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Intrinsic Characterizations of Orthogonal Separability for Natural Hamiltonians with Scalar Potentials on Pseudo-Riemannian Spaces

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Abstract

Orthogonal separability of finite-dimensional Hamiltonians is characterized by using various geometrical concepts, including Killing tensors, moving frames, the Nijenhuis tensor, bi-Hamiltonian and quasi-bi-Hamiltonian representations. In addition, a complete classification of separable metrics defined in two-dimensional locally flat Lorenzian spaces is presented.

1 Introduction

This paper continues our study [1, 2] of the problem of Liouville-integrability of a general Hamiltonian system which exploits an intrinsic characterization of orthogonal separability due to Benenti [6] by the method of moving frames. We consider a Hamiltonian system defined by a general Hamiltonian of the form

$$H = \frac{1}{2} g^{ij}(\mathbf{q}) p_i p_j + V(\mathbf{q}), \quad i, j = 1, \ldots n, \tag{1.1}$$

where g^{ij} denote the components of the contravariant metric tensor on a pseudo-Riemannian n-dimensional base manifold $\hat{M}$, V is a scalar field on $\hat{M}$ and $\mathbf{q} = (q^1, \ldots, q^n)$ denote local (position) coordinates, while $\mathbf{p} = (p_1, \ldots, p_n)$ are the corresponding conjugate (momenta) coordinates. This setting implies that the Hamiltonian vector field X_H corresponding to (1.1) is defined with respect to the canonical Poisson bi-vector $\mathbf{P_0} = \sum_{i=1}^n \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i}$ in the usual way:

$$X_H = [\mathbf{P_0}, H] \tag{1.2}$$

Here and below, unless otherwise indicated, [,] denotes the Schouten bracket [3] which generalizes the usual Lie bracket of vector fields.

In [1, 2] we combined the classical method of moving frames (see Cartan [4], Olver [5] as well as the relevant references therein) with the Benenti intrinsic criterion [6] of orthogonal separability of the system (1.1). As a result we were able to classify separable coordinates of the system (1.1) in Riemannian spaces of constant curvature. See [1, 2] for more details. In this paper we shall employ this idea to classify separable coordinates in pseudo-Rimannian spaces of constant curvature. We note that this problem has been studied by other methods (see Benenti and Rastelli [7], Kalnins [8], Miller [9] and Rastelli [10]). We shall compare our results with the results obtained previously. In addition, we characterize orthogonal separability by means of appropriate bi-Hamiltonian and quasi-bi-Hamiltonian representations for the system (1.1).

2 Geometrical preliminaries

The essence of the method of moving frames can be briefly described as follows. At each point $p \in \tilde{M}$ in a given n-dimensional pseudo-Riemannian manifold $(\tilde{M}, \mathbf{g})$, we replace the natural basis of the cotangent space $T\tilde{M}_p^*$: $(dq^1, \dots, dq^n)$ arising from a coordinate system $(q^1, \dots, q^n)$, by a basis of n pointwise linearly independent one-forms (co-vectors) $E^1, \dots, E^n \in T\tilde{M}_p^*$. The advantage of such an arrangement is the freedom to adopt the basis to the geometrical situation. In the considerations that follow the natural choice is that in which the metric tensor $\mathbf{g}$ takes its algebraic canonical form. In other words, with respect to the basis $E^a, a = 1, \dots, n$ we have

$$g_{ab} = \mathrm{diag}(1, \dots, 1, -1, \dots, -1). \tag{2.1}$$

The co-frame of one-forms $E^1, \dots, E^n$ is said to be *rigid* in this case. One can now proceed to study the relations between the one-forms $E^a \in T\tilde{M}_p^*$, their exterior derivatives dE^a and the dual basis $(E_1, \dots, E_n)$ of the tangent space $T\tilde{M}_p$ *independently of local coordinates*. Thus, we can consider an open set $A \ni p$ and *(orthonormal) moving co-frame* $E^1, \dots, E^n$ of one-forms defined in A, for which the metric tensor $\mathbf{g}$ takes the form (2.1). We note that the elements of the moving co-frame E^a and their counterparts E_a are connected with the natural basis associated with local coordinates $(q^1, \dots, q^n)$ about $p \in A$ as follows:

$$E^a = h^a{}_i dq^i, \quad E_a = h_a{}^i \frac{\partial}{\partial q^i}. \tag{2.2}$$

The structure functions $C^c{}_{ab}$ are defined by

$$[E_a, E_b] = C^c{}_{ab} E_c \quad \text{or} \quad dE^a = -\frac{1}{2} C^a{}_{bc} E^b \wedge E^c. \tag{2.3}$$

Now by (2.2) $C^c{}_{ab} = h^c{}_i (h_a{}^j h_{b,j}{}^i - h_b{}^j h_{a,j}{}^i)$, $a, b, c, i, j = 1, \dots, n$. Here and below $_{,i}$ denotes the usual partial derivative with respect to the ith coordinate. We introduce the connection coefficients Γ corresponding to the Levi-Civita connection ∇ associated with g_{ab} as follows:

$$\nabla_{E_a} E_b = \Gamma_{ab}{}^c E_c, \quad \nabla_{E_c} E^b = -\Gamma_{cd}{}^b E^d.$$

The vanishing of the torsion tensor of ∇ is expressed by

$$\Gamma_{bc}{}^a - \Gamma_{cb}{}^a - C^a{}_{bc} = 0, \tag{2.4}$$

while the curvature tensor of ∇ is given by

$$R^a{}_{bcd} = E_c\Gamma_{db}{}^a + \Gamma_{db}{}^e\Gamma_{ce}{}^a - E_d\Gamma_{cb}{}^a - \Gamma_{cb}{}^e\Gamma^a_{de} - C^e{}_{cd}\Gamma_{eb}{}^a. \tag{2.5}$$

We now define a one-form valued matrix $\omega^a{}_b$ called the *connection one-form* by

$$\omega^a{}_b := \Gamma_{cb}{}^a E^c. \tag{2.6}$$

Further, we define

$$\omega_{ab} := g_{ac}\omega^c{}_b.$$

On account of the above the connection one-forms, ω_{ab} are obviously skew-symmetric. The condition (2.4) and the definition (2.5) may be expressed in the language of differential forms as

$$dE^a + \omega^a{}_b \wedge E^b = 0, \tag{2.7}$$

and

$$d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = \Theta^a{}_b, \tag{2.8}$$

where $\wedge$ is exterior multiplication, d the exterior derivative and $\Theta^a{}_b := (1/2)R^a{}_{bcd}E^c \wedge E^d$ is the *curvature two-form.*

Finally, the equations satisfied by a valence two, symmetric, covariant Killing tensor $\mathbf{K}$ can be written in frame components as

$$K_{(ab;c)} = 0, \tag{2.9}$$

where ; denotes the covariant derivative defined by

$$K_{ab;c} := E_c K_{ab} - K_{db}\Gamma_{ca}{}^d - K_{ad}\Gamma_{cb}{}^d. \tag{2.10}$$

We shall use these formulas to study orthogonal separability of the Hamiltonian systems (1.1) defined in pseudo-Rimannian manifolds of arbitrary curvature.

3 Orthogonal separability

Recall that coordinates $(q^1, \dots, q^n)$ are called *separable* if the Hamilton-Jacobi equation

$$\frac{1}{2}g^{ij}\frac{\partial W}{\partial q^i}\frac{\partial W}{\partial q^j} + V(\mathbf{q}) = E$$

corresponding to (1.1) admits a complete solution in the following separable form:

$$W(\mathbf{q}, \mathbf{c}) = W_1(q^1, \mathbf{c}) + \cdots + W_n(q^n, \mathbf{c}),$$

where $\mathbf{c} = c_1, \dots, c_n$ are the constants of integration. Moreover, if the metric $\mathbf{g}$ of (1.1) is diagonal in these coordinates, they are also said to be *orthogonal* and the system defined by the Hamiltonian (1.1) is said to be *orthogonally separable* (see, for example, [6] for more details). The next theorem provides a coordinate-free criterion of orthogonal integrability of a Hamiltonian system defined by (1.1).

Theorem 1 (Benenti). *A Hamiltonian system defined by* (1.1) *is orthogonally separable iff there exists a valence two Killing tensor* $\mathbf{K}$ *with pointwise simple and real eigenvalues, orthogonally integrable eigenvectors and such that* $d(\hat{\mathbf{K}}dV) = 0$*, where the linear operator* $\hat{\mathbf{K}}$ *is given by* $\hat{\mathbf{K}} := \mathbf{K}\mathbf{g}$ *(or in the index form* $\hat{K}^i{}_j := K^{i\ell}g_{\ell j}$*).*

Note the existence of a Killing tensor satisfying the condition of Theorem 1 implies the existence of a second first integral of (1.1) of the following form.

$$F = K^{ij}(\mathbf{q})p_ip_j + U(\mathbf{q}), \tag{3.1}$$

We propose a criterion for orthogonal separability of a general Hamiltonian system (1.1) in Cartesian coordinates.

Theorem 2. *The following statements are equivalent.*

(a) The Hamiltonian system defined by (1.1) *is orthogonally separable with respect to Cartesian coordinates* $(x^1, \dots, x^n, y^1, \dots, y^n)$*.*

(b) The associated pseudo-Riemannian manifold $(\tilde{M}, \mathbf{g})$ *admits a valence two covariant Killing tensor* $\mathbf{K}$ *with pointwise simple and real eigenvalues and vanishing Nijenhuis tensor* $\mathbf{N}_{\hat{\mathbf{K}}}$*, where* $\hat{\mathbf{K}} := \mathbf{K}\mathbf{g}$*.*

(c) The Hamiltonian system defined by (1.1) *admits a bi-Hamiltonian representation*

$$X_{H,F} = [P_1, H] = [P_2, F] \tag{3.2}$$

with respect to two constant Poisson bi-vectors P_1 *and* P_2 *defined by*

$$P_1 := \sum_{i=1}^{n} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial y^i}, \quad P_2 := \sum_{i=1}^{n} \frac{1}{\lambda_i}\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial y^i}, \tag{3.3}$$

where λ_i*,* $i = 1, \dots, n$ *are constant and* F *is given by* (3.1) *with* $\mathbf{K}$ *having real and distinct eigenvalues.*

Proof. The equivalence $(a) \Leftrightarrow (b)$ was proven in [1, 2] based on the properties of the Nijenhuis tensor. Let us prove now the equivalence $(a) \Leftrightarrow (c)$. The part $(a) \Rightarrow (c)$ is straightforward: If a Hamiltonian system defined by (1.1) is orthogonally separable with respect to Cartesian coordinates $(x^1, \dots, x^n, y^1, \dots, y^n)$ then by Theorem 1 and the equivalence $(a) \Leftrightarrow (b)$ we conclude that in these coordinates the metric $\mathbf{g}$ of (1.1) and the valence two Killing tensor $\mathbf{K}$ of (3.1) take the forms $\mathbf{g} = \mathrm{diag}(\epsilon_1, \dots, \epsilon_n)$ and $\mathbf{K} = \mathrm{diag}(\epsilon_1\lambda_1, \dots, \epsilon_n\lambda_n)$ respectively, where $\epsilon_i = \pm 1, i = 1 \dots n$ and $\lambda_1, \dots, \lambda_n$ are real and distinct constants. The result then follow by a direct substitutuion. Assume (c), that is that in some coordinates $(x^1, \dots, x^n, y^1, \dots, y^n)$ the Hamiltonian system defined by (1.1) admits a bi-Hamiltonian representation given by (3.2) and (3.3). Substituting (1.1), (3.1) and (3.3) into (3.2), we arrive at two sets of conditions for the components of $\mathbf{g}$ and $\mathbf{K}$. The first is $g^{ij} = \lambda_i K^{ij} = \lambda_j K^{ij}, i \neq j,\ i, j = 1, \dots, n$. Which means that $g^{ij} = K^{ij} = 0, i \neq j$, that is the separability is orthogonal. The second set of conditions concerns the diagonal elements of the matrices defining $\mathbf{g}$ and $\mathbf{K}$: $g^{ii} = 1/\lambda_i K^{ii}$, or $K^{ii} = \lambda_i g^{ii}$, $i = 1, \dots, n$. Taking into account that both $\mathbf{K}$ and $\mathbf{g}$ are diagonal in this case along with the characteristic equation $|K^{ij} - \lambda g^{ij}| = 0$, we conclude that the constants $\lambda_1, \dots, \lambda_n$ are the eigenvalues of the valence Killing tensor $\mathbf{K}$ of (3.1) and the result follows from $(a) \Leftrightarrow (b)$. This completes the proof. ■

In general, however, the situation is more complex and finding a complete set of separable coordinates is a very complicated problem. To approach this problem we employ the method of moving frames, described in the preceeding section, along with Theorem 1 as follows. In an orthonormal frame of eigenvectors of the Killing tensor $\mathbf{K}$ of (3.1), both the Killing tensor and metric of (1.1) are diagonal:

$$g_{ab} = \mathrm{diag}(\epsilon_1, \ldots, \epsilon_n), \quad \epsilon_a^2 = 1, \quad a = 1, \ldots, n, \tag{3.4}$$

$$K_{ab} = \lambda_a g_{ab}, \tag{3.5}$$

where λ_a, $a = 1, \ldots, n$ are the eigenvalues of $\mathbf{K}$. Using the Frobenius theorem and integrability of the eigenvectors $E_1, \ldots, E_n$ of $\mathbf{K}$, we derive the separable coordinates $E^a = f_a dx^a$, or alternatively, $E_a = (f_a)^{-1}\frac{\partial}{\partial x^a}$. The connection coefficients take the form $\Gamma_{aab} = -\epsilon_a (f_a f_b)^{-1}\frac{\partial f_a}{\partial x^b}$, while the metric of (1.1) is as follows.

$$ds^2 = \epsilon_1 f_1^2 (dx^1)^2 + \cdots + \epsilon_n f_n^2 (dx^n)^2. \tag{3.6}$$

The Killing tensor equations are found [13] to be

$$\frac{\partial \lambda_a}{\partial x^a} = 0, \tag{3.7}$$

$$\frac{\partial \lambda_a}{\partial x^b} = (\lambda_a - \lambda_b)\frac{\partial}{\partial x^b}(\ln f_a^2), \tag{3.8}$$

with the corresponding integrability conditions [13]:

$$\frac{\partial^2}{\partial x^a \partial x^b}(\ln f_a^2) + \frac{\partial}{\partial x^a}(\ln f_b^2)\frac{\partial}{\partial x^b}(\ln f_a^2) = 0, \quad a \neq b, \tag{3.9}$$

$$\begin{aligned} &\frac{\partial}{\partial x^b \partial x^c}(\ln f_a^2) - \frac{\partial}{\partial x^b}(\ln f_a^2)\frac{\partial}{\partial x^c}(\ln f_a^2) + \\ &\frac{\partial}{\partial x^b}(\ln f_a^2)\frac{\partial}{\partial x^c}(\ln f_b^2) + \frac{\partial}{\partial x^c}(\ln f_a^2)\frac{\partial}{\partial x^b}(\ln f_c^2) \quad = \quad 0, \quad a \neq b \neq c \neq a. \end{aligned} \tag{3.10}$$

For the the potential V of (1.1), using the condition $d(\mathbf{K}dV) = 0$ of Theorem 1, we derive the following equation in coordinates $x^1, \ldots, x^n$:

$$V_{,[a}\lambda_{[a,b]} + \lambda_{[a}V_{,[a,b]} = 0,$$

which after removing derivatives $\lambda_{a,b}$, using the Killing equation, yields

$$V_{,ab} + V_{,a}(\ln f_a^2)_{,b} + V_{,b}(\ln f_b^2)_{,a} = 0 \tag{3.11}$$

with the corresponding equation for the function U of (3.1): $U_{,a} = \lambda_a V_{,a}$ (see [6] for more details). We observe that in the orthogonal case the eigenvalues for the Killing tensor, the metric functions, the potential V and the function U are independent of the signature of the pseudo-Riemannian metric $\mathbf{g}$ of (1.1).

4 Two dimensional pseudo-Riemannian manifolds

In this section we apply the above analysis to classify the separable cases for the Hamiltonian system (1.1) defined in a pseudo-Riemannian manifold $(\tilde{M}, \mathbf{g})$ of dimension two. We proceed by solving the integrability conditions (3.10) to obtain the form of the metric in canonical separable coordinates and the Killing tensor equations (3.7) and (3.8) to obtain λ_1 and λ_2 in separable coordinates, seeking the general solution. Next we solve (3.11) to obtain the forms of the separable potential and the corresponding equation for the function U of the first integral. Finally, we write down the form of the general separable Hamiltonian (1.1) and corresponding second first integral F (3.1). To study the separable cases in locally flat pseudo-Riemannian spaces we employ the following formula for the Riemann curvature tensor in frame components.

$$R_{1212} = E_1\Gamma_{221} - E_2\Gamma_{121} - \epsilon_1(\Gamma_{112})^2 - \epsilon_2(\Gamma_{221})^2. \tag{4.1}$$

Solving the integrability conditions (3.10) along with the Killing tensor equations (3.7), (3.8), we use the derived eigenvalues λ_1 and λ_2 of $\mathbf{K}$ (as its invariants) to classify the separable coordinates. It is natural to consider the following Separable Cases (SC). SC1: λ_1 and λ_2 both constant; SC2: λ_1 constant, λ_2 non-constant (or, alternatively, the other way around) and, finally, SC3: λ_1 and λ_2 both non-constant. Performing the analysis outlined above, we arrive at the following formulas in each case for the corresponding metric $\mathbf{g}$ of (1.1), Killing tensor $\mathbf{K}$ of (3.1), Hamiltonian (1.1) and second first integral (3.1) as well as the Riemann curvature tensor in SC2 and SC3.

SC1.
Metric:

$$ds^2 = \epsilon_1 du^2 + \epsilon_2 dv^2. \tag{4.2}$$

Killing tensor:

$$K_{ab} = \mathrm{diag}(\epsilon_1\lambda_1, \epsilon_2\lambda_2). \tag{4.3}$$

Hamiltonian:

$$H = \frac{1}{2}(\epsilon_1 p_u^2 + \epsilon_2 p_v^2) + C(u) + D(v). \tag{4.4}$$

First integral:

$$F = \epsilon_1\lambda_1 p_u^2 + \epsilon_2\lambda_2 p_v^2 + 2(\lambda_1 C(u) + \lambda_2 D(v)). \tag{4.5}$$

SC2.
Metric:

$$ds^2 = \epsilon_1 du^2 + \epsilon_2 g^2(u) dv^2. \tag{4.6}$$

Killing tensor:

$$K_{ab} = \mathrm{diag}(\epsilon_1\lambda_1, \epsilon_2(\lambda_1 + m g^2(u))). \tag{4.7}$$

Hamiltonian:

$$H = \frac{1}{2}(\epsilon_1 p_u^2 + \epsilon_2 g^{-2}(u) p_v^2) + g^{-2}(u) D(v) + C(u). \tag{4.8}$$

First integral:

$$F = \epsilon_2 p_v^2 + 2D(v). \tag{4.9}$$

Riemann curvature tensor:

$$R_{1212} = -\epsilon_2 \frac{g_{uu}}{g}. \tag{4.10}$$

SC3 (General Case).
Metric:

$$ds^2 = (A(u) + B(v))(\epsilon_1 du^2 + \epsilon_2 dv^2). \tag{4.11}$$

Note that the metric (4.11) has the general *Liouville form*.
Killing tensor:

$$K_{ab} = \mathrm{diag}(\epsilon_1 B(v), -\epsilon_2 A(u)). \tag{4.12}$$

Hamiltonian:

$$H = (A(u) + B(v))^{-1}\Big[\frac{1}{2}(\epsilon_1 p_u^2 + \epsilon_2 p_v^2) + C(u) + D(v)\Big]. \tag{4.13}$$

First integral:

$$F = (A(u) + B(v))^{-1}\Big[\epsilon_1 B(v) p_u^2 - \epsilon_2 A(u) p_v^2 + 2(B(v)C(u) - A(u)B(v))\Big]. \tag{4.14}$$

Riemann curvature tensor:

$$R_{1212} = -\frac{(A(u) + B(v))(\epsilon_2 A''(u) + \epsilon_1 B''(v)) - \epsilon_2 (A'(u))^2 - \epsilon_1 (B'(v))^2}{2(A(u) + B(v))^3}. \tag{4.15}$$

It was shown in [11] that separability in the Cartesian coordinates when $\epsilon_1 = \epsilon_2 = 1$ was equivalent to the existence of certain bi-Hamiltonian and Lax representations of fixed types. The first part of this result was generalized in Theorem 2 for arbitrary n. For the case of two degrees of freedom the above result can be generalized for the arbitrary case without making any assumption on either the type of separable coordinates or signature of the corresponding metric. Recall that the bi-Hamiltonian property (3.2) is in general a very restrictive condition which does not hold true in general for any system of coordinates for a Liouville-integrable sysem defined by (1.1). Recall also that a Hamiltonian system is said to be quasi-bi-Hamiltonian (QBH) if its vector field $X_{H,F}$ enjoys the following representations

$$X_{H,F} = [P_1, H] = \frac{1}{\rho}[P_2, F], \tag{4.16}$$

where P_1 and P_2 are compatible Poisson bi-vectors and ρ is a function. If $\rho = -\prod_{i=1}^{n} \mu_i$, where μ_i, $i = 1, \ldots, n$ are eigenvalues of the operator $A = P_2 P_1^{-1}$, assuming A is of minimal degeneracy (i.e., it has exactly n distinct eigenvalues) and P_1 is invertible, the QBH system (4.16) is called *Pfaffian* (see, for instance [12]). We use this concept to complete the characterization of orthogonally separable Hamiltonian systems with two degrees of freedom in the following theorem.

Theorem 3. *The following statements are equivalent.*

(a) The pseudo-Riemannian manifold $(\tilde{M}, \mathbf{g})$ *defined by* (4.17) *admits a valence two Killing tensor* $\mathbf{K}$ *with real and distinct eigenavlues.*

(b) There exist coordinates (u, v) *with respect to which the metric takes the form* (4.11)

(c) The Hamiltonian system defined by

$$H = \frac{1}{2} g^{ij}(\mathbf{q}) p_i p_j + V(\mathbf{q}), \quad i, j = 1, 2 \tag{4.17}$$

in the pseudo-Riemannian manifold $(\tilde{M}, \mathbf{g})$ *can be integrated by separation of variables.*

(d) The completely integarable Hamiltonian system defined by (4.17) *with a second first integral of the type* (3.1) *defines a QBH of the Pfaffian type* (4.16) *with* $\rho = -A(u)B(v)$*, where* $A(u)$*,* $B(v)$ *are the eigenvalues of the linear operator* $\hat{\mathbf{K}} = \mathbf{K}\mathbf{g}$*.*

Proof. The $(b) \Leftrightarrow (c)$ was proven first in 1881 using local coordinates by Morera [14], who also extracted the four separable systems of coordinates in the Euclidean flat space. The $(a) \Leftrightarrow (c)$ part is simply a restatement of Theorem 1 and $(a) \Leftrightarrow (b)$ follows from the above considerations. The implication $(c) \Rightarrow (d)$ follows from Proposition 2 [12]. Indeed, it is shown in [12] that there exists a system of coordinates with respect to which a general Pfaffian quasi-bi-Hamiltonian system with two degrees of freedom admits the Gantmakher form [15] and is separable as such. It follows from the above, that the Hamiltonian systems defined by (1.1) in this case are Liouville, since both H and F are quadratic in momenta. The converse statement can be verified by a straghtforward calculation using the formula (4.16) for $P_1 = \partial_u \wedge \partial_{p_u} + \partial_v \wedge \partial_{p_v}$ and $P_2 = A(u)\partial_u \wedge \partial_{p_u} + B(v)\partial_v \wedge \partial_{p_v}$, along with (4.13) and (4.14). ■

The following corollary characterizes admissible coordinates in a locally flat Riemannian space via Pfaffian quasi-bi-Hamiltonian structures.

Corollary 1. *Let a Hamiltonian system with two degrees of freedom defined by* (1.1) *in a locally flat Riemannian space be Pfaffian quasi-bi-Hamiltonian in some system of coordinates* (u, v) *with respect to the following Poisson bi-vectors:* $P_1 = \partial_u \wedge \partial_{p_u} + \partial_v \wedge \partial_{p_v}$ *and* $P_2 = A(u)\partial_u \wedge \partial_{p_u} + B(v)\partial_v \wedge \partial_{p_v}$ *with* $\rho = -A(u)B(v)$*. Then this system of coordinates is one of the following three types: Polar, parabolic or elliptic-hyperbolic. Moreover, in the degenerate case when the system is bi-Hamiltonian defined by the constant Poisson bi-vectors with* $-B(v) = A(u) = 1$ *in* P_2 *, then the system of coordinate* (u, v) *is Cartesian.*

5 Two-dimensional locally flat pseudo-Riemannian spaces

In this section we give a complete classification of the canonical separable coordnate systems and the corresponding metrics for the case when $R_{1212} = 0$. We also present the coordinate transformations relating these coordinates to Cartesian coordinates. The forms of the Hamiltonian and first integral in each case will not be explicitly written, since they can be obtained directlty from the relevant formulas of the preceeding section, once the metric has been derived.

Taking into account the condition $R_{1212} = 0$ in the general case, we arrive at the following separable cases. We have used the available coordinate freedom to eliminate inessential constants of integration in order to obtain the simplest possible canonical forms for the metric in each case.
SC1.
Metric:

$$ds^2 = \epsilon_1 du^2 + \epsilon_2 dv^2. \tag{5.1}$$

SC2.

$$ds^2 = \epsilon^2 du^2 + \epsilon_2 u^2 dv^2. \tag{5.2}$$

Locally flat Riemannian space: $\epsilon_1 = \epsilon_2 = 1$.
Metric:

$$ds^2 = du^2 + u^2 dv^2. \tag{5.3}$$

Coordinate transformation:

$$x = u \cos v, \quad y = u \sin v, \quad u > 0, 0 < v \leq 2\pi. \tag{5.4}$$

Locally flat Lorentzian space: $\epsilon_1 = -\epsilon_2 = 1$.
Metric:

$$ds^2 = du^2 - u^2 dv^2. \tag{5.5}$$

Coordinate transformation:

$$t = u \cosh v, \quad x = u \sinh v. \tag{5.6}$$

We note that the case λ_1 is non-constant, λ_2 is constant is equivalent to SC3 in both locally flat Riemannian and Lorentzian spaces.

SC3.
Separating the equation (4.15), we get

$$\frac{A'''}{A'} = -\epsilon_1 \epsilon_2 \frac{B'''}{B'} = \alpha,$$

where α is constant. The Riemannian and Lorenzian cases must now be considered separately.

Locally flat Riemannian space. Without loss of generality we may set $\alpha = \mu^2$, where $\mu \geq 0$. The case $\mu = 0$ leads to the following metric and coordinate transformation.
Metric:

$$ds^2 = (u^2 + v^2)(du^2 + dv^2). \tag{5.7}$$

Coordinate transformation (two parabolic coordinate systems):

$$x = \frac{1}{2}(u^2 - v^2), \quad y = uv. \tag{5.8}$$

The case of $\mu > 0$ yields, after solving (4.15).
Metric:

$$ds^2 = a^2(\cosh^2 u - \cos^2 v)(du^2 + dv^2). \tag{5.9}$$

Coordinate transformation (two elliptic-hyperbolic coordinate systems):

$$x = a\cosh u \cos v, \quad y = a \sinh u \sin v. \tag{5.10}$$

Locally flat Lorentzian case. In this case we must set $\alpha = \epsilon\mu^2$, where $\epsilon^2 = 1$ and $\mu \geq 0$. This transforms the equation (4.15) as follows.

$$\frac{A'''}{A'} = \frac{B'''}{B'} = \epsilon\mu^2.$$

Now the case of $\mu = 0$ leads to the following metrics and coordinate transformations.
Metric:

$$ds^2 = (u + v)(du^2 - dv^2). \tag{5.11}$$

Coordinate transformation:

$$t = \frac{1}{4}(u + v)^2 + \frac{1}{2}(u - v), \quad x = -\frac{1}{4}(u + v)^2 + \frac{1}{2}(u - v). \tag{5.12}$$

Metric:

$$ds^2 = (u^2 - v^2)(du^2 - dv^2). \tag{5.13}$$

Coordinate transformation:

$$t = \frac{1}{2}(u^2 + v^2), \quad x = uv. \tag{5.14}$$

When $\mu > 0, \epsilon = -1$, we derive
Metric:

$$ds^2 = b(\sin u + \sin v)(du^2 + dv^2). \tag{5.15}$$

Coordinate transformation:

$$t = 2\sqrt{2b}\cos\left(\frac{u}{2} + \frac{\pi}{4}\right)\cos\left(\frac{v}{2} + \frac{\pi}{4}\right), \quad x = 2\sqrt{2b}\sin\left(\frac{u}{2} + \frac{\pi}{4}\right)\sin\left(\frac{v}{2} + \frac{\pi}{4}\right). \tag{5.16}$$

Finally, for $\mu > 0, \epsilon = 1$, we derive the following remaining inequivalent metrics.
Metric:

$$ds^2 = (e^u + e^v)(du^2 - dv^2). \tag{5.17}$$

Coordinate transformation:

$$t + x = 2\sinh\left(\frac{1}{2}(u - v)\right), \quad t - x = 4e^{\frac{1}{2}(u+v)}. \tag{5.18}$$

Metric:

$$ds^2 = (e^u - e^v)(du^2 - dv^2). \tag{5.19}$$

Coordinate transformation:

$$t + x = 2\cosh\left(\frac{1}{2}(u - v)\right), \quad t - x = 4e^{\frac{1}{2}(u+v)}. \tag{5.20}$$

Metric:

$$ds^2 = b(\sinh u + \sinh v)(du^2 - dv^2), \quad b > 0 \tag{5.21}$$

Coordinate transformation:

$$t + x = 2\sqrt{2b}\cosh\left(\frac{1}{2}(u + v)\right), \quad t - x = 2\sqrt{2b}\sinh\left(\frac{1}{2}(u - v)\right). \tag{5.22}$$

Metric:

$$ds^2 = b(\cosh u + \cosh v)(du^2 - dv^2), \quad b > 0. \tag{5.23}$$

Coordinate transformation:

$$t = 2\sqrt{2b}\sinh\frac{u}{2}\cosh\frac{v}{2}, \quad x = 2\sqrt{2b}\cosh\frac{u}{2}\sinh\frac{v}{2}. \tag{5.24}$$

Metric:

$$ds^2 = b(\cosh u - \cosh v)(du^2 - dv^2), \quad b > 0. \tag{5.25}$$

Coordinate transformation:

$$t = 2\sqrt{2b}\cosh\frac{u}{2}\cosh\frac{v}{2}, \quad x = 2\sqrt{2b}\sinh\frac{u}{2}\sinh\frac{v}{2}. \tag{5.26}$$

This completes the classification. We note that our classification differs somewhat from the classification obtained previously by Kalnins [8] employing other methods. More specifically, the forms of the metrics (5.21), (5.23) and (5.25) found here contain an arbitrary positive parameter b, which, we believe, is essential and cannot be dispensed with. It is analogous to the parameter a that appears in the flat metric (5.9) in the Riemannian case. Recall that a is interpretable as the (half) distance between the foci of the ellipses of the elliptic-hyperbolic coordinate system.

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References

[1] Bruce A T, McLenaghan R G, Smirnov R G, A geometrical approach to the problem of integrability of Hamiltonian systems by the method of separation of variables, *J. Geom. Phys.*, in press.

[2] Bruce A T, McLenaghan R G, Smirnov R G, Benenti's theorem and the method of moving frames, *Rep. Math. Phys.*, in press.

[3] Schouten J A, Über Differentalkomitanten zweier kontravarianter Grössen, *Proc. Kon. Ned. Akad. Amsterdam* **43** (1940) 449–452.

[4] Cartan E, Leçons sur la Géométrie des Espaces de Riemann, Gauthier-Villars, 1951.

[5] Olver P J, Moving frames and joint differential invariants, *Regular and Chaotic Mechanics*, **4** (1999), 3–18.

[6] Benenti S, Intrinsic characterization of the variable separation in the Hamilton-Jacobi equation, *J. Math. Phys.* **38** (1997) 6578–6602.

[7] Benenti S, Rastelli G, Sistemi di coordinate separabili nel piano euclideo, *Mem. Acad. Sci. Torino* **15** (1991), 1–21.

[8] Kalnins E G, On the separation of variable of the Laplace equation in two- and three-dimensional Minkowski space, *SIAM J. Math. Anal.* **6** (1975), 340–374.

[9] Miller, W Jr, Symmertry and Separation of Variables, Addison-Wesley, 1977.

[10] Rastelli, G, Singular points of the orthogonal separable coordinates in the hyperbolic plane, *Rend. Sem. Mat. Univ. Politec. Torino* **52** (1994), 407–434.

[11] McLenaghan R G, Smirnov R G, A class of two-dimensional Liouville integrable Hamiltonian systems, *J. Math. Phys.* **41**, No. 10 (2000) 6879–6889.

[12] Brouzet R, Caboz R, Rabenivo J, Ravoson V, Two degrees of freedom quasi-bi-Hamiltonian systems, *J. Phys A: Math. Gen.* **29** (1996) 2069–2076.

[13] Eisenhart L P, Riemannian Geometry, Princeton University Press, Princeton, 1926.

[14] Morera G, Sulla separazione delle variabili nelle equazioni del moto di un punto materiale su una superfice, *Atti. Sci. di Torino* **16** (1881) 276–295.

[15] Gantmakher F, Lectures in Analytical Mechanics, MIR, 1970.