of bipyramid is presented that contains two identified copies of each of edges  $b_i$  and  $b_{i+1}$ . One can see that here a triple “2nd case” from Section 2 takes place (which, of course, yields again the 5th case). Namely, the rôle of face  $ABC$  from Fig. 1 is played in Fig. 5 by each of faces  $C_0B_iB_{i+1}$ ,  $B_{i+1}C_1C_0$  and  $C_1B_{i+1}B_{i+2}$ . This gives

$$\begin{aligned} (F|c)_{i,i+1} &= \frac{1}{6} \left( \frac{V_{C_1B_iB_{i+1}B_{i+2}}}{V_{C_0C_1B_{i+1}B_i}V_{C_0B_iB_{i+1}B_{i+2}}} \right. \\ &\quad \left. + \frac{V_{C_0C_1B_{i+2}B_i}}{V_{C_0C_1B_{i+1}B_i}V_{C_0C_1B_{i+2}B_{i+1}}} + \frac{V_{C_0B_iB_{i+1}B_{i+2}}}{V_{C_0C_1B_{i+2}B_{i+1}}V_{C_1B_iB_{i+1}B_{i+2}}} \right). \end{aligned} \quad (3.12)$$

Using the fact that points  $B_j$  and  $C_j$  are located in two parallel planes, we can again simplify the expression (3.12) to

$$(F|c)_{i,i+1} = \frac{2}{6V_i} + \frac{2}{6V_{i+1}}. \quad (3.13)$$

Perhaps these two types of matrix elements can be considered as most complicated. Now we write out *all* matrix elements of  $F|c$ .

### Submatrix $F|c$ for $L(7, 1)$ equals $1/6 \times$

$$\begin{pmatrix} -\frac{4}{V_1} - \frac{1}{V_0} - \frac{1}{V_2} & \frac{2}{V_1} + \frac{2}{V_2} & -\frac{1}{V_2} & 0 & 0 \\ \frac{2}{V_1} + \frac{2}{V_2} & -\frac{4}{V_2} - \frac{1}{V_1} - \frac{1}{V_3} & \frac{2}{V_2} + \frac{2}{V_3} & -\frac{1}{V_3} & 0 \\ -\frac{1}{V_2} & \frac{2}{V_2} + \frac{2}{V_3} & -\frac{4}{V_3} - \frac{1}{V_2} - \frac{1}{V_4} & \frac{2}{V_3} + \frac{2}{V_4} & -\frac{1}{V_4} \\ 0 & -\frac{1}{V_3} & \frac{2}{V_3} + \frac{2}{V_4} & -\frac{4}{V_4} - \frac{1}{V_3} - \frac{1}{V_5} & \frac{2}{V_4} + \frac{2}{V_5} \\ 0 & 0 & -\frac{1}{V_4} & \frac{2}{V_4} + \frac{2}{V_5} & -\frac{4}{V_5} - \frac{1}{V_4} - \frac{1}{V_6} \end{pmatrix}. \quad (3.14)$$

Submatrix  $F|_c$  for  $L(7, 2)$  equals  $1/6 \times$

$$\begin{pmatrix} -\Sigma_{0123} & -\frac{1}{V_2} + \frac{1}{V_1} + \frac{1}{V_3} & \frac{1}{V_2} + \frac{1}{V_3} & -\frac{1}{V_3} & -\frac{1}{V_0} \\ -\frac{1}{V_2} + \frac{1}{V_1} + \frac{1}{V_3} & -\Sigma_{1234} & -\frac{1}{V_3} + \frac{1}{V_2} + \frac{1}{V_4} & \frac{1}{V_3} + \frac{1}{V_4} & -\frac{1}{V_4} \\ \frac{1}{V_2} + \frac{1}{V_3} & -\frac{1}{V_3} + \frac{1}{V_2} + \frac{1}{V_4} & -\Sigma_{2345} & -\frac{1}{V_4} + \frac{1}{V_3} + \frac{1}{V_5} & \frac{1}{V_4} + \frac{1}{V_5} \\ -\frac{1}{V_3} & \frac{1}{V_3} + \frac{1}{V_4} & -\frac{1}{V_4} + \frac{1}{V_3} + \frac{1}{V_5} & -\Sigma_{3456} & -\frac{1}{V_5} + \frac{1}{V_4} + \frac{1}{V_6} \\ -\frac{1}{V_0} & -\frac{1}{V_4} & \frac{1}{V_4} + \frac{1}{V_5} & -\frac{1}{V_5} + \frac{1}{V_4} + \frac{1}{V_6} & -\Sigma_{4560} \end{pmatrix}. \quad (3.15)$$

Here we had to introduce a brief notation

$$\Sigma_{ijkl} \stackrel{\text{def}}{=} \frac{1}{V_i} + \frac{1}{V_j} + \frac{1}{V_k} + \frac{1}{V_l}$$

in order that the matrix could be placed on the page.

Now we will be busy with the differential form (3.6). *Before* calculating it we state the following

**Proposition.** *The differential form (3.6) equals*

$$|\text{const} \cdot \rho d\rho \wedge \sigma d\sigma \wedge ds \wedge d\alpha| \quad (3.16)$$

(with notations taken from (3.1)).

**Proof.** First, we note that the rank of form (3.6) is equal to  $\text{card} \bar{C}$ , that is 4 in our case. Thus, if expressed through  $\rho$ ,  $\sigma$ ,  $s$  and  $\alpha$ , it inevitably takes the form

$$|f(\rho, \sigma, s, \alpha) d\rho \wedge d\sigma \wedge ds \wedge d\alpha|.$$

The form of function  $f$  is easily obtained from reasoning of the same type as in Section 5 of paper [1]: differential form (3.6) must be represented as the exterior product of the volume form for point  $C$ , i.e.  $dx_C \wedge dy_C \wedge dz_C = \sigma d\sigma \wedge ds \wedge d\alpha$  by a factor not depending on the location of point  $C$  with respect to point  $B$  and symmetry axis  $z$ . This must also hold for point  $B$ . Thus, we are led unambiguously to formula (3.16) for the form (3.6). The proposition is proved.  $\blacksquare$

In the case we are considering, it is natural to take the constant entering in (3.16) as our invariant (which, we recall, we put in correspondence to the *pair*  $(M, \varphi)$ , where  $\varphi$  is a homomorphism from  $\pi_1(M)$  to  $E_3$ ).

In the numerator of formula (3.6) we see the exterior product  $\bigwedge_{\bar{C}} l dl$  which in our case, as one can check, has the form (we write it out for arbitrary  $p$  and  $q$ )

$$\begin{aligned} l_a dl_a \wedge l_{b_0} dl_{b_0} \wedge l_{b_{p-1}} dl_{b_{p-1}} \wedge l_c dl_c \\ = 8R \sin^2 \frac{\pi k}{p} \sin \frac{\pi q k}{p} \cos \left( \alpha + \frac{\pi k}{p} \right) \rho d\rho \wedge \sigma d\sigma \wedge ds \wedge d\alpha \end{aligned}$$

( $R$  was introduced in formula (3.9)). This shows that the **dependence of the denominator of formula (3.6) on  $\alpha$  must, too, reduce to a factor  $\cos(\alpha + \pi k/p)$** , in order that this factor could cancel out and we get a formula of type (3.16). We hope that, with

this remark, our calculations of the expression (3.6), for which we actively used the Maple program (the most difficult step was the calculation of determinants of matrices (3.14) and (3.15)), will be repeatable.

Here are the results for our invariant, i.e. value “const” from formula (3.16), which we will now call  $I_k$  because it depends on the integer  $k$  from formulas (3.1) that determines the homomorphism  $\varphi$ . We recall that it is enough to consider  $k = 1, 2$  or  $3$  for  $L(7, q)$ . So, with the precision of ten significant digits,

$$I_1(L(7, 1)) = 0.08100567416, \quad (3.17)$$

$$I_2(L(7, 1)) = 0.8540328192, \quad (3.18)$$

$$I_3(L(7, 1)) = 2.064961508, \quad (3.19)$$

$$I_1(L(7, 2)) = 0.2630237713, \quad (3.20)$$

$$I_2(L(7, 2)) = 1.327985278, \quad (3.21)$$

$$I_3(L(7, 2)) = 0.4089909518. \quad (3.22)$$

It is obvious that  $L(7, 1)$  cannot be homeomorphic to  $L(7, 2)$ . Exact expressions for values of  $I_k$  in terms of trigonometric functions are presented in Discussion, formula (4.1), together with a conjecture about their form for an arbitrary lens space.

## 4 Discussion

Invariants of the stated type can be calculated for any 3-dimensional closed orientable manifolds. It looks very plausible that they can be also generalized to manifolds with boundary and not necessarily orientable. It looks clear that the differential form (3.16) whose very appearance is connected with the presence of one symmetry axis in the image in  $\mathbb{R}^3$  of the universal covering of manifold pre-triangulation will have to be replaced with other forms, as soon as the fundamental group of the manifold is different from  $\mathbb{Z}_p$  (recall also the differential form [1, (33)] that corresponded to the trivial homomorphism  $\varphi$ ).

As for the lens spaces with arbitrary  $p$  and  $q$ , we have the following conjecture about the values of invariant  $I_k$  for them:

$$I_k(L(p, q)) = \frac{16}{p} \sin^2 \frac{\pi k}{p} \sin^2 \frac{\pi q k}{p}, \quad k = 1, \dots, \left\lfloor \frac{p}{2} \right\rfloor. \quad (4.1)$$

Approximate values of exactly these numbers were presented in formulas (3.17)–(3.22). Note that formula (4.1) agrees with the known fact [2, 3] that lens spaces  $L(p, q_1)$  and  $L(p, q_2)$  are homeomorphic if (and only if)

$$q_2 = \pm q_1^{\pm 1} \pmod{p}.$$

Indeed, if  $q_2 = \pm q_1 \pmod{p}$ , then, obviously,  $I_k(L(p, q_1)) = I_k(L(p, q_2))$ . If, on the other hand,  $q_2 = \pm q_1^{-1} \pmod{p}$ , then  $I_k(L(p, q_1)) = I_{kq_1}(L(p, q_2))$ , i.e. the set of numbers corresponding to the space is the same.

Our invariants may be interesting also because they depend not only on a manifold  $M$  but also on a homomorphism  $\varphi: \pi_1(M) \rightarrow E_3$ . In particular, they make it possible to distinguish the generators of fundamental groups for lens spaces.

Now we mention the following unsolved and intriguing problems.

- In preprint [4] a “higher analogue of pentagon equation” is presented — an algebraic relation corresponding to a  $3 \rightarrow 3$  re-building of a cluster of 4-simplices (expressed, again, in terms of Euclidean metric values assigned to those simplices). It would be very interesting to construct a 4-manifold invariant on this basis. If our classical invariants work for higher-dimensional manifolds, which looks plausible, this would be an advantage with respect to the known quantum invariants because, for these latter, it is quite unclear how to generalize them nontrivially to higher dimensions. Maybe, equations of pentagon type that arise in connection with multidimensional topology will give an idea of how to construct integrable models in multidimensional *mathematical physics* as well.
- In paper [5], we have constructed one more classical solution of pentagon equation which we called  $SL(2)$ -solution. We must recognize that we were not yet able to “globalize” it, that is build on its basis invariants of 3-manifolds, not to say of passing to higher dimensions.
- The third intriguing question is whether we can invent some quantum objects of which our classical solutions (presented in [1, 4] and [5]) are quasiclassical limits.

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### Note added in proof

For a given  $p$ , formula (4.1) gives, essentially, the squared *Reidemeister torsion* of  $L(p, q)$  multiplied by some constant. We thank Yuka Taylor for this intriguing observation.

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