Rational Solutions of an Extended Lotka-Volterra Equation

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Abstract

A series of rational solutions are presented for an extended Lotka-Volterra equation. These rational solutions are obtained by using Hirota's bilinear formalism and Bäcklund transformation. The crucial step is the use of nonlinear superposition formula.

The so-called extended Lotka-Volterra equation is [1]

$$\frac{d}{dt}\prod_{i=0}^{m-1} a_{n-\frac{m-1}{2}+i} = \prod_{i=0}^{k-1} a_{n+\frac{m-1}{2}+i-(k-1)} - \prod_{i=0}^{k-1} a_{n-\frac{m-1}{2}+i} \tag{1}$$

$$(m = 1, 2, \cdots; k = 1, 2, \cdots; m \neq k)$$

or

$$\frac{d}{dt}\prod_{i=0}^{m-1} a_{n-\frac{m-1}{2}+i} = \left(\prod_{i=0}^{-k-1} a_{n+\frac{m+1}{2}+i}\right)^{-1} - \left(\prod_{i=0}^{-k-1} a_{n-\frac{m+1}{2}+i+k+1}\right)^{-1}. \tag{2}$$

$$(m = 1, 2, \cdots; -k = 1, 2, \cdots)$$

In particular, if $m = 1$ in (1), equation (1) can be transformed into

$$\frac{d}{dt}N_n = \sum_{r=1}^{k-1}(N_{n-r} - N_{n+r})N_n \tag{3}$$

by the variable transformation

$$N_n = \prod_{i=0}^{k-2} a_{n+i-\frac{k}{2}+1}.$$

Obviously, when $k = 2$ in (3), equation (3) becomes the Lotka-Volterra equation

$$\frac{d}{dt}N_n = (N_{n-1} - N_{n+1})N_n. \tag{4}$$

The Lotka-Volterra equation expresses a sequence of ecologocal prey-predator process [2]. It also finds applications in other areas such as plasma physics [3]. As to the integrability of (3) and (4), it is known that (3) and (4) are completely integrable. For examples, (3) and (4) have N-soliton solutions [4,5] and infinite number of conserved quantities [2,6,7]. Bogoyavlensky [8] has found the Lax form for (3). In [9], a recursion operator for (3) with k=3 was given and higher symmetries were presented. A higher order version of (4) was also considered in [10].

In this short paper, we shall consider (1) and (2). By the transformation

$$a_n = \frac{f_{n-\frac{k+1}{2}} f_{n+\frac{k+1}{2}}}{f_{n-\frac{k-1}{2}} f_{n+\frac{k-1}{2}}},$$

equation (1) or (2) can be transformed into the following bilinear equation [1]

$$\left[D_t \sinh\left(\frac{m}{2}D_n\right) - 2\sinh\left(\frac{k}{2}D_n\right)\sinh\left(\frac{1}{2}(m-k)D_n\right)\right] f_n \cdot f_n = 0. \tag{5}$$

Here the Hirota's bilinear differential operator D_t^l and the bilinear difference operator $\exp(\delta D_n)$ are defined by

$$D_t^l\, a \cdot b \equiv \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^l a(t)b(t')|_{t'=t},$$

$$\exp(\delta D_n)a(n) \cdot b(n) \equiv \exp\left[\delta\left(\frac{\partial}{\partial n} - \frac{\partial}{\partial n'}\right)\right] a(n)b(n')|_{n'=n} = a(n+\delta)b(n-\delta).$$

Equation (5) is reduced to the Lotka-Volterra equation or a differential-difference analogue of the KdV equation [4,11] corresponding to the choices of $2m = k$ or $m = -k$. In [12], we have presented a series of rational solutions for the differential-difference analogue of the KdV equation. Thus it is natural and interesting to search for rational solutions of the extended Lotka-Volterra equation (5). It is noted that in [5], a Bäcklund transformation and nonlinear superposition formula for (5) was given. As a result, N-soliton conjecture by Narita is confirmed. Here in order to derive a series of rational solutions of (1) or (2), we focus on the following special Bäcklund transformation for (5):

$$\begin{aligned}\exp&\left(\frac{1}{2}(m-k)D_n\right) f_n \cdot f_n' \\ &= \left[\frac{m-k}{m}\exp\left(\frac{1}{2}(m+k)D_n\right) + \frac{k}{m}\exp\left(\frac{1}{2}(k-m)D_n\right)\right] f_n \cdot f_n',\end{aligned} \tag{6a}$$

$$\left[D_t - \frac{m-k}{m}\exp(kD_n) + \frac{m-k}{m}\right] f_n \cdot f_n' = 0. \tag{6b}$$

We shall represent the transformation (6) symbolically by $f_n \longrightarrow f_n'$.

Henceforth, we denote $f_n(t) \equiv f(n,t) \equiv f(n) \equiv f$. We obtain the following result

Proposition. *Let f_0, f_1 and f_{12} be three solutions of eqn.(5) and $f_0 \longrightarrow f_1 \longrightarrow f_{12}$ with $f_0, f_1, f_{12} \neq 0$. Then there exists a f_2 given by*

$$\exp\left(\frac{k}{2}D_n\right) f_0 \cdot f_{12} = c\frac{m-k}{m} \sinh\left(\frac{k}{2}D_n\right) f_1 \cdot f_2, \tag{7}$$

$$\exp\left(\frac{m}{2}D_n\right) f_0 \cdot f_{12} = c\frac{k}{m}\frac{m-k}{m} \sinh\left(\frac{m}{2}D_n\right) f_1 \cdot f_2 \tag{8}$$

such that f_2 is a new solution of (5) and $f_0 \longrightarrow f_2 \longrightarrow f_{12}$, where c is a nonzero constant.

Proof. First we choose a particular solution F from (7) and (8), i.e., F satisfies

$$\exp\left(\frac{k}{2}D_n\right) f_0 \cdot f_{12} = c\frac{m-k}{m} \sinh\left(\frac{k}{2}D_n\right) f_1 \cdot F, \tag{9}$$

$$\exp\left(\frac{m}{2}D_n\right) f_0 \cdot f_{12} = c\frac{k}{m}\frac{m-k}{m} \sinh\left(\frac{m}{2}D_n\right) f_1 \cdot F. \tag{10}$$

We have, by using (9),(A1),(A2) and $f_0 \longrightarrow f_1 \longrightarrow f_{12}$, that

$$\begin{aligned}
&\sinh\left(\frac{k}{2}D_n\right)\left[D_t f_1 \cdot F - \frac{2}{c}\exp(kD_n) f_0 \cdot f_{12}\right] \cdot f_1^2 \\
&= D_t \left[\sinh\left(\frac{k}{2}D_n\right) f_1 \cdot F\right] \cdot \left[\cosh\left(\frac{k}{2}D_n\right) f_1 \cdot f_1\right] \\
&\qquad - \frac{2}{c}\sinh\left(\frac{k}{2}D_n\right)\left[\exp(kD_n) f_0 \cdot f_{12}\right] \cdot f_1^2 \\
&= \frac{m}{c(m-k)} D_t \left[\exp\left(\frac{k}{2}D_n\right) f_0 \cdot f_{12}\right] \cdot \left[\exp\left(-\frac{k}{2}D_n\right) f_1 \cdot f_1\right] \\
&\qquad - \frac{2}{c}\sinh\left(\frac{k}{2}D_n\right)\left[\exp(kD_n) f_0 \cdot f_{12}\right] \cdot f_1^2 \\
&= \frac{m}{c(m-k)} \exp\left(\frac{k}{2}D_n\right)\{[D_t f_0 \cdot f_1] \cdot f_1 f_{12} - f_0 f_1 \cdot [D_t f_1 \cdot f_{12}]\} \\
&\qquad - \frac{2}{c}\sinh\left(\frac{k}{2}D_n\right)\left[\exp(kD_n) f_0 \cdot f_{12}\right] \cdot f_1^2 \\
&= \frac{1}{c} \exp\left(\frac{k}{2}D_n\right)\{[\exp(kD_n) f_0 \cdot f_1] \cdot f_1 f_{12} - f_0 f_1 \cdot [\exp(kD_n) f_1 \cdot f_{12}]\} \\
&\qquad - \frac{2}{c}\sinh\left(\frac{k}{2}D_n\right)\left[\exp(kD_n) f_0 \cdot f_{12}\right] \cdot f_1^2 \\
&= 0
\end{aligned}$$

which implies that

$$D_t f_1 \cdot F - \frac{2}{c}\exp(kD_n) f_0 \cdot f_{12} = g(t) f_1^2, \tag{11}$$

where $g(t)$ is a suitable function of t. Now we choose $f_2 = F + f_1 \int^t g(t')dt'$. Then (11) becomes

$$D_t f_1 \cdot f_2 - \frac{2}{c}\exp(kD_n) f_0 \cdot f_{12} = 0. \tag{12}$$

Obviously, f_2 so obtained satisfies (7) and (8). Using (7), (8) and (12), we can deduce that

$$\begin{aligned}
&\exp\left(\frac{1}{2}(m-k)D_n\right) f_0\cdot f_2\\
&\quad=\left[\frac{m-k}{m}\exp\left(\frac{1}{2}(m+k)D_n\right)+\frac{k}{m}\exp\left(\frac{1}{2}(k-m)D_n\right)\right] f_0\cdot f_2,\\
&\left[D_t-\frac{m-k}{m}\exp(kD_n)+\frac{m-k}{m}\right] f_0\cdot f_2=0.
\end{aligned}$$

In fact, we have, by using (7), (8), (12), (A3), (A4) and $f_0 \longrightarrow f_1$,

$$\begin{aligned}
&-\left[\exp\left(\frac{1}{2}(m+k)D_n\right) f_0\cdot f_1\right]\left[\exp\left(\frac{1}{2}(m-k)D_n\right)\right.\\
&\qquad\left.-\frac{m-k}{m}\exp\left(\frac{1}{2}(m+k)D_n\right)-\frac{k}{m}\exp\left(\frac{1}{2}(k-m)D_n\right)\right] f_0\cdot f_2\\
&=\left[\exp\left(\frac{1}{2}(m+k)D_n\right) f_0\cdot f_2\right]\left[\exp\left(\frac{1}{2}(m-k)D_n\right)\right.\\
&\qquad\left.-\frac{m-k}{m}\exp\left(\frac{1}{2}(m+k)D_n\right)-\frac{k}{m}\exp\left(\frac{1}{2}(k-m)D_n\right)\right] f_0\cdot f_1\\
&\qquad-\left[\exp\left(\frac{1}{2}(m+k)D_n\right) f_0\cdot f_1\right]\left[\exp\left(\frac{1}{2}(m-k)D_n\right)\right.\\
&\qquad\left.-\frac{m-k}{m}\exp\left(\frac{1}{2}(m+k)D_n\right)-\frac{k}{m}\exp\left(\frac{1}{2}(k-m)D_n\right)\right] f_0\cdot f_2\\
&=2\exp\left(\frac{m}{2}D_n\right)\left[\exp\left(\frac{k}{2}D_n\right) f_0\cdot f_0\right]\cdot\left[\sinh\left(\frac{k}{2}D_n\right) f_1\cdot f_2\right]\\
&\qquad-2\frac{k}{m}\exp\left(\frac{k}{2}D_n\right)\left[\exp\left(\frac{m}{2}D_n\right) f_0\cdot f_0\right]\cdot\left[\sinh\left(\frac{m}{2}D_n\right) f_1\cdot f_2\right]\\
&=\frac{2m}{c(m-k)}\left\{\exp\left(\frac{m}{2}D_n\right)\left[\exp\left(\frac{k}{2}D_n\right) f_0\cdot f_0\right]\cdot\left[\exp\left(\frac{k}{2}D_n\right) f_0\cdot f_{12}\right]\right.\\
&\qquad\left.-\exp\left(\frac{k}{2}D_n\right)\left[\exp\left(\frac{m}{2}D_n\right) f_0\cdot f_0\right]\cdot\left[\exp\left(\frac{m}{2}D_n\right) f_0\cdot f_{12}\right]\right\}\\
&=0
\end{aligned}$$

and

$$
\begin{aligned}
&-\left[\left(D_t - \frac{m-k}{m}\exp(kD_n) + \frac{m-k}{m}\right) f_0 \cdot f_2\right] f_1 \\
&= \left[\left(D_t - \frac{m-k}{m}\exp(kD_n) + \frac{m-k}{m}\right) f_0 \cdot f_1\right] f_2 \\
&\qquad - \left[\left(D_t - \frac{m-k}{m}\exp(kD_n) + \frac{m-k}{m}\right) f_0 \cdot f_2\right] f_1 \\
&= -f_0(n) D_t f_1 \cdot f_2 + 2\frac{m-k}{m}\exp\left(-\frac{k}{2}\frac{\partial}{\partial n}\right)\left[f_0\left(n + \frac{3k}{2}\right)\sinh\left(\frac{k}{2}D_n\right) f_1 \cdot f_2\right] \\
&= f_0(n)[-D_t f_1 \cdot f_2 + \frac{2}{c}\exp(kD_n) f_0 \cdot f_{12}] \\
&= 0
\end{aligned}
$$

which imply that $f_0 \longrightarrow f_2$. Similarly, we can show that $f_2 \longrightarrow f_{12}$. Thus we have completed the proof of Proposition. ■

Remark 1. It is noted that the process of generating solutions is not carried out in the conventional manor. Usually we use f_0, f_1 and f_2 to generate f_{12}. But here we start with three solutions f_0, f_1 and f_{12} to derive f_2. The reason for this is the following: For a soliton equation, soliton solutions could be linked by a Bäcklund transformation with Bäcklund parameters while rational solutions are usually linked by a Bäcklund transformation without Bäcklund parameters. Thus in soliton solutions case, starting from a seed solution f_0 we can easily obtain f_1 and f_2 due to Bäcklund parameters. But in rational solutions case it is difficult to find three seed rational solutions f_0, f_1 and f_2 such that $f_0 \longrightarrow f_i$ $(i = 1, 2)$. Instead in this case it is easier to find three seed solutions f_0, f_1 and f_{12} such that $f_0 \longrightarrow f_1 \longrightarrow f_{12}$.

Remark 2. Just from proposition, it is unclear how to explicitly calculate desired f_2 because f_2 determined by (7) and (8) is not unique. However from the proof of proposition, it is easy to know that the desired f_2 could be found via the following steps. First of all, we choose a special solution F such that (9) and (10) hold. Secondly we calculate $g(t)$ from (11). Then the desired f_2 is given by $f_2 = F + f_1 \int^t g(t')dt'$.

Remark 3. Under the conditions of the Proposition, there exists some relation between

(7) and (8). In fact, by using (A5), (A6) and $f_0 \longrightarrow f_1 \longrightarrow f_{12}$, we have

$$
\begin{aligned}
&2\sinh\left(\frac{k}{2}D_n\right)\left[\exp\left(\frac{m}{2}D_n\right)f_0\cdot f_{12}\right.\\
&\qquad\left.-c\frac{k}{m}\frac{m-k}{m}\sinh\left(\frac{m}{2}D_n\right)f_1\cdot f_2\right]\cdot\left[\exp\left(\frac{m}{2}D_n\right)f_1\cdot f_1\right]\\
&=\left[e^{\frac{m+k}{2}D_n}f_0\cdot f_1\right]\left[e^{\frac{m-k}{2}D_n}f_1\cdot f_{12}\right]-\left[e^{\frac{m-k}{2}D_n}f_0\cdot f_1\right]\left[e^{\frac{m+k}{2}D_n}f_1\cdot f_{12}\right]\\
&\qquad-2c\frac{k}{m}\frac{m-k}{m}\sinh\left(\frac{m}{2}D_n\right)\left[\sinh\left(\frac{k}{2}D_n\right)f_1\cdot f_2\right]\cdot\left[e^{\frac{k}{2}D_n}f_1\cdot f_1\right]\\
&=\frac{k}{m}\left[e^{\frac{m+k}{2}D_n}f_0\cdot f_1\right]\left[e^{\frac{k-m}{2}D_n}f_1\cdot f_{12}\right]-\frac{k}{m}\left[e^{\frac{k-m}{2}D_n}f_0\cdot f_1\right]\left[e^{\frac{m+k}{2}D_n}f_1\cdot f_{12}\right]\\
&\qquad-2c\frac{k}{m}\frac{m-k}{m}\sinh\left(\frac{m}{2}D_n\right)\left[\sinh\left(\frac{k}{2}D_n\right)f_1\cdot f_2\right]\cdot\left[e^{\frac{k}{2}D_n}f_1\cdot f_1\right]\\
&=2\frac{k}{m}\sinh\left(\frac{m}{2}D_n\right)\left[\exp\left(\frac{k}{2}D_n\right)f_0\cdot f_{12}\right]\cdot\left[\exp\left(\frac{k}{2}D_n\right)f_1\cdot f_1\right]\\
&\qquad-2c\frac{k}{m}\frac{m-k}{m}\sinh\left(\frac{m}{2}D_n\right)\left[\sinh\left(\frac{k}{2}D_n\right)f_1\cdot f_2\right]\cdot\left[e^{\frac{k}{2}D_n}f_1\cdot f_1\right]\\
&=2\frac{k}{m}\sinh\left(\frac{m}{2}D_n\right)\left[\exp\left(\frac{k}{2}D_n\right)f_0\cdot f_{12}\right.\\
&\qquad\left.-c\frac{m-k}{m}\sinh\left(\frac{k}{2}D_n\right)f_1\cdot f_2\right]\cdot\left[\exp\left(\frac{k}{2}D_n\right)f_1\cdot f_1\right].
\end{aligned}
$$

As an application of Proposition, we can obtain a hierarchy of polynomial solutions of (5). For example, if we choose

$$f_0 = n+\frac{k}{m}(m-k)t+A_1, \quad f_1 = 1, \quad f_{12} = n+\frac{k}{m}(m-k)t+A_2$$

with A_i being arbitrary constants, then it is easily verified that $n+\frac{k}{m}(m-k)t+A_i$ and 1 are solutions of (5) and

$$n+\frac{k}{m}(m-k)t+A_1 \longrightarrow 1 \longrightarrow n+\frac{k}{m}(m-k)t+A_2.$$

Furthermore, we can show that if $A_2 = A_1+\frac{2}{3}(m+k)$, then

$$
\begin{aligned}
f_2 &= \left(n+\frac{k}{m}(m-k)t\right)^3+(3A_1+m+k)\left(n+\frac{k}{m}(m-k)t\right)^2\\
&\quad+[3A_1^2+2A_1(m+k)+km]\left(n+\frac{k}{m}(m-k)t\right)\\
&\quad+\frac{k}{m}(m-k)[k(m-k)-3A_1^2-2A_1(m+k)]t+c_0
\end{aligned}
$$

satisfies (7),(8) and (12) with $c=-\frac{2}{3}\frac{m}{k(m-k)}$, where c_0 is an arbitrary constant. Thus we have

$$
\begin{aligned}
n+\frac{k}{m}(m-k)t+A_1 \longrightarrow &\left(n+\frac{k}{m}(m-k)t\right)^3+(3A_1+m+k)\left(n+\frac{k}{m}(m-k)t\right)^2 \\
&+[3A_1^2+2A_1(m+k)+km]\left(n+\frac{k}{m}(m-k)t\right) \\
&+\frac{k}{m}(m-k)[k(m-k)-3A_1^2-2A_1(m+k)]t+c_0 \\
&\longrightarrow n+\frac{k}{m}(m-k)t+A_1+\frac{2}{3}(m+k)
\end{aligned}
$$

from which it follows that

$$
\begin{aligned}
&\left(n+\frac{k}{m}(m-k)t\right)^3-(m+k)\left(n+\frac{k}{m}(m-k)t\right)^2+km\left(n+\frac{k}{m}(m-k)t\right) \\
&\quad+\frac{k^2(m-k)^2}{m}t+A_3 \longrightarrow n+\frac{k}{m}(m-k)t \\
&\quad\longrightarrow \left(n+\frac{k}{m}(m-k)t\right)^3-(m+k)\left(n+\frac{k}{m}(m-k)t\right)^2 \\
&\quad+km\left(n+\frac{k}{m}(m-k)t\right)+\frac{k^2(m-k)^2}{m}t+A_4.
\end{aligned}
$$

Next, we assume that

$$
\begin{aligned}
f_0 = &\left(n+\frac{k}{m}(m-k)t\right)^3-(m+k)\left(n+\frac{k}{m}(m-k)t\right)^2 \\
&+km\left(n+\frac{k}{m}(m-k)t\right)+\frac{k^2(m-k)^2}{m}t+A_3, \qquad (13)\\
f_{12} = &\left(n+\frac{k}{m}(m-k)t\right)^3+(m+k)\left(n+\frac{k}{m}(m-k)t\right)^2 \\
&+km\left(n+\frac{k}{m}(m-k)t\right)+\frac{k^2(m-k)^2}{m}t+A_4, \qquad (14)\\
f_1 = &\, n+\frac{k}{m}(m-k)t. \qquad (15)
\end{aligned}
$$

Then we seek a solution in the form

$$
\begin{aligned}
f_2 = &\left(n+\frac{k}{m}(m-k)t\right)^6+a_1(t)\left(n+\frac{k}{m}(m-k)t\right)^5+a_2(t)\left(n+\frac{k}{m}(m-k)t\right)^4 \\
&+a_3(t)\left(n+\frac{k}{m}(m-k)t\right)^3+a_4(t)\left(n+\frac{k}{m}(m-k)t\right)^2 \\
&+a_5(t)\left(n+\frac{k}{m}(m-k)t\right)+a_6(t) \qquad (16)
\end{aligned}
$$

such that (7), (8) and (12) hold. A direct calculation shows that

$$A_4 = A_3 + \frac{2}{3}(m+k)\left[\frac{1}{3}km - \frac{2}{15}(m^2+k^2)\right], \tag{17}$$

$$c = -\frac{2}{5}\frac{m}{k(m-k)},\ a_1 = 0,\ a_2 = \frac{5}{3}(km - k^2 - m^2), \tag{18}$$

$$a_3 = 5A_3 + \frac{5}{3}(m+k)\left[\frac{1}{3}km - \frac{2}{15}(m^2+k^2)\right] + 5\frac{k^2(m-k)^2}{m}t, \tag{19}$$

$$a_4 = \frac{4}{9}k^4 - \frac{8}{9}k^3m + \frac{4}{3}k^2m^2 - \frac{8}{9}km^3 + \frac{4}{9}m^4, \tag{20}$$

$$a_5 = \frac{k}{m}(m-k)\left(\frac{11}{3}k^4 - \frac{22}{3}k^3m + \frac{13}{3}k^2m^2 - \frac{2}{3}km^3\right)t + c_1, \tag{21}$$

$$\begin{aligned} a_6 = {} & t\left(-\frac{14}{9}k^6 + \frac{4}{9}\frac{k^7}{m} + \frac{10}{9}k^5m + \frac{10}{9}k^4m^2 - \frac{14}{9}k^3m^3 + \frac{4}{9}k^2m^4\right) - 5\frac{k^4}{m^2}(m-k)^4t^2 \\ & - 10A_3\frac{k^2(m-k)^2}{m}t - 5A_3^2 - \frac{10}{3}A_3(m+k)\left[\frac{1}{3}km - \frac{2}{15}(m^2+k^2)\right], \end{aligned} \tag{22}$$

where c_1 is an arbitrary constant.

We now explain how to choose seed polynomial solutions f_0, f_1, f_{12} at each step such that the above process goes on to generate a series of polynomial solutions for (5) step by step. In practice, these seed solutions could be found from last step.

In fact, if we have four seed solutions from last step: $\hat{f}_0, \hat{f}_1, \hat{f}_2, \hat{f}_{12}$ such that $\hat{f}_0 \longrightarrow \hat{f}_1 \longrightarrow \hat{f}_{12}, \hat{f}_0 \longrightarrow \hat{f}_2 \longrightarrow \hat{f}_{12}$, then we try to find a polynomial P from the relation $\hat{f}_2(n+A) + B\hat{f}_1(n+A) \longrightarrow \hat{f}_{12}(n+A)$ with A, B being arbitrary constants such that $P \longrightarrow \hat{f}_0$. By doing so, we now choose $f_0 = P, f_1 = \hat{f}_0, f_{12} = \hat{f}_2$ as seed solutions of current step. Then using Proposition, we can find a new f_2. Thus we may deduce a series of polynomial solutions for (5).

In the appendex B, we will show you some calculation detail of how to choose seed solutions for next step when we start with (13)-(16) with coefficients given by (17)-(21).

In conclusion, we have obtained rational solutions of the extended Lotka-Volterra equation. The method used here is Hirota's bilinear formalism and Bäcklund transformation. The crucial step is the use of nonlinear superposition formula. It is noted that these obtained rational solutions are linked via the Bäcklund transformation (6) which is a special case of a Bäcklund transformation with parameters [5]. Thus it enables us to obtain other new solutions of the equation under consideration by combining soliton solutions with rational solutions.

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Appendix A. Hirota bilinear operator identities.

The following bilinear operator identities hold for arbitrary functions a, b, c and d.

$$\sinh(\delta D_n)(D_t a \cdot b) \cdot a^2 = D_t[\sinh(\delta D_n)a \cdot b] \cdot [\cosh(\delta D_n)a \cdot a]. \tag{A1}$$

$$2\sinh(\delta D_n)[\exp(2\delta D_n)a \cdot b] \cdot c^2 = e^{\delta D_n}\{[e^{2\delta D_n}a \cdot c] \cdot cb - ac \cdot [e^{2\delta D_n}c \cdot b]\}. \tag{A2}$$

$$[e^{\delta_1 D_n}a \cdot b][e^{\delta_2 D_n}a \cdot c] = e^{\frac{1}{2}(\delta_1+\delta_2)D_n}[e^{\frac{1}{2}(\delta_1-\delta_2)D_n}a \cdot a] \cdot [e^{\frac{1}{2}(\delta_1-\delta_2)D_n}c \cdot b]. \tag{A3}$$

$$(D_t a \cdot b)c - (D_t a \cdot c)b = -aD_t b \cdot c. \tag{A4}$$

$$\begin{aligned}&\sinh(\delta_1 D_n)[\sinh(\delta_2 D_n)a \cdot b] \cdot [\exp(\delta_2 D_n)a \cdot a]\\ &\qquad = \sinh(\delta_2 D_n)[\sinh(\delta_1 D_n)a \cdot b] \cdot [\exp(\delta_1 D_n)a \cdot a].\end{aligned} \tag{A5}$$

$$\begin{aligned}&2\sinh(\delta_1 D_n)[\exp(\delta_2 D_n)a \cdot b] \cdot [\exp(\delta_2 D_n)c \cdot c]\\ &\qquad = [e^{(\delta_2+\delta_1)D_n}a \cdot c][e^{(\delta_2-\delta_1)D_n}c \cdot b] - [e^{(\delta_2-\delta_1)D_n}a \cdot c][e^{(\delta_2+\delta_1)D_n}c \cdot b].\end{aligned} \tag{A6}$$

Appendix B.

From (13)-(16) with (17)-(22), we have, by using MATHEMATICA, that

$$
\begin{aligned}
&\left(n-\frac{2}{3}(m+k)+\frac{k}{m}(m-k)t\right)^6+\frac{5}{3}(km-k^2-m^2)\left(n-\frac{2}{3}(m+k)+\frac{k}{m}(m-k)t\right)^4\\
&\quad+\left\{\frac{4}{27}(k+m)(2k-m)(2m-k)+\frac{5}{3}(m+k)\left[\frac{1}{3}km-\frac{2}{15}(m^2+k^2)\right]\right.\\
&\quad\left.+5\frac{k^2(m-k)^2}{m}t\right\}\left(n-\frac{2}{3}(m+k)+\frac{k}{m}(m-k)t\right)^3\\
&\quad+\left(\frac{4}{9}k^4-\frac{8}{9}k^3m+\frac{4}{3}k^2m^2-\frac{8}{9}km^3+\frac{4}{9}m^4\right)\left(n-\frac{2}{3}(m+k)+\frac{k}{m}(m-k)t\right)^2\\
&\quad+\left[\frac{k}{m}(m-k)\left(\frac{11}{3}k^4-\frac{22}{3}k^3m+\frac{13}{3}k^2m^2-\frac{2}{3}km^3\right)t+A_5\right]\\
&\quad\times\left(n-\frac{2}{3}(m+k)+\frac{k}{m}(m-k)t\right)\\
&\quad+t\left(-\frac{14}{9}k^6+\frac{4}{9}\frac{k^7}{m}+\frac{10}{9}k^5m+\frac{10}{9}k^4m^2-\frac{14}{9}k^3m^3+\frac{4}{9}k^2m^4\right)-5\frac{k^4}{m^2}(m-k)^4t^2\\
&\quad-\frac{8}{27}(k+m)(2k-m)(2m-k)\frac{k^2(m-k)^2}{m}t-5\left[\frac{4}{135}(k+m)(2k-m)(2m-k)\right]^2\\
&\quad-\frac{10}{3}\left[\frac{4}{135}(k+m)(2k-m)(2m-k)\right](m+k)\left[\frac{1}{3}km-\frac{2}{15}(m^2+k^2)\right]\\
&\longrightarrow\left(n+\frac{k}{m}(m-k)t\right)^3-(m+k)\left(n+\frac{k}{m}(m-k)t\right)^2\\
&\quad+km\left(n+\frac{k}{m}(m-k)t\right)+\frac{k^2(m-k)^2}{m}t
\end{aligned}
$$

and

$$
\begin{aligned}
&\left(n+\frac{k}{m}(m-k)t\right)^3-(m+k)\left(n+\frac{k}{m}(m-k)t\right)^2+km\left(n+\frac{k}{m}(m-k)t\right)+\frac{k^2(m-k)^2}{m}t\\
&\longrightarrow\left(n+\frac{k}{m}(m-k)t\right)^6+\frac{5}{3}(km-k^2-m^2)\left(n+\frac{k}{m}(m-k)t\right)^4\\
&\quad+\left\{\frac{5}{3}(m+k)\left[\frac{1}{3}km-\frac{2}{15}(m^2+k^2)\right]+5\frac{k^2(m-k)^2}{m}t\right\}\left(n+\frac{k}{m}(m-k)t\right)^3\\
&\quad+\left(\frac{4}{9}k^4-\frac{8}{9}k^3m+\frac{4}{3}k^2m^2-\frac{8}{9}km^3+\frac{4}{9}m^4\right)\left(n+\frac{k}{m}(m-k)t\right)^2\\
&\quad+\left[\frac{k}{m}(m-k)\left(\frac{11}{3}k^4-\frac{22}{3}k^3m+\frac{13}{3}k^2m^2-\frac{2}{3}km^3\right)t+A_6\right]\left(n+\frac{k}{m}(m-k)t\right)\\
&\quad+t\left(-\frac{14}{9}k^6+\frac{4}{9}\frac{k^7}{m}+\frac{10}{9}k^5m+\frac{10}{9}k^4m^2-\frac{14}{9}k^3m^3+\frac{4}{9}k^2m^4\right)-5\frac{k^4}{m^2}(m-k)^4t^2
\end{aligned}
$$

Now we take

$$
\begin{aligned}
f_1 = & \left(n + \frac{k}{m}(m-k)t\right)^3 - (m+k)\left(n + \frac{k}{m}(m-k)t\right)^2 \\
& + km\left(n + \frac{k}{m}(m-k)t\right) + \frac{k^2(m-k)^2}{m}t
\end{aligned}
$$

$$
\begin{aligned}
f_0 = & \left(n - \frac{2}{3}(m+k) + \frac{k}{m}(m-k)t\right)^6 \\
& + \frac{5}{3}(km - k^2 - m^2)\left(n - \frac{2}{3}(m+k) + \frac{k}{m}(m-k)t\right)^4 \\
& + \left\{\frac{4}{27}(k+m)(2k-m)(2m-k) + \frac{5}{3}(m+k)\left[\frac{1}{3}km - \frac{2}{15}(m^2+k^2)\right]\right. \\
& \left. +5\frac{k^2(m-k)^2}{m}t\right\}\left(n - \frac{2}{3}(m+k) + \frac{k}{m}(m-k)t\right)^3 \\
& + \left(\frac{4}{9}k^4 - \frac{8}{9}k^3m + \frac{4}{3}k^2m^2 - \frac{8}{9}km^3 + \frac{4}{9}m^4\right)\left(n - \frac{2}{3}(m+k) + \frac{k}{m}(m-k)t\right)^2 \\
& + \left[\frac{k}{m}(m-k)\left(\frac{11}{3}k^4 - \frac{22}{3}k^3m + \frac{13}{3}k^2m^2 - \frac{2}{3}km^3\right)t\right. \\
& \left. +A_5\right]\left(n - \frac{2}{3}(m+k) + \frac{k}{m}(m-k)t\right) \\
& + t\left(-\frac{14}{9}k^6 + \frac{4}{9}\frac{k^7}{m} + \frac{10}{9}k^5m + \frac{10}{9}k^4m^2 - \frac{14}{9}k^3m^3 + \frac{4}{9}k^2m^4\right) - 5\frac{k^4}{m^2}(m-k)^4t^2 \\
& - \frac{8}{27}(k+m)(2k-m)(2m-k)\frac{k^2(m-k)^2}{m}t - 5\left[\frac{4}{135}(k+m)(2k-m)(2m-k)\right]^2 \\
& - \frac{10}{3}\left[\frac{4}{135}(k+m)(2k-m)(2m-k)\right](m+k)\left[\frac{1}{3}km - \frac{2}{15}(m^2+k^2)\right]
\end{aligned}
$$

$$
\begin{aligned}
f_{12} = & \left(n + \frac{k}{m}(m-k)t\right)^6 + \frac{5}{3}(km - k^2 - m^2)\left(n + \frac{k}{m}(m-k)t\right)^4 \\
& + \left\{\frac{5}{3}(m+k)\left[\frac{1}{3}km - \frac{2}{15}(m^2+k^2)\right] + 5\frac{k^2(m-k)^2}{m}t\right\}\left(n + \frac{k}{m}(m-k)t\right)^3 \\
& + \left(\frac{4}{9}k^4 - \frac{8}{9}k^3m + \frac{4}{3}k^2m^2 - \frac{8}{9}km^3 + \frac{4}{9}m^4\right)\left(n + \frac{k}{m}(m-k)t\right)^2 \\
& + \left[\frac{k}{m}(m-k)\left(\frac{11}{3}k^4 - \frac{22}{3}k^3m + \frac{13}{3}k^2m^2 - \frac{2}{3}km^3\right)t + A_6\right]\left(n + \frac{k}{m}(m-k)t\right) \\
& + t\left(-\frac{14}{9}k^6 + \frac{4}{9}\frac{k^7}{m} + \frac{10}{9}k^5m + \frac{10}{9}k^4m^2 - \frac{14}{9}k^3m^3 + \frac{4}{9}k^2m^4\right) - 5\frac{k^4}{m^2}(m-k)^4t^2
\end{aligned}
$$

as seed solutions of next step where A_5 and A_6 are arbitrary constants. Thus using Proposition, we can further find new polynomial solution f_2.

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