

# Bilinear Forms of Integrable Lattices Related to Toda and Lotka-Volterra Lattices

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## Abstract

Hirota's bilinear technique is applied to some integrable lattice systems related to the Bäcklund transformations of the 2DToda, Lotka-Volterra and relativistic Lotka-Volterra lattice systems, which include the modified Lotka-Volterra lattice system, the modified relativistic Lotka-Volterra lattice system, and the generalized Blaszk-Marciniak lattice systems. Determinant solutions are constructed through the resulting bilinear forms, especially for the modified relativistic Lotka-Volterra lattice system and a two-dimensional Blaszk-Marciniak lattice system.

## 1 Introduction

Nonlinear integrable lattice systems have been studied extensively in various sciences. Among the most remarkable and well-studied integrable lattice systems are the Toda lattice (TL) system,

$$\dot{a}_n = a_n(b_{n-1} - b_n), \quad \dot{b}_n = a_n - a_{n+1}, \quad (1.1)$$

and the Lotka-Volterra (LV) lattice system

$$\dot{u}_n = u_n(v_n - v_{n+1}), \quad \dot{v}_n = v_n(u_{n-1} - u_n), \quad (1.2)$$

where the dot denotes the differentiation with respect to the time variable  $t$ .

It is well-known that the algebraic structures in the solution space are very important in investigating nonlinear integrable systems in all continuous, discrete and full discrete cases. In this aspect, the so-called  $\tau$ -function is one of the most fundamental objects which characterize nonlinear integrable systems [1, 2]. It is known that the TL system (1.1) is transformed into

$$\ddot{\tau}_n \tau_n - (\dot{\tau}_n)^2 = \tau_{n+1} \tau_{n-1} - \tau_n^2, \quad (1.3)$$

under the dependent variable transformation

$$a_n = \frac{d^2}{dt^2} \log \tau_n + 1 = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}, \quad (1.4a)$$

$$b_n = \frac{d}{dt} \log \frac{\tau_n}{\tau_{n+1}} \quad (1.4b)$$

and the LV lattice system (1.2) is transformed into

$$\dot{F}_n G_n - F_n \dot{G}_n + F_n G_n = c_1 F_{n+1} G_{n-1}, \quad (1.5a)$$

$$F_n \dot{G}_{n-1} - \dot{F}_n G_{n-1} + F_n G_{n-1} = c_2 F_{n-1} G_n, \quad (1.5b)$$

under the dependent variable transformation

$$u_n = \frac{d}{dt} \log \left( \frac{F_n}{G_n} \right) + 1 = c_1 \frac{F_{n+1} G_{n-1}}{F_n G_n}, \quad (1.6a)$$

$$v_n = \frac{d}{dt} \log \left( \frac{G_{n-1}}{F_n} \right) + 1 = c_2 \frac{F_{n-1} G_n}{F_n G_{n-1}}. \quad (1.6b)$$

The equations (1.3) and (1.5) are all bilinear, and normally called Hirota's bilinear forms of (1.1) and (1.2), respectively. Hirota's bilinear forms provide us with a powerful tool for constructing a broad class of exact solutions for nonlinear integrable systems, both continuous and discrete. Moreover, from the  $\tau$ -function theory of the KP hierarchy or through the Wronskian (Casoratian) technique [1]-[6], we know that many solutions of nonlinear integrable systems can be expressed in terms of determinants. It is also worth noting that lots of important features of nonlinear integrable systems, for example, infinitely many symmetries and conserved densities, can be shown by use of the elementary properties of the  $\tau$ -function.

Let us now mention some other integrable lattice systems. The relativistic Toda lattice (RTL) equation,

$$\begin{aligned} \ddot{q}_n &= (\dot{q}_{n-1} + c)(\dot{q}_n + c) \frac{g^2 \exp(q_{n-1} - q_n)}{1 + g^2 \exp(q_{n-1} - q_n)} \\ &\quad - (\dot{q}_n + c)(\dot{q}_{n+1} + c) \frac{g^2 \exp(q_n - q_{n+1})}{1 + g^2 \exp(q_n - q_{n+1})}, \end{aligned} \quad (1.7)$$

with  $c$  being the light speed and  $g$  being the coupling constant, was furnished by Ruijsenaars [7], which becomes the ordinary TL system in the limit  $c \rightarrow \infty$ . An approach for constructing relativistic generalizations of integrable lattice systems was proposed by Gibbons and Kupershmidt [8, 9], and it is applicable to the whole lattice KP hierarchy. Suris and Ragnisco proposed a systematic procedure for finding integrable relativistic deformations for integrable lattice systems, based on the integrable time discretization, and presented the relativistic Lotka-Volterra (RLV) lattice system [10]:

$$\dot{u}_n = u_n (v_n - v_{n+1} + \alpha u_n v_n - \alpha u_{n+1} v_{n+1}), \quad (1.8a)$$

$$\dot{v}_n = v_n (u_{n-1} - u_n + \alpha u_{n-1} v_{n-1} - \alpha u_n v_n). \quad (1.8b)$$

where  $\alpha$  is the coupling constant. By using the same idea as in [10], the modified relativistic Lotka-Volterra (mRLV) lattice system was also constructed [11]:

$$\dot{y}_n = y_n(1 + \epsilon y_n) \left( \frac{z_n + \alpha y_n z_n}{1 - \epsilon \alpha y_n z_n} - \frac{z_{n+1} + \alpha y_{n+1} z_{n+1}}{1 - \epsilon \alpha y_{n+1} z_{n+1}} \right), \quad (1.9a)$$

$$\dot{z}_n = z_n(1 + \epsilon z_n) \left( \frac{y_{n-1} + \alpha y_{n-1} z_{n-1}}{1 - \epsilon \alpha y_{n-1} z_{n-1}} - \frac{y_n + \alpha y_n z_n}{1 - \epsilon \alpha y_n z_n} \right). \quad (1.9b)$$

where  $\alpha$  and  $\epsilon$  are constants. The integrable time discretization of the RLV system was also performed by Suris and Ragnisco [10], and the bilinear structure and the determinant solution of the RLV system were investigated by Maruno and Oikawa [12]. This article will consider all these same questions for the mRLV system.

Blaszak and Marciniak proposed new integrable lattice systems by using a different formalism, an  $r$ -matrix formalism [13, 14]. They investigated the bi-hamiltonian structure of lattice hierarchies and the Miura-like gauge transformation between lattice hierarchies. One of the integrable lattice systems that they presented is the three field lattice system

$$\frac{\partial}{\partial t} A_n = C_{n+1} - C_{n-1}, \quad (1.10a)$$

$$\frac{\partial}{\partial t} B_n = A_{n-1} C_{n-1} - A_n C_n, \quad (1.10b)$$

$$\frac{\partial}{\partial t} C_n = C_n(B_n - B_{n+1}), \quad (1.10c)$$

which is called the Blaszak-Marciniak (BM) lattice system. For this three field BM lattice system, the bilinear form and soliton solutions were constructed by Hu and Zhu [15], and the recursion structure and the master symmetry algebra were furnished under a systematical skeleton by Fuchssteiner and Ma [16, 17]. Moreover, Hu and Tam proposed a two-dimensional generalization of the BM lattice system [18]

$$\frac{\partial}{\partial y} A_n = C_{n+1} - C_{n-1}, \quad (1.11a)$$

$$\frac{\partial}{\partial y} B_n = A_{n-1} C_{n-1} - A_n C_n, \quad (1.11b)$$

$$\frac{\partial}{\partial x} C_n = C_n(B_n - B_{n+1}), \quad (1.11c)$$

and showed that their bilinear equations are equal to the bilinear equations of Leznov's two-dimensional ultra-Toda lattice system [19]. This generalization is similar to the two-dimensional generalization of the Toda lattice system. However, the bilinear equations presented in [18] are quite complicated, and it is difficult to connect their bilinear equations with the determinant identity being fundamental in the Sato theory. So we need to search for some other bilinearization and determinant identity for the generalized BM lattice system and establish its relationship with other integrable lattice systems.

The contents of this article are twofold. On the one hand, we show an interesting relation among the modified Lotka-Volterra lattice system, the modified relativistic Lotka-Volterra lattice system, and the full-discrete equations of the Lotka-Volterra and the relativistic Lotka-Volterra lattice system, through exposing their bilinear forms. On the other

hand, we consider the bilinear form (or the determinant identity) for the generalized (i.e., two- and three-dimensional) Blaszk-Marciniak lattice systems. Our main concerns are bilinear forms, determinant solutions, and the relationship with other integrable lattice systems. The bilinear form (or the determinant identity) is of fundament in the mathematical theory of solitons and integrable systems. Throughout the article, we also expose a few properties for the modified Lotka-Volterra lattice system, the modified relativistic Lotka-Volterra lattice system, and the generalized Blaszk-Marciniak lattice systems, based on their bilinear forms.

## 2 Modified Lotka-Volterra lattice system

To exhibit the relation between the discrete and modified equations, let us first consider the discrete time Lotka-Volterra (dLV) lattice system [20, 11, 21]

$$u_n^{t+1}(1 + hv_{n+1}^{t+1}) = u_n^t(1 + hv_n^t), \quad v_n^{t+1}(1 + hu_n^{t+1}) = v_n^t(1 + hu_{n-1}^t). \quad (2.1)$$

The bilinear equations for the above dLV lattice system,

$$c_1 F_{n+1}^{t+1} G_{n-1}^t = c_2 G_n^t F_n^{t+1} - \frac{1}{h} G_n^{t+1} F_n^t, \quad (2.2a)$$

$$c_3 F_{n-1}^t G_n^{t+1} = c_4 F_n^t G_{n-1}^{t+1} - \frac{1}{h} F_n^{t+1} G_{n-1}^t, \quad (2.2b)$$

are obtained by use of the dependent variable transformation

$$u_n^t = c_1 \frac{F_{n+1}^{t+1} G_{n-1}^t}{G_n^{t+1} F_n^t} = c_2 \frac{G_n^t F_n^{t+1}}{G_n^{t+1} F_n^t} - \frac{1}{h}, \quad (2.3a)$$

$$v_n^t = c_3 \frac{G_n^{t+1} F_{n-1}^t}{F_n^{t+1} G_{n-1}^t} = c_4 \frac{F_n^t G_{n-1}^{t+1}}{F_n^{t+1} G_{n-1}^t} - \frac{1}{h}, \quad (2.3b)$$

where the  $c_i$ 's are arbitrary constants. This dependent variable transformation (2.3) can be easily obtained through the singularity confinement test [22].

The modified Lotka-Volterra (mLV) lattice system [11, 21, 23]

$$\dot{y}_n = y_n(1 + \epsilon y_n)(z_n - z_{n+1}), \quad \dot{z}_n = z_n(1 + \epsilon z_n)(y_{n-1} - y_n), \quad (2.4)$$

is transformed into two semi-discrete bilinear equations

$$\dot{G}_{n-1}^k F_n^k - G_{n-1}^k \dot{F}_n^k = \epsilon c_2 c_3 G_n^k F_{n-1}^k + G_{n-1}^k F_n^k, \quad (2.5a)$$

$$\dot{F}_{n-1}^k G_{n-1}^k - F_{n-1}^k \dot{G}_{n-1}^k = \epsilon c_1 c_4 F_n^k G_{n-2}^k + F_{n-1}^k G_{n-1}^k, \quad (2.5b)$$

and two full discrete bilinear equations

$$c_1 F_{n+1}^{k+1} G_{n-1}^k = c_2 G_n^k F_n^{k+1} - \frac{1}{\epsilon} G_n^{k+1} F_n^k, \quad (2.6a)$$

$$c_3 G_n^{k+1} F_{n-1}^k = c_4 F_n^k G_{n-1}^{k+1} - \frac{1}{\epsilon} F_n^{k+1} G_{n-1}^k, \quad (2.6b)$$

if we introduce the following dependent variable transformation

$$y_n = c_1 \frac{F_{n+1}^{k+1} G_{n-1}^k}{G_{n+1}^{k+1} F_n^k} = c_2 \frac{G_n^k F_n^{k+1}}{G_{n+1}^{k+1} F_n^k} - \frac{1}{\epsilon}, \quad (2.7a)$$

$$z_n = c_3 \frac{G_n^{k+1} F_{n-1}^k}{F_n^{k+1} G_{n-1}^k} = c_4 \frac{F_n^k G_{n-1}^{k+1}}{F_n^{k+1} G_{n-1}^k} - \frac{1}{\epsilon}. \quad (2.7b)$$

We point out that the transformation (2.7) is equal to the transformation (2.3) if we set  $k = t, \epsilon = h$ . Therefore, the systems (2.1) and (2.4) belong to the same class of integrable lattice systems, and they are related to the Bäcklund transformation of the LV lattice system. In fact, from the bilinear equations, we know that the independent variables  $t$  and  $k$  in the dLV and mLV lattice systems are parameters of the Bäcklund transformation of the LV lattice system.

### 3 Modified Relativistic Lotka-Volterra lattice system

In the previous section, we found an interesting relation between the dLV lattice system and the mLV lattice system. Now let us consider the question of whether the RLV lattice system also has such an interesting relation.

The RLV lattice system [i.e., (1.8)],

$$\dot{u}_n = u_n (v_n - v_{n+1} + \alpha u_n v_n - \alpha u_{n+1} v_{n+1}), \quad (3.1a)$$

$$\dot{v}_n = v_n (u_{n-1} - u_n + \alpha u_{n-1} v_{n-1} - \alpha u_n v_n), \quad (3.1b)$$

and the discrete-time relativistic Lotka-Volterra (dRLV) lattice system,

$$u_n^{t+1} \frac{1 + h v_{n+1}^{t+1}}{1 - h \alpha u_{n+1}^{t+1} v_{n+1}^{t+1}} = u_n^t \frac{1 + h v_n^t}{1 - h \alpha u_n^t v_n^t}, \quad (3.2a)$$

$$v_{n+1}^{t+1} \frac{1 + h u_{n+1}^{t+1}}{1 - h \alpha u_{n+1}^{t+1} v_{n+1}^{t+1}} = v_{n+1}^t \frac{1 + h u_n^t}{1 - h \alpha u_n^t v_n^t}, \quad (3.2b)$$

are proposed by Suris and Ragnisco [10, 11, 21].

The transformation can be taken as

$$u_n^t = -\frac{1}{h} + c_1 \frac{F_{n+1}^{t-1} G_n^t}{F_{n+1}^t G_n^{t-1}} = \beta_1 \frac{F_n^{t-1} G_{n+1}^t}{F_{n+1}^t G_n^{t-1}}, \quad (3.3a)$$

$$v_n^t = -\frac{1}{h} + c_2 \frac{F_n^t \bar{G}_{n+1}^t}{F_n^{t-1} \bar{G}_{n+1}^{t+1}} = \beta_2 \frac{F_{n+1}^t \bar{G}_n^t}{F_n^{t-1} \bar{G}_{n+1}^{t+1}}, \quad (3.3b)$$

where the  $c_i$ 's and  $\beta_i$ 's are arbitrary constants [12]. Substituting (3.3) into (3.2), we readily obtain the following bilinear equations,

$$-(1/h) F_{n+1}^t G_n^{t-1} + c_1 F_{n+1}^{t-1} G_n^t = \beta_1 F_n^{t-1} G_{n+1}^t, \quad (3.4a)$$

$$-(1/h) F_n^{t-1} \bar{G}_{n+1}^{t+1} + c_2 F_n^t \bar{G}_{n+1}^t = \beta_2 F_{n+1}^t \bar{G}_n^t, \quad (3.4b)$$

$$G_n^{t-1} \bar{G}_{n+1}^{t+1} - \gamma G_n^t \bar{G}_{n+1}^t = (\beta_1 \beta_2 h \alpha) G_{n+1}^t \bar{G}_n^t, \quad (3.4c)$$

where  $\gamma$  is an arbitrary constant. Conversely, we can readily obtain the dRLV lattice system (3.2) from the bilinear equations (3.4). Thus, solutions of the bilinear equations (3.4) generate solutions of the dRLV lattice system (3.2). The Casorati determinant solution can be obtained directly from these bilinear equations [12].

From the continuous limit of the discrete dependent variable transformation, the following continuous dependent variable transformation is suggested,

$$u_n = c_1 + \frac{d}{dt} \left( \log \frac{G_n}{F_{n+1}} \right) = \beta_1 \frac{F_n G_{n+1}}{F_{n+1} G_n}, \quad (3.5a)$$

$$v_n = c_2 + \frac{d}{dt} \left( \log \frac{F_n}{\bar{G}_{n+1}} \right) = \beta_2 \frac{F_{n+1} \bar{G}_n}{F_n \bar{G}_{n+1}}. \quad (3.5b)$$

Substituting this transformation into the RLV lattice system (3.1), we obtain the bilinear equations

$$F_{n+1} \dot{G}_n - \dot{F}_{n+1} G_n + c_1 F_{n+1} G_n = \beta_1 F_n G_{n+1}, \quad (3.6a)$$

$$\dot{F}_n \bar{G}_{n+1} - F_n \dot{\bar{G}}_{n+1} + c_2 F_n \bar{G}_{n+1} = \beta_2 F_{n+1} \bar{G}_n, \quad (3.6b)$$

$$G_n \dot{\bar{G}}_{n+1} - \dot{G}_n \bar{G}_{n+1} - (\beta_1 \beta_2 \alpha) G_{n+1} \bar{G}_n = \gamma G_n \bar{G}_{n+1}, \quad (3.6c)$$

where  $\gamma$  is an arbitrary function of  $t$ . We need, however, to take  $\gamma(t)$  as  $-(\beta_1 \beta_2 \alpha)$ , if we want to get soliton solutions.

The mRLV lattice system [i.e., (1.9)]

$$\dot{y}_n = y_n (1 + \epsilon y_n) \left( \frac{z_n + \alpha y_n z_n}{1 - \epsilon \alpha y_n z_n} - \frac{z_{n+1} + \alpha y_{n+1} z_{n+1}}{1 - \epsilon \alpha y_{n+1} z_{n+1}} \right), \quad (3.7a)$$

$$\dot{z}_n = z_n (1 + \epsilon z_n) \left( \frac{y_{n-1} + \alpha y_{n-1} z_{n-1}}{1 - \epsilon \alpha y_{n-1} z_{n-1}} - \frac{y_n + \alpha y_n z_n}{1 - \epsilon \alpha y_n z_n} \right), \quad (3.7b)$$

is transformed into two semi-discrete bilinear equations

$$F_{n+1} \dot{G}_n - \dot{F}_{n+1} G_n = c_2 \beta_1 F_n G_{n+1} + c_3 F_{n+1} G_n, \quad (3.8a)$$

$$\dot{F}_n \tilde{G}_{n+1} - F_n \dot{\tilde{G}}_{n+1} = c_1 \beta_2 F_{n+1} \tilde{G}_n + c_4 F_n \tilde{G}_{n+1}, \quad (3.8b)$$

and three full discrete bilinear equations

$$-(1/\epsilon) F_{n+1}^k G_n^{k-1} + c_1 F_{n+1}^{k-1} G_n^k = \beta_1 F_n^{k-1} G_{n+1}^k, \quad (3.9a)$$

$$-(1/\epsilon) F_n^{k-1} \tilde{G}_{n+1}^k + c_2 F_n^k \tilde{G}_{n+1}^{k-1} = \beta_2 F_{n+1}^k \tilde{G}_n^{k-1}, \quad (3.9b)$$

$$G_n^{k-1} \tilde{G}_{n+1}^k - \gamma G_n^k \tilde{G}_{n+1}^{k-1} = (\beta_1 \beta_2 \epsilon \alpha) G_{n+1}^k \tilde{G}_n^{k-1}, \quad (3.9c)$$

with  $\gamma$  being an arbitrary constant, if we introduce the following dependent variable transformation

$$y_n = \beta_1 \frac{F_n^{k-1} G_{n+1}^k}{F_{n+1}^k G_n^{k-1}} = -\frac{1}{\epsilon} + c_1 \frac{F_{n+1}^{k-1} G_n^k}{F_{n+1}^k G_n^{k-1}}, \quad (3.10a)$$

$$z_n = \beta_2 \frac{F_{n+1}^k \tilde{G}_n^{k-1}}{F_n^{k-1} \tilde{G}_{n+1}^k} = -\frac{1}{\epsilon} + c_2 \frac{F_n^k \tilde{G}_{n+1}^{k-1}}{F_n^{k-1} \tilde{G}_{n+1}^k}. \quad (3.10b)$$

We note that (3.10) is equal to (3.3) if we set  $\tilde{G}_n^k = \tilde{G}_n^{t+1}$ ,  $k = t$ ,  $\epsilon = h$ . This fact shows that the mRLV lattice system and the dRLV lattice system are related to the Bäcklund transformations of the RLV lattice system, and both of them belong to the same class of integrable lattice systems. In fact, from the bilinear equations, we know that the independent variables  $t$  and  $k$  in the dRLV and mRLV lattice systems are parameters of the Bäcklund transformation of the RLV lattice system.

The  $N$ -soliton solution of the mRLV lattice system (3.7) can then be expressed by the Casorati determinant as

$$y_n = \beta_1 \frac{F_n^{k-1} G_{n+1}^k}{F_{n+1}^k G_n^{k-1}}, \quad z_n = \beta_2 \frac{F_{n+1}^k \tilde{G}_n^{k-1}}{F_n^{k-1} \tilde{G}_{n+1}^k}, \quad (3.11)$$

where  $F_n^k$ ,  $G_n^k$  and  $\tilde{G}_n^k$  are given by

$$F_n^k = \begin{vmatrix} \phi_1^{(k)}(n) & \phi_1^{(k+1)}(n) & \cdots & \phi_1^{(k+N-1)}(n) \\ \phi_2^{(k)}(n) & \phi_2^{(k+1)}(n) & \cdots & \phi_2^{(k+N-1)}(n) \\ \vdots & \vdots & \cdots & \vdots \\ \phi_N^{(k)}(n) & \phi_N^{(k+1)}(n) & \cdots & \phi_N^{(k+N-1)}(n) \end{vmatrix}, \quad (3.12a)$$

$$G_n^k = \begin{vmatrix} \psi_1^{(k)}(n) & \psi_1^{(k+1)}(n) & \cdots & \psi_1^{(k+N-1)}(n) \\ \psi_2^{(k)}(n) & \psi_2^{(k+1)}(n) & \cdots & \psi_2^{(k+N-1)}(n) \\ \vdots & \vdots & \cdots & \vdots \\ \psi_N^{(k)}(n) & \psi_N^{(k+1)}(n) & \cdots & \psi_N^{(k+N-1)}(n) \end{vmatrix}, \quad (3.12b)$$

$$\tilde{G}_n^k = \begin{vmatrix} \tilde{\psi}_1^{(k)}(n) & \tilde{\psi}_1^{(k+1)}(n) & \cdots & \tilde{\psi}_1^{(k+N-1)}(n) \\ \tilde{\psi}_2^{(k)}(n) & \tilde{\psi}_2^{(k+1)}(n) & \cdots & \tilde{\psi}_2^{(k+N-1)}(n) \\ \vdots & \vdots & \cdots & \vdots \\ \tilde{\psi}_N^{(k)}(n) & \tilde{\psi}_N^{(k+1)}(n) & \cdots & \tilde{\psi}_N^{(k+N-1)}(n) \end{vmatrix}. \quad (3.12c)$$

In the above formulas, the functions involved are defined by

$$\phi_i^{(k)}(n) = e^{p_i t} p_i^k (1 - ap_i)^{-n} (1 - d_1/p_i)^{-l-1} (1 - d_2/p_i)^{-m} + e^{q_i t} q_i^k (1 - aq_i)^{-n} (1 - d_1/q_i)^{-l-1} (1 - d_2/q_i)^{-m}, \quad (3.13a)$$

$$\psi_i^{(k)}(n) = e^{p_i t} p_i^k (1 - ap_i)^{-n} (1 - d_1/p_i)^{-l} (1 - d_2/p_i)^{-m-1} + e^{q_i t} q_i^k (1 - aq_i)^{-n} (1 - d_1/q_i)^{-l} (1 - d_2/q_i)^{-m-1}, \quad (3.13b)$$

$$\tilde{\psi}_i^{(k)}(n) = e^{p_i t} p_i^{k-1} (1 - ap_i)^{-n} (1 - d_1/p_i)^{-l} (1 - d_2/p_i)^{-m} + e^{q_i t} q_i^{k-1} (1 - aq_i)^{-n} (1 - d_1/q_i)^{-l} (1 - d_2/q_i)^{-m}, \quad (3.13c)$$

where the  $p_i$ 's are arbitrary constants and

$$q_i = (-d_1 + d_2 + bd_1 d_2 - bd_2 p_i) / (b(d_2 - p_i)), \quad (3.14a)$$

$$b = \alpha_1 h a, \quad d_1 = \frac{1}{h \alpha_2 a}, \quad d_2 = \frac{1}{\gamma a}, \quad (3.14b)$$

$$g = \frac{1}{\alpha_1 - \beta_1} = \frac{1}{\alpha_2 - \beta_2}, \quad \alpha = \frac{1 - \gamma}{\beta_1 \beta_2 h}. \quad (3.14c)$$

The variables  $l$  and  $m$  are two free parameters of the Bäcklund transformations. The proof can be given by using the Laplace expansion of determinants, which is a standard technique in the bilinear theory.

It is interesting to note that this determinant solution is also a solution of the dRLV lattice system shown in [12].

## 4 Generalized Blaszk-Marciniak lattice systems

Let us start from the two-dimensional Blaszk-Marciniak (2DBM) lattice system, i.e., (1.11). We just change the dependent variables of the original 2DBM lattice system from lower case to upper case, in order to clearly exhibit a relation with the Toda lattice system. The 2DBM lattice system [i.e., (1.11)]

$$\frac{\partial}{\partial y} A_n = C_{n+1} - C_{n-1}, \quad (4.1a)$$

$$\frac{\partial}{\partial y} B_n = A_{n-1} C_{n-1} - A_n C_n, \quad (4.1b)$$

$$\frac{\partial}{\partial x} C_n = C_n (B_n - B_{n+1}), \quad (4.1c)$$

is decomposed into

$$D_x D_y \tau_n \cdot \tau_n = 2\lambda \tau_{n+1} \tau_{n-1} - 2\lambda \tau_n^2, \quad (4.2a)$$

$$D_z \tau_{n+1} \cdot \tau_{n-1} = \tau_n^2 - \mu \tau_{n+1} \tau_{n-1}, \quad (4.2b)$$

$$D_y D_z \tau_n \cdot \tau_n = -\frac{2}{\lambda} \tau_{n+1} \tau_{n-1} + \frac{2}{\lambda} \tau_n^2, \quad (4.2c)$$

if we introduce the following dependent variable transformation

$$A_n = \frac{1}{2} \frac{D_x D_y \tau_{n+1} \cdot \tau_{n+1}}{\tau_n \tau_{n+2}} = \lambda - \lambda \frac{\tau_{n+1}^2}{\tau_n \tau_{n+2}}, \quad (4.3a)$$

$$B_n = \left( \ln \frac{\tau_n}{\tau_{n+1}} \right)_x, \quad (4.3b)$$

$$C_n = \frac{\tau_{n+2} \tau_n}{\tau_{n+1}^2} = 1 - \frac{\lambda}{2} \frac{D_y D_z \tau_{n+1} \cdot \tau_{n+1}}{\tau_{n+1}^2}, \quad (4.3c)$$

where  $\lambda$  and  $\mu$  are arbitrary parameters and  $z$  is the auxiliary variable, and  $D_x, D_y$  and  $D_z$  are Hirota's D-operators defined by

$$\begin{aligned} D_x^n D_y^m D_z^l f \cdot g \\ = (\partial_x - \partial_{x'})^n (\partial_y - \partial_{y'})^m (\partial_z - \partial_{z'})^l f(x, y, z) g(x', y', z')|_{x'=x, y'=y, z'=z}. \end{aligned} \quad (4.4)$$

The bilinear equations (4.2) are different from ones in Hu and Tam [18]. It is convenient to construct the determinant solution by using this bilinear form. We note that (4.2a) and (4.2c) are the bilinear forms for the two-dimensional Toda lattice system, and (4.2b) comes from the bilinear form for the Bäcklund transformation of the two-dimensional Toda lattice system. So we can readily construct the determinant solution of the 2DBM lattice



system by using the standard Wronskian (Casoratian) technique in Hirota's bilinear theory [4, 24]. The  $\tau$ -function can be written as the following determinant

$$\tau_n = \begin{vmatrix} \phi_1(n) & \phi_1(n+1) & \cdots & \phi_1(n+N-1) \\ \phi_2(n) & \phi_2(n+1) & \cdots & \phi_2(n+N-1) \\ \cdots & \cdots & \cdots & \cdots \\ \phi_N(n) & \phi_N(n+1) & \cdots & \phi_N(n+N-1) \end{vmatrix}, \quad (4.5)$$

where the functions involved satisfy

$$\frac{\partial \phi_i(n)}{\partial x} = \frac{1}{\lambda} \phi_i(n+1), \quad \frac{\partial \phi_i(n)}{\partial y} = -\phi_i(n-1), \quad \frac{\partial \phi_i(n)}{\partial z} = \lambda \phi_i(n-1), \quad 1 \leq i \leq N. \quad (4.6)$$

Soliton solutions of the one-dimensional BM lattice system (1.10) can be constructed by the reduction procedure from the  $\tau$ -function (4.5). The Miura transformation to Toda's variables  $a_n$  and  $b_n$ :

$$B_n = b_n, \quad C_n = a_{n+1}, \quad (4.7)$$

can be derived immediately by observing the  $\tau$ -function. This transformation is valid in both the 1D case and the 2D case.

Naturally, the Blaszk-Marciniak lattice system (1.10) can also be generalized into the following three-dimensional version:

$$\frac{\partial}{\partial z} A_n = C_{n+1} - C_{n-1}, \quad (4.8a)$$

$$\frac{\partial}{\partial y} B_n = A_{n-1} C_{n-1} - A_n C_n, \quad (4.8b)$$

$$\frac{\partial}{\partial x} C_n = C_n (B_n - B_{n+1}). \quad (4.8c)$$

Replacing  $A_n$ ,  $B_n$  and  $C_n$  in (4.8) with

$$A_n = \frac{1}{2} \frac{D_x D_y \tau_{n+1} \cdot \tau_{n+1}}{\tau_n \tau_{n+2}} = \lambda - \lambda \frac{\tau_{n+1}^2}{\tau_n \tau_{n+2}}, \quad (4.9a)$$

$$B_n = \left( \ln \frac{\tau_n}{\tau_{n+1}} \right)_x, \quad (4.9b)$$

$$C_n = \frac{\tau_{n+2} \tau_n}{\tau_{n+1}^2} = 1 - \frac{\lambda}{2} \frac{D_x D_z \tau_{n+1} \cdot \tau_{n+1}}{\tau_{n+1}^2}, \quad (4.9c)$$

we obtain the following three bilinear equations

$$D_x D_y \tau_n \cdot \tau_n = 2\lambda \tau_{n+1} \tau_{n-1} - 2\lambda \tau_n^2, \quad (4.10a)$$

$$D_x \tau_{n+1} \cdot \tau_{n-1} = \tau_n^2 - \mu \tau_{n+1} \tau_{n-1}, \quad (4.10b)$$

$$D_x D_z \tau_n \cdot \tau_n = -\frac{2}{\lambda} \tau_{n+1} \tau_{n-1} + \frac{2}{\lambda} \tau_n^2, \quad (4.10c)$$

where  $\lambda$  and  $\mu$  are arbitrary parameters. Similarly, we can easily construct the determinant solution of the above 3DBM lattice system by using the Wronskian technique.

It follows from the above result that we can transform the other two two-dimensional generalizations of the BM lattice system into bilinear form. The one is

$$\frac{\partial}{\partial x} A_n = C_{n+1} - C_{n-1}, \quad (4.11a)$$

$$\frac{\partial}{\partial y} B_n = A_{n-1} C_{n-1} - A_n C_n, \quad (4.11b)$$

$$\frac{\partial}{\partial x} C_n = C_n (B_n - B_{n+1}). \quad (4.11c)$$

Replacing  $A_n$ ,  $B_n$  and  $C_n$  in (4.11) with

$$A_n = \frac{1}{2} \frac{D_x D_y \tau_{n+1} \cdot \tau_{n+1}}{\tau_n \tau_{n+2}} = \lambda - \lambda \frac{\tau_{n+1}^2}{\tau_n \tau_{n+2}}, \quad (4.12a)$$

$$B_n = \left( \ln \frac{\tau_n}{\tau_{n+1}} \right)_x, \quad (4.12b)$$

$$C_n = \frac{\tau_{n+2} \tau_n}{\tau_{n+1}^2} = 1 - \frac{\lambda}{2} \frac{D_x D_z \tau_{n+1} \cdot \tau_{n+1}}{\tau_{n+1}^2}, \quad (4.12c)$$

we obtain the following two bilinear equations

$$D_x D_y \tau_n \cdot \tau_n = 2\lambda \tau_{n+1} \tau_{n-1} - 2\lambda \tau_n^2, \quad (4.13a)$$

$$D_z \tau_{n+1} \cdot \tau_{n-1} = \tau_n^2 - \mu \tau_{n+1} \tau_{n-1}, \quad (4.13b)$$

$$D_x D_z \tau_n \cdot \tau_n = -\frac{2}{\lambda} \tau_{n+1} \tau_{n-1} + \frac{2}{\lambda} \tau_n^2, \quad (4.13c)$$

where  $\lambda$  and  $\mu$  are arbitrary parameters and  $z$  is the auxiliary variable. The other one is

$$\frac{\partial}{\partial z} A_n = C_{n+1} - C_{n-1}, \quad (4.14a)$$

$$\frac{\partial}{\partial x} B_n = A_{n-1} C_{n-1} - A_n C_n, \quad (4.14b)$$

$$\frac{\partial}{\partial x} C_n = C_n (B_n - B_{n+1}). \quad (4.14c)$$

Replacing  $A_n$ ,  $B_n$  and  $C_n$  in (4.14) with

$$A_n = \frac{1}{2} \frac{D_x^2 \tau_{n+1} \cdot \tau_{n+1}}{\tau_n \tau_{n+2}} = \lambda - \lambda \frac{\tau_{n+1}^2}{\tau_n \tau_{n+2}}, \quad (4.15a)$$

$$B_n = \left( \ln \frac{\tau_n}{\tau_{n+1}} \right)_x, \quad (4.15b)$$

$$C_n = \frac{\tau_{n+2} \tau_n}{\tau_{n+1}^2} = 1 - \frac{\lambda}{2} \frac{D_x D_z \tau_{n+1} \cdot \tau_{n+1}}{\tau_{n+1}^2}, \quad (4.15c)$$

we obtain the following three bilinear equations

$$D_x^2 \tau_n \cdot \tau_n = 2\lambda \tau_{n+1} \tau_{n-1} - 2\lambda \tau_n^2, \quad (4.16a)$$

$$D_x \tau_{n+1} \cdot \tau_{n-1} = \tau_n^2 - \mu \tau_{n+1} \tau_{n-1}, \quad (4.16b)$$

$$D_x D_z \tau_n \cdot \tau_n = -\frac{2}{\lambda} \tau_{n+1} \tau_{n-1} + \frac{2}{\lambda} \tau_n^2, \quad (4.16c)$$

where  $\lambda$  and  $\mu$  are arbitrary parameters. The detailed construction of determinant solutions of these two systems will also be discussed elsewhere.

## 5 Concluding Remarks

We have presented the bilinear forms for some integrable lattice systems related to the Bäcklund transformations of the Toda, Lotka-Volterra and relativistic Lotka-Volterra lattice systems. We also have shown that solutions of these integrable lattice systems can be constructed by using determinants.

On the one hand, our results especially show that the mRLV lattice system is related to the dRLV lattice system in the  $\tau$ -function level. We also realized that the mRLV lattice system and the dRLV lattice system belong to the same class of integrable lattice systems, and they form the Bäcklund transformation of the RLV lattice system. This mysterious fact was pointed out first by Suris in the Lax formalism [11]. We remark that we can furnish a similar relation between the discrete relativistic Toda lattice system and the modified relativistic Toda lattice system.

On the other hand, our results show that the 2DBM lattice system (4.1) is decomposed into three bilinear equations (i.e., two two-dimensional Toda lattice systems and its Bäcklund transformation). This situation is similar to that in the relativistic Toda lattice system [25]. Moreover, the BM lattice system was generalized to the three-dimensional situation, and the resulting lattice system was decomposed into three bilinear equations. It is very easy to construct various determinant solutions of the 3DBM lattice system by using the Wronskian technique, starting from our bilinear form. The other two two-dimensional generalizations of the BM lattice system are just two reductions of the 3DBM lattice system (4.8).

We remark that it also should be interesting to make integrable time discretizations for the BM lattice system. We guess that it is a good way to start from our bilinear equations to construct full-discrete BM lattice systems. Our bilinear form would be a resource of our future research to present the time discretization of the BM lattice system and the determinant solution.

Finally, we point out that there is an integrable four field BM lattice system [13], for which the bilinear form and the Bäcklund transformation were constructed [26]. For the generalized four field BM lattice system, a natural exploitation also should be done for the bilinear form, the determinant solution, and the relation with other integrable lattice systems.

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