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Hamiltonian Structure and Linear Stability of Solitary Waves of the Green-Naghdi Equations

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Abstract

We investigate linear stability of solitary waves of a Hamiltonian system. Unlike weakly nonlinear water wave models, the physical system considered here is nonlinearly dispersive, and contains nonlinearity in its highest derivative term. This results in more detailed asymptotic analysis of the eigenvalue problem in presence of a large parameter. Combining the technique of singular perturbation with the Evans function, we show that the problem has no eigenvalues of positive real part and solitary waves of small amplitude are linearly stable.

1 Introduction

It has long been an issue to model and understand the full water wave problem due to its broad applications to coastal engineering, and fluid mechanics. The full water wave problem is imposed as a fully nonlinear system. A great deal of effort has been made to directly tackle it both numerically and analytically, the problem is still not completely well understood due to the complexity of its nonlinearity. Approximate model equations of this problem have been developed to understand its physical ramifications. One primary approach was linear approximation under the assumption of a small perturbation from a quiescent state. While using a higher order approximation, weakly nonlinear models have been developed in the parameter regime of small amplitude and long wave length. Among them are the well-known Korteweg-de Vries (KdV) and Boussinesq equations [19]. The derivation of these equations confirmed the existence of solitary waves for the the full water wave problem, as a consequence, leading to the development of theories on solitons, integrability and inverse scattering transform [1]. Despite both physical and mathematical importance of the weakly nonlinear approximation for the full water wave problem, it has limitations to model higher nonlinear phenomena, including high-amplitude waves and wave breaking. Efforts have been made to obtain higher nonlinear model equations. Among them, the Green-Naghdi (GN) equations [10], [11], [6], [7]

$$\eta_t + (u\eta)_x = 0, \quad u_t + uu_x + \eta_x = \frac{1}{3\eta}\big(\eta^2 \frac{d}{dt}(\eta u_x)\big)_x, \tag{1.1}$$

and its alternations for variable depth of fluid were derived for both free surface and inter-facial surface waves in the regime of long wave length but relatively large amplitude compared with the depth of the fluid. Here, $\frac{d}{dt}(\eta u_x) = (\eta u_x)_t + u(\eta u_x)_x$, η and u represent the surface disturbance and the horizontal velocity, respectively. Here, we shall analytically investigate linear stability of solitary waves of the GN equations. It is well-known that solitary waves of the KdV equation are orbitally stable and this has been proved by using a variational method due to the fact that its solitary waves are minimizers of its Hamiltonian functional [4]. However, this is not a common property for the full water wave problem [5], a class of Boussinesq equations [14], and the GN equations. As a matter of fact, the second variation of their Hamiltonian functional subject to certain constraints are indefinite. Therefore, variational approach for the stability analysis of these systems may not be applicable. Therefore, we use the techniques of the Evans function and singular perturbation to investigate eigenvalue problems for solitary waves of the GN equations. Evans [9] has used this method for the stability issue of the impulses in nerve axon equation. Later, this method was further developed by Jones *et al,* [2], [3], and Pego and Weinstein [13], [14] to apply it to a wide range of nonlinear evolution equations, including the weakly nonlinear KdV equation and Boussinesq equations. However, compared with weakly nonlinear models, the higher nonlinearity possessed by the GN equations demands more detailed analysis on this system. To deal with singular perturbation problem in presence of a large eigenvalue parameter, we shall decompose operators to separate slow and fast flows in the dynamical system. Then the singular perturbation method [8], [18] is applied to investigating the fast flows. As a result, we show that the dynamical system has no homoclinic orbits in presence of a large eigenvalue parameter. The fact that the KdV equation is a second order approximation of the GN equations and analyticity of the Evans function are used to eigenvalue analysis in a neighbourhood of zero.

2 Hamiltonian structure of the GN equations

The GN equations have a Hamiltonian structure of the form

$$\begin{pmatrix} m_t \\ \eta_t \end{pmatrix} = J \begin{pmatrix} \frac{\delta H}{\delta m} \\ \frac{\delta H}{\delta \eta} \end{pmatrix},$$

where the Hamiltonian functional H takes the form

$$H = \frac{1}{2} \int \left(\eta u^2 + \frac{1}{3} \eta^3 u_x^2 + (\eta - 1)^2 \right) dx,$$

and the Hamiltonian operator J is expressed as

$$J = - \begin{pmatrix} \partial m + m\partial & \eta\partial \\ \partial\eta & 0 \end{pmatrix},$$

$m = \mathcal{L}u = \eta u - \frac{1}{3}\left(\eta^3 u_x\right)_x$ and ∂ is the derivative with respect to the spatial variable x. One-parameter symmetry groups of the GN equations include the space translation $(x+\epsilon, t, \eta, u)$, the time translation $(x, t+\epsilon, \eta, u)$, the Galilean boost $(x+\epsilon t, t, \eta, u+\epsilon)$ and the scaling $e^\epsilon(e^\epsilon x, t, e^\epsilon \eta, u)$. Using the characteristics of the first three symmetry groups and the Hamiltonian operator J, one may recover the following conserved quantities. $Q = \int m\big(1-\frac{1}{\eta}\big)dx$, H and $\int(tm - x(\eta-1))\,dx$, respectively. The last symmetry does not yield a conservation law with respect to J. In addition, $\int \frac{m}{\eta}\,dx$ and $\int(\eta-1)dx$ are distinguished functionals for J.

A direct calculation shows that each solitary wave solution $(m(x-ct), \eta(x-ct))$ of the GN equations is a critical point of the functional $H - c\,Q$, *i.e.* the identities

$$\frac{\delta H}{\delta m} - c\frac{\delta Q}{\delta m} = 0, \qquad \frac{\delta H}{\delta \eta} - c\frac{\delta Q}{\delta \eta} = 0,$$

hold at the solitary wave, where

$$\eta = 1 + (c^2-1)\,\mathrm{sech}^2\Big(\frac{\sqrt{3(c^2-1)}\,(x-ct)}{2c}\Big), \quad u = c\big(1-\frac{1}{\eta}\big), \quad m = \mathcal{L}u,$$

for any constant c with $|c|>1$. Since the second variational derivative $H'' - cQ''$ is a self-adjoint operator, it follows from Weyl's essential spectrum theorem [17] that the essential spectrum of the operator $H'' - c\,Q''$ evaluated at a solitary wave coincides with that of its asymptotic operator $H''_\infty - c\,Q''_\infty$ as $|x|\to\infty$. Because

$$H''_\infty - c\,Q''_\infty = \begin{pmatrix} \mathcal{L}_0^{-1} & -c \\ -c & I \end{pmatrix},$$

with $\mathcal{L}_0^{-1} = (I - \partial^2/3)^{-1}$, it follows that the essential spectrum of $H'' - cQ''$ consists of the intervals $[\frac{1-\sqrt{1+4c^2}}{2}, 1-|c|]$ and $[\frac{1+\sqrt{1+4c^2}}{2}, 1+|c|]$. Hence, the operator $H'' - c\,Q''$ has a negative, infinite-dimensional spectral space. This fact fails to satisfy one of the basic assumptions on $H - c\,Q$ to be used for nonlinear stability analysis, *i.e.* $H'' - cQ''$ has at most a finite-dimensional, negative spectral space [4], [12]. Therefore, as the first step to consider the stability issue, we investigate linear stability of solitary waves.

3 Main Results

Assume that the depth of the fluid flow is $h=1$ so that the surface disturbance η and the horizontal velocity u satisfy the condition $\eta\to 1$ and $u\to 0$ as $|x|\to\infty$. One may also multiply the first equation of (1.1) by u and the second equation by η, and then adding the resulting equations together. By letting $w = u\eta$, we obtain the equivalent system

$$\begin{aligned} &\eta_t + w_x = 0, \\ &w_t = -\big(\tfrac{w^2}{\eta}\big)_x - \eta\eta_x + \tfrac{1}{3}\Big(\eta^2\tfrac{d}{dt}\big(\eta(\tfrac{w}{\eta})_x\big)\Big)_x. \end{aligned} \tag{3.1}$$

The system (3.1) will be used to conduct linear stability analysis of solitary waves.

A solitary wave solution $(w(x-ct), \eta(x-ct))$ of the system (3.1) takes the form of sech-functions such that

$$\begin{aligned} &w = c(c^2-1)\,\mathrm{sech}^2\big(\tfrac{\sqrt{3(c^2-1)}\,(x-ct)}{2c}\big), \\ &\eta = 1 + \tfrac{w}{c} \end{aligned}$$

for any fixed constant c with $|c| > 1$. Then we use the standard expression

$$\tilde{w} = w_c(x-ct) + e^{\lambda t} w(x-ct), \qquad \tilde{\eta} = \eta_c(x-ct) + e^{\lambda t}\eta(x-ct)$$

to derive the eigenvalue problem as a system of the following two ordinary differential equations.

$$\begin{aligned} \lambda w &= \big(J_1 w + J_2 w' + J_3 w'' + J_4 \eta + J_5 \eta' + J_6 \eta''\big)', \\ \lambda \eta &= c\,\eta' - w', \end{aligned} \tag{3.2}$$

where $'$ represents the derivative with respect to $\xi = x - ct$, and J_k's are functions of the solitary wave and its derivatives such that

$$\begin{aligned} J_1 &= c - \tfrac{2w_c}{\eta_c} - \tfrac{\lambda}{3}\eta_c\eta_c' - \tfrac{2}{3}\eta_c' w_c' - \tfrac{2}{3} w_c \eta_c'' + \tfrac{2}{3}\eta_c w_c'' + \tfrac{2w_c(\eta_c')^2}{3\eta_c}, \\ J_2 &= \tfrac{\lambda}{3}\eta_c^2 + \tfrac{c\,\eta_c\eta_c'}{3} - \tfrac{2\eta_c' w_c}{3}, \qquad J_3 = \tfrac{2\eta_c w_c}{3} - \tfrac{c}{3}\eta_c^2, \\ J_4 &= -\tfrac{2c\,\eta_c w_c''}{3} + \tfrac{w_c^2}{\eta_c^2} - \eta_c + \tfrac{c}{3}\eta_c' w_c' + \tfrac{2}{3} w_c w_c'' - \tfrac{w_c^2(\eta_c')^2}{3\eta_c^2}, \\ J_5 &= \tfrac{c\eta_c w_c'}{3} - \tfrac{2}{3} w_c w_c' + \tfrac{2w_c^2\eta_c'}{3\eta_c}, \qquad J_6 = -\tfrac{1}{3}w_c^2. \end{aligned}$$

Lemma 1. *Let (w,η) be a solution of the system (3.2). Then there are an constant $N > 0$, functions u_k, for $1 \le k \le 4$, $\varDelta_1$ and $\varDelta_2$, depending on the solitary wave solution (w_c, η_c) and λ, such that whenever $|\lambda| \ge N$, the linear operator $\mathcal{E}_\lambda$, defined by*

$$\mathcal{E}_\lambda f = -\big(a_2 + \frac{s_1}{\lambda^2\rho}\big) f'' - \big(a_1 - \frac{s_2 b}{\lambda^3\rho} + \frac{u_1}{\lambda}\big) f' + \big(a_0 + 1 - \frac{u_2}{\lambda}\big) f,$$

is invertible on the function space

$$\{f;\ \sup_{|x|<\infty} e^{a|x|}|f^{(j)}(x)| < \infty, j = 0,1,2\}$$

for any fixed $a \ge 0$, and the first equation of the system (3.2) can be decomposed as

$$\begin{aligned} \lambda\mathcal{E}_\lambda w = {} & \mathcal{E}_\lambda\Big(\frac{ca_2 - \frac{2w_c\eta_c}{3}}{a_2 + \frac{s_1}{\lambda^2\rho}} w' + \frac{w_c^2}{3\big(a_2 + \frac{s_1}{\lambda^2\rho}\big)}\eta' + \frac{c}{\lambda\rho\big(a_2 + \frac{s_1}{\lambda^2\rho}\big)}\eta\Big) + \\ & - \mathcal{E}_\lambda\Big(\frac{\varDelta_1 + J_2^* + J_3' + u_3/\lambda}{a_2 + \frac{s_1}{\lambda^2\rho}} w + \frac{\varDelta_2 + J_5 + J_6' + u_4/\lambda}{a_2 + \frac{s_1}{\lambda^2\rho}}\eta\Big) + \frac{1}{\lambda^2}\mathcal{U}, \end{aligned}$$

where $a_0 = \frac{1}{3}(\eta_c\eta_c')'$, $a_1 = \frac{1}{3}\eta_c\eta_c'$ and $a_2 = \frac{1}{3}\eta_c^2$, and the coefficients b, ρ and s_k, for $k = 1,2,3$, satisfy the equations

$$\begin{aligned} & 1 - s_1 = \frac{a_2 s_2 b\, c}{\lambda^2\big(a_2 + \frac{s_1}{\lambda^2\rho}\big)}, \quad c - \frac{ca_2}{a_2 + \frac{s_1}{\lambda^2\rho}} = \frac{(1-s_2)b}{\lambda^2\rho}, \quad b = \frac{-c}{a_2 + \frac{s_1}{\lambda^2\rho}}, \\ & s_3 = \frac{s_2}{\lambda^2\rho(a_2 + \frac{s_1}{\lambda^2\rho})}, \qquad \rho = 1 + \frac{(1-s_3)bc}{\lambda^2}. \end{aligned}$$

In addition, $\mathcal{U}$ is a linear function of $y = (w,\eta)$, y' and y'', together with the expressions u_k, $\varDelta_1$ and $\varDelta_2$, satisfying the inequalities

$$\begin{aligned} & |\mathcal{U}| \le M\gamma^2 e^{-\gamma|x|}(\|y\| + \|y'\| + \|y''\|), \quad |u_k| \le M\gamma^2 e^{-\gamma|x|}, \\ & |\varDelta_1| \le M\gamma^2 e^{-\gamma|x|}, \quad |\varDelta_2| \le M\gamma^2 e^{-\gamma|x|}. \end{aligned}$$

for $\gamma = 1 - c^{-2}$ and some constant M independent of λ for any $|\lambda| \ge N$.

It follows from the above Lemma that the system (3.2) may be expressed as

$$\begin{pmatrix} w' \\ \eta' \end{pmatrix} = \mathcal{A}_1 \begin{pmatrix} w \\ \eta \end{pmatrix} - \frac{3\big(a_2 + \frac{s_1}{\lambda^2\rho}\big)\mathcal{E}_\lambda^{-1}(\mathcal{U})}{\lambda^2 c^2} \begin{pmatrix} c \\ 1 \end{pmatrix}, \tag{3.3}$$

where

$$\begin{aligned} \mathcal{A}_1 = \lambda \begin{pmatrix} \frac{3a_2}{c} & -\frac{w_c^2}{c^2} \\ \frac{3a_2}{c^2} & \frac{3a_2}{c} - \frac{2w_c\eta_c}{c^2} \end{pmatrix} + \frac{3}{\lambda c^2 \rho} \begin{pmatrix} cs_1 & -c^2 \\ s_1 & -c \end{pmatrix} + \\ + \frac{3}{c^2} \begin{pmatrix} c(\Delta_1 + J_2^* + J_3') & c(\Delta_2 + J_5 + J_6') \\ \Delta_1 + J_2^* + J_3' & \Delta_2 + J_5 + J_6' \end{pmatrix} + \frac{3}{c^2\lambda} \begin{pmatrix} cu_3 & cu_4 \\ u_3 & u_4 \end{pmatrix}. \end{aligned}$$

The kernels of the operators $\mathcal{E}_\lambda$ and $\partial I - \mathcal{A}_1$ correspond to slow flows and fast flows asymptotically in the dynamical system (3.2). Since $\mathcal{E}_\lambda$ is invertible, one sees that slow flows are not homoclinic orbits. Furthermore, applying the singular perturbation method [18], we obtain the following estimates for fundamental solution of the system $Y' = \mathcal{A}_1 Y$.

Theorem 1. *Let $X = X(\xi)$ be the fundamental solution of the system $Y' = \mathcal{A}_1 Y$. Then there is a constant $M > 0$ independent of λ of sufficiently large magnitude such that the inequality $|X(\xi)X^{-1}(s)| \le |\lambda| M$ holds for $c > 0$, and any ξ and s with $-\infty < \xi \le s < \infty$. If $c < 0$, then the inequality $|X(\xi)X^{-1}(s)| \le |\lambda| M$ is valid for any ξ and s with $-\infty < s \le \xi < \infty$.*

Applying the above estimates of the fundamental solution X to the equations (3.3), one may conclude that the dynamical system (3.2) has no homoclinic orbits when $|\lambda|$ becomes sufficiently large.

Next, we substitute the KdV scaling

$$s = \gamma(x - ct), \quad \tau = c\gamma^3 t, \quad \eta = 1 + \gamma^2 v_1 + \gamma^4 v_2 + \cdots, \quad w = c(\gamma^2 u_1 + \gamma^4 u_2 + \cdots)$$

into the system (3.1), from which we derive the second order approximation

$$\begin{aligned} & v_{1s} - u_{1s} = 0, \quad v_{1\tau} - v_{2s} + u_{2s} = 0, \\ & u_{1\tau} - u_{2s} + v_{2s} - v_{1s} = -(u_1^2)_s - v_1 v_{1s} - \tfrac{1}{3} u_{1sss}. \end{aligned}$$

It follows that u_1 is a solution of the KdV equation.

$$u_{1\tau} - \frac{1}{2}u_{1s} + \frac{3}{4}(u_1^2)_s + \frac{1}{6}u_{1sss} = 0.$$

Then one may use this fact to show that the Evans function of the linearized KdV equation about its solitary waves also approximates that of the GN equations in the KdV scaling. Using the technique in [15], we show that in the regime of the KdV approximation, the Evans function of the GN equations does not vanish in a neighbourhood of zero except the zero itself. Combining the above results, we draw the conclusion.

Theorem 2. *For any $\gamma > 0$ sufficiently small, the problem (3.1) has no other eigenvalues except $\lambda = 0$ that has a geometric multiplicity of one and algebraic multiplicity of two.*

Let $z_t = \mathcal{A}z$ be the linearized system of the GN equations (3.1). The linear stability analysis of (3.1) relies on properties of the spectrum of the semigroup $e^{\mathcal{A}t}$. Based on the previous eigenvalue analysis for the system $\lambda y = \mathcal{A}y$ and the result by Prüss [16], one may show that $e^{\mathcal{A}t}$ has no unstable eigenvalues by verifying the following properties of the operator $\mathcal{A}$.

I. $\mathcal{A}$ generates a C_0 semigroup on the Banach space X to be specified below.

II. For any complex valued λ with $\Re\lambda \geq 0$ and $\lambda \neq 0$, λ belongs to the resolvent set of $\mathcal{A}$, and $\lambda = 0$ is a simple eigenvalue with an algebraic multiplicity two.

III. For the set of all λ outside any small neighbourhood of $\lambda = 0$, the operators $(\lambda - \mathcal{A})^{-1}$ are uniformly bounded on X for $\Re\lambda \geq 0$.

Here we use a weighted norm to define the Banach space X that consists of all functions (f, g) such that $e^{ax}\mathcal{S}(f, g)$ is in $H \times L^2$ with $0 < a < \min\{\gamma/2, \sqrt{3}\}$ if $c > 0$, or $-\min\{\gamma/2, \sqrt{3}\} < a < 0$ if $c < 0$. This is a technique similar to that used in [15] to shift essential spectrum of $\mathcal{A}$ to the left side of imaginary axis. Hence, we have proved linear stability of solitary waves of the GN equations.

Theorem 3. *When $|c| > 1$ and $\gamma = \sqrt{1 - c^{-2}}$ is sufficiently small, the corresponding solitary waves of the GN equations are linearly modulational stable in the sense that the initial data z_0 of the system $z_t = \mathcal{A}z$ modulo the generalized kernel of $\mathcal{A}$ satisfies the inequality $\|z\|_X \leq Me^{-\beta t}\|z_0\|_X$ for some fixed $\beta \geq 0$ and any $t > 0$.*

The nonlinear stability analysis for solitary waves of the GN equations depends on well-posedness of the system. This is also an issue to be considered by the author.

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