

# Invariant Sets and Explicit Solutions to a Third-Order Model for the Shearless Stratified Turbulent Flow

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## Abstract

We study dynamics of the shearless stratified turbulent flows. Using the method of differential constraints we find a class of explicit solutions to the problem under consideration and establish that the differential constraint obtained coincides with the well-known Zeman–Lumley model for stratified flows.

## 1 Introduction

The results of this article continue our earlier studies of the problem of interaction and mixing between two semi-infinite turbulent flow fields of different scales given in [4, 5] wherein it was proposed a concept based on the method of differential constraints [23, 19] for examining the closure procedure for momentum equations in Parametric Turbulent Models. The key idea of this approach can be formulated shortly as follows: the algebraic expressions for the  $n$ -order moments of statistical characteristics of turbulent flow are determined as the equations of invariant sets (manifolds) of the corresponding differential equations. As an illustrative example we have shown how this concept can be applied for finding a selfsimilar solution to the above-mentioned problem in the case of nonstratified flow.

The aim of the present article is to extend the proposed method to the general case including stable and unstable stratification of the flow. At first, we find a differential constraint generated by the model under consideration and then we construct a reduction that enables us to rewrite that model in a more simple form on the invariant set obtained both for stable and unstable stratifications. This makes it possible to find explicit solutions. The analysis of those solutions shows that the influence of stratification on statistical characteristics of turbulent flow is essential. The time scales of turbulence for the all cases are founded in explicit forms and we show that their behaviors are different. As an application to the Theory of Parametric Turbulent Models, we indicate that the differential constraint obtained coincides with the well-known Zeman–Lumley model [24].

The motivation for the study is that, in many practical situations, a turbulent flow is embedded in a surrounding field of turbulence of different intensity. The shearless turbulent flows appears in decaying grid turbulence in which the mean velocity is constant throughout. Those flows include both the homogeneously-damped grid turbulence and shearless turbulent mixing layer which is formed beyond a composite grid.

## 2 The governing equations

The directions of turbulent-flow investigations based on constructing closed equations of turbulent transport are used intensely in modelling. The approach based on two-parametric models of turbulence gained wide application, as well as one based on second-order closure models (for example, the  $K - \epsilon$  model), which are effective from the computational viewpoint and yields results whose accuracy is sufficient for practical applications. That models are based on parametrization of third-order moments of the gradient type. However, the use of that models for description of turbulent transport in stratified flows gives a qualitatively incorrect result (in some cases) see, for example [16]. The anisotropic character of the buoyancy effect on the structure of turbulence is manifested in appearance of the long wavelength spectrum of turbulent oscillations [18]. This spectral range corresponds to large-scale vortex structures containing the main portion of turbulent energy. According to experimental and theoretical studies, the following large-scale vortex structure is formed in stratified flows: turbulent spots in the case of stable stratification and coherent structures in the case of unstable stratification which are mainly responsible for turbulent transport. The effect of intermediacy and asymmetry of vertical turbulent transport caused by the influence of large-scale vortex structures makes the probability distributions of turbulent fluctuations significantly non-Gaussian. The turbulent structure in these flows is usually described by third-order closure models where the triple correlations (asymmetry) are calculated from differential transport equations. There is considerable number of references connected with the use of third-order turbulent models. However, as a rule these models employ Millionshchikov's quasnormality hypothesis for the parametrization of diffusion processes in equations for triple correlations and, according to this hypothesis, all cumulants of fourth- and higher-order may be negligibly small in comparison with the corresponding correlation functions. As a consequence, in the former case the triple-correlation equations are of the first-order without a dumping mechanism for triple correlations that leads to physically contradictory results [16]. The approach proposed in [10, 11] allows us to overcome this obstacle; the technique also includes a physically reasonable way for constructing approximate algebraic parametrizations of higher moments. Using these observations, we present a third-order model of turbulence to describe correctly the shearless turbulent mixing layer in the framework stated above.

To determine the values of the horizontal components  $e_h$  of the turbulent kinetic energy  $e = e_h + 1/2\langle w^2 \rangle$  and the rate of dissipation  $\epsilon$  and the one-point velocity correlation  $\langle w^2 \rangle$  of the second-order, we make use of the differential equations:

$$\frac{\partial e_h}{\partial t} = -\frac{\partial \langle e_h w \rangle}{\partial z} - \frac{c_1}{\tau} \left[ e_h - \frac{2}{3} E \right] - \frac{2}{3} \epsilon,$$

$$\frac{\partial \epsilon}{\partial t} = \frac{\partial}{\partial z} \left[ c_d \tau \langle w^2 \rangle \frac{\partial \epsilon}{\partial z} \right] + \frac{c_{\epsilon_1}}{\tau} \beta g \langle w \theta \rangle - c_{\epsilon_2} \frac{\epsilon}{\tau},$$

$$\frac{\partial \langle w^2 \rangle}{\partial t} = - \frac{\partial \langle w^3 \rangle}{\partial z} + 2\beta g \langle w \theta \rangle - \frac{c_1}{\tau} \left[ \langle w^2 \rangle - \frac{2}{3} E \right] - \frac{2}{3} \epsilon.$$

Here  $E$ ,  $\tau = E/\epsilon$  are the kinetic energy and the time scale of turbulence respectively. The volumetric expansion coefficient is  $\beta = 1/\Theta$ ;  $\Theta$  and  $\theta$  are the mean and variance potential temperatures respectively. The constants involved in the model with the lower case letters are denoted by  $c_{**}$ . The system of equations is nonclosed and we complete it by the transport equation for the triple correlation of the vertical velocity fluctuation:

$$\frac{\partial \langle w^3 \rangle}{\partial t} = - \frac{\partial C}{\partial z} - 3 \langle w^2 \rangle \frac{\partial \langle w^2 \rangle}{\partial z} + 3\beta g \langle w^2 \theta \rangle - c_2 \frac{\langle w^3 \rangle}{\tau},$$

where  $C = \langle w^3 \rangle - 3 \langle w^2 \rangle^2$  is the cumulant of velocity fluctuations. To obtain a closed model of turbulent transport that does not imply equality to zero of the fourth-order cumulants, the closure procedure is performed at the level of the fifth moments, i.e. it is assumed that the fifth-order cumulants are equal to zero. Thus, to obtain the distribution of the third-order cumulants (correlations), the latter are calculated from differential transport equations: the fourth-order cumulants are determined approximately (from algebraic expressions), and the fifth-order cumulants are assumed to be equal to zero, since their contribution is negligibly small. Results [12] of numerical simulation of vertical turbulent transport in a convective boundary layer confirm the validity of this approach. To express the fourth-order cumulant  $C$ , we can use the equality

$$C = - \frac{\tau}{c_3} \left[ 6 \langle w^3 \rangle \frac{\partial \langle w^2 \rangle}{\partial z} + 4 \langle w^2 \rangle \frac{\partial \langle w^3 \rangle}{\partial z} \right].$$

The triple correlation  $\langle w^2 \theta \rangle$ , the vertical heat flux  $\langle w \theta \rangle$  and the temperature dispersion  $\langle \theta^2 \rangle$  are approximated by the following algebraic relationships [11]:

$$\langle w^2 \theta \rangle = - \frac{\tau}{c_4} \left[ \langle w^3 \rangle \frac{\partial \Theta}{\partial z} - 2\beta g \langle w \theta^2 \rangle \right], \quad \langle w \theta^2 \rangle = - \frac{\tau}{c_5} \langle w^2 \rangle \frac{\partial \langle \theta^2 \rangle}{\partial z},$$

$$\langle w \theta \rangle = - \frac{\tau}{c_{\theta_1}} \langle w^2 \rangle \frac{\partial \Theta}{\partial z} \equiv - \frac{\tau N^2}{\beta g c_{\theta_1}} \langle w^2 \rangle, \quad N^2 = \beta g \frac{\partial \Theta}{\partial z},$$

$$\langle \theta^2 \rangle = - \frac{\tau}{c_{\theta_1} r} \langle w \theta \rangle \frac{\partial \Theta}{\partial z} = - \frac{\tau N^2}{\beta g r} \langle w \theta \rangle = \left( \frac{\tau N^2}{\beta g} \right)^2 \frac{\langle w^2 \rangle}{c_{\theta_1} r},$$

where  $r = \tau/\tau_\theta$ ,  $\tau_\theta$  is the time scale of potential temperature variance,  $N$  is the Brunt–Vaisala frequency. On using the balance approximation between exchange mechanism and dissipation, the equation for the horizontal component  $e_h$  of the turbulent kinetic energy may be simplified and may be written as follows:

$$-c_1 \left[ e_h - \frac{2}{3} \left( e_h + \frac{\langle w^2 \rangle}{2} \right) \right] = \frac{2}{3} \left( e_h + \frac{\langle w^2 \rangle}{2} \right).$$

Hence,

$$e_h = \frac{c_1 - 1}{c_1 + 2} \langle w^2 \rangle, \quad E = \frac{3c_1}{2(c_1 + 2)} \langle w^2 \rangle, \quad \tau = \frac{3c_1}{2(c_1 + 2)} \frac{\langle w^2 \rangle}{\epsilon}.$$

As a result of the above simplifications, the model of the shearless mixing turbulence layer of third-order is represented by the following system of differential equations

$$\begin{aligned}\frac{\partial \langle w^2 \rangle}{\partial t} &= -\frac{\partial \langle w^3 \rangle}{\partial z} - \frac{c_1}{c_1 + 2} \frac{\langle w^2 \rangle}{\tau^*}, \\ \frac{\partial \langle w^3 \rangle}{\partial t} &= \frac{\partial}{\partial z} \left[ \frac{\tau}{c_3} \left( 6 \langle w^3 \rangle \frac{\partial \langle w^2 \rangle}{\partial z} + 4 \langle w^2 \rangle \frac{\partial \langle w^3 \rangle}{\partial z} \right) \right] - 3 \langle w^2 \rangle \frac{\partial \langle w^2 \rangle}{\partial z} - c_2 \frac{\langle w^3 \rangle}{\tau^{**}}, \\ \frac{\partial \epsilon}{\partial t} &= \frac{\partial}{\partial z} \left[ c_d \tau \langle w^2 \rangle \frac{\partial \epsilon}{\partial z} \right] - c_{\epsilon_2} \frac{\epsilon}{\tau^{***}}.\end{aligned}$$

Here

$$\tau^* \approx \frac{\tau}{1 + \frac{6(c_1+2)}{(c_1+4)c_{\theta_1}\tau^2 N^2}}, \quad \tau^{**} \approx \frac{\tau}{1 + \frac{2}{c_2 c_4 \tau^2 N^2}}, \quad \tau^{***} \approx \frac{\tau}{1 + \frac{c_{\epsilon_1}(c_1+c_2)}{c_{\epsilon_2}(c_1+1)c_{\theta_1}\tau^2 N^2}}.$$

It follows from the formula for  $C$  [11] that the contribution of the second term in the algebraic model for the cumulant  $C$  is essential. Also, we can assume that (see [22])

$$\tau^* = \tau^{**} = \tau^{***} = \frac{\tau}{1 + \frac{\pi}{18}\tau^2 N^2} = \tau_w.$$

Thus the governing equations are:

$$\frac{\partial \langle w^2 \rangle}{\partial t} = -\frac{\partial \langle w^3 \rangle}{\partial z} - \alpha (1 + a\hat{\tau}^2 N^2) \frac{\langle w^2 \rangle}{\hat{\tau}}, \quad (2.1)$$

$$\frac{\partial \langle w^3 \rangle}{\partial t} = \frac{\partial}{\partial z} \left[ \kappa \hat{\tau} \langle w^2 \rangle \frac{\partial \langle w^3 \rangle}{\partial z} \right] - 3 \langle w^2 \rangle \frac{\partial \langle w^2 \rangle}{\partial z} - \gamma (1 + a\hat{\tau}^2 N^2) \frac{\langle w^3 \rangle}{\hat{\tau}}, \quad (2.2)$$

$$\frac{\partial \epsilon}{\partial t} = \frac{\partial}{\partial z} \left[ \delta \hat{\tau} \langle w^2 \rangle \frac{\partial \epsilon}{\partial z} \right] - \varrho (1 + a\hat{\tau}^2 N^2) \frac{\epsilon}{\hat{\tau}}, \quad (2.3)$$

where  $\hat{\tau} = \langle w^2 \rangle / \epsilon$  and  $\alpha = 2/3$ ,  $\kappa = 6c_1/c_3(c_1+2)$ ,  $\gamma = 2c_2(c_1+1)/3c_1$ ,  $\delta = 3c_1 c_d / 2(c_1+2)$ ,  $\varrho = 2c_{\epsilon_2}(c_1+2)/3c_1$ ,  $a = c_1^2 \pi / 18(c_1+2)^2$ .

In addition, we indicate the equation for  $\hat{\tau}$

$$\begin{aligned}\frac{\partial \hat{\tau}}{\partial t} &= -\frac{\hat{\tau}}{\langle w^2 \rangle} \left[ \frac{\partial \langle w^3 \rangle}{\partial z} + \delta \hat{\tau} \langle w^2 \rangle \frac{\partial^2 \langle w^2 \rangle}{\partial z^2} + \delta \hat{\tau} \left( \frac{\partial \langle w^2 \rangle}{\partial z} \right)^2 \right] \\ &\quad + \delta \langle w^2 \rangle \hat{\tau} \frac{\partial^2 \hat{\tau}}{\partial z^2} + 2\delta \hat{\tau} \frac{\partial \hat{\tau}}{\partial z} \frac{\partial \langle w^2 \rangle}{\partial z} - \delta \left( \frac{\partial \hat{\tau}}{\partial z} \right)^2 \langle w^2 \rangle + (\varrho - \alpha) (1 + a\hat{\tau}^2 N^2)\end{aligned} \quad (2.4)$$

which can be obtained from (2.1), (2.3). This equation will be crucial for the further study of properties of system (2.1)–(2.3).

### 3 Invariant sets

In this section we show that system (2.1)–(2.3) admits an invariant set (manifold) of the form

$$D = \{ \langle w^2 \rangle, \langle w^3 \rangle, \hat{\tau} : \mathcal{H}^1(\langle w^2 \rangle, \langle w^3 \rangle, \hat{\tau}) \equiv \langle w^3 \rangle + \delta \hat{\tau} \langle w^2 \rangle \langle w^2 \rangle_z = 0 \}. \quad (3.1)$$

A possible application of the existence of an invariant set (manifold) generated by the differential equation for the triple correlation of the vertical velocity fluctuations consists in constructing explicit solutions to the model obtained. Here “explicit” means solutions which can be found by means of ordinary differential equations (or algebraic systems).

Let us briefly present the special terminology of Symmetry Analysis (see [9, 13] for more details).

Consider a system of evolution equations  $\mathcal{F}$

$$u_t^i = \mathcal{F}^i \left( t, x_1, \dots, x_n, u^1, \dots, u_\lambda^k, \dots \right)$$

where  $i = 1, \dots, m$ ,  $u_\lambda^k = \partial^\lambda u^k / \partial x_1^{\lambda_1} \dots \partial x_n^{\lambda_n}$ .

A set (manifold)  $\mathcal{H}$  given by equations

$$h_i \left( t, x_1, \dots, x_n, \dots, u^1, \dots, u^m, \dots, u_\lambda^k, \dots \right) = 0$$

is said to be the invariant set (manifold) of system  $\mathcal{F}$  if

$$\begin{aligned} V_{\mathcal{F}}(h^i) \Big|_{[\mathcal{H}]_0} &= 0, \\ V_{\mathcal{F}} &= \frac{\partial}{\partial t} + \sum_{i=1}^m \mathcal{F}^i \frac{\partial}{\partial u^i} + \sum_{i=1}^m D^\alpha(\mathcal{F}^i) \frac{\partial}{\partial u_\alpha^i}, \end{aligned}$$

where  $\alpha = (\alpha_1, \dots, \alpha_n)$ ,  $D^\alpha = D_{x_1}^{\alpha_1} \dots D_{x_n}^{\alpha_n}$ .

The invariant condition can be written in the following equivalent form

$$D_t(h_i) \Big|_{[\mathcal{F}]_0} \Big|_{[\mathcal{H}]_0} = 0.$$

Here  $[\mathcal{F}]_0$  is  $\infty$ -prolongation (see [13]) of  $\mathcal{F}$  with respect to  $x_1, \dots, x_n$ . The set  $[\mathcal{H}]_0$  is determined by analogy.

To show that  $D$  is invariant under the flow generated by equation (2.2), we prove that the operator  $\mathcal{H}^1(\langle w^2 \rangle, \langle w^3 \rangle, \hat{\tau}) \equiv \langle w^3 \rangle + \delta \hat{\tau} \langle w^2 \rangle \langle w^2 \rangle_z$  preserves the sign on the set of sufficiently smooth solutions to system (2.1)–(2.3).

We introduce into consideration the following sets

$$D^+ = \{p, q, w \in C^\infty(\mathbb{R}) : \mathcal{H}^1(p, q, w) \geq 0\}, \quad (3.2)$$

$$D^- = \{p, q, w \in C^\infty(\mathbb{R}) : \mathcal{H}^1(p, q, w) \leq 0\}. \quad (3.3)$$

**Definition 1.** Operator  $\mathcal{H}^1$  is said to be sign-invariant of system (2.1)–(2.3) if

$$\begin{aligned} (\langle w^2(\cdot, t_0) \rangle, \langle w^3(\cdot, t_0) \rangle, \hat{\tau}(\cdot, t_0)) &\in D^+(D^-) \\ \Rightarrow (\langle w^2(\cdot, t) \rangle, \langle w^3(\cdot, t) \rangle, \hat{\tau}(\cdot, t)) &\in D^+(D^-), \quad t > t_0. \end{aligned}$$

**Remark 1.** For quasilinear parabolic equations derivation of suitable sign-invariants plays an important role for existence, uniqueness, regularity, existence of explicit solutions and other problems (see, for example, [14, 3]).

It is convenient to determine the invariant set  $D$  under the flow generated by equation (2.2) via the sign-invariant sets  $D^+$  and  $D^-$  as their intersection. The invariance condition takes the form

$$\left. \frac{\partial}{\partial t} \mathcal{H}^1 \right|_{\mathcal{H}^1=0} = 0.$$

As the first result that uses the above notion we have:

**Theorem 1.** *Let  $\{(\langle w^2 \rangle, \langle w^3 \rangle, \epsilon)\}$  be a set of sufficiently smooth solutions of (2.1)–(2.3) such that the functions  $\hat{\tau} = \langle w^2 \rangle / \epsilon$  satisfy the relations*

$$\frac{\partial \hat{\tau}}{\partial z} = 0, \quad \frac{\partial \hat{\tau}}{\partial t} = (2\alpha - \gamma) (1 + a\hat{\tau}^2 N^2) + \frac{3}{\delta}. \quad (3.4)$$

Assume that  $\kappa = \delta$ . Then operator  $\mathcal{H}^1$  is a sign-invariant of system (2.1)–(2.3).

**Proof.** Calculating the time derivative, we obtain

$$\frac{\partial}{\partial t} \mathcal{H}^1 = \frac{\partial \langle w^3 \rangle}{\partial t} + \delta \frac{\partial}{\partial t} \left[ \hat{\tau} \langle w^2 \rangle \frac{\partial \langle w^3 \rangle}{\partial z} \right]. \quad (3.5)$$

Using equation (2.2) and the assumption that  $\partial \hat{\tau} / \partial z = 0$ , we can rewrite (3.5) as follows

$$\begin{aligned} \frac{\partial}{\partial t} \mathcal{H} &= \frac{\partial}{\partial z} \left[ \kappa \hat{\tau} \langle w^2 \rangle \frac{\partial \langle w^3 \rangle}{\partial z} \right] - 3 \langle w^2 \rangle \frac{\partial \langle w^2 \rangle}{\partial z} - \gamma \frac{\langle w^3 \rangle}{\hat{\tau}} (1 + a\hat{\tau}^2 N^2) \\ &+ \delta \frac{\partial \hat{\tau}}{\partial t} \frac{\partial \langle w^2 \rangle}{\partial z} \langle w^2 \rangle + \delta \hat{\tau} \frac{\partial \langle w^2 \rangle}{\partial t} \frac{\partial \langle w^2 \rangle}{\partial z} + \delta \hat{\tau} \langle w^2 \rangle \frac{\partial^2 \langle w^2 \rangle}{\partial t \partial z}. \end{aligned}$$

Replacing the derivatives  $\partial \langle w^2 \rangle / \partial t$  and  $\partial^2 \langle w^2 \rangle / \partial t \partial z$  by their representations from equation (2.1) (taking into account that  $\partial \hat{\tau} / \partial z = 0$ ), we have

$$\begin{aligned} \frac{\partial}{\partial t} \mathcal{H}^1 &= \kappa \hat{\tau} \frac{\partial \langle w^2 \rangle}{\partial z} \frac{\partial \langle w^3 \rangle}{\partial z} + \kappa \hat{\tau} \langle w^2 \rangle \frac{\partial^2 \langle w^3 \rangle}{\partial z^2} - 3 \langle w^2 \rangle \frac{\partial \langle w^2 \rangle}{\partial z} - \gamma \frac{\langle w^3 \rangle}{\hat{\tau}} (1 + a\hat{\tau}^2 N^2) \\ &+ \delta \frac{\partial \hat{\tau}}{\partial t} \frac{\partial \langle w^2 \rangle}{\partial z} \langle w^2 \rangle + \delta \hat{\tau} \left[ -\frac{\partial \langle w^3 \rangle}{\partial z} - \alpha \frac{\langle w^2 \rangle}{\hat{\tau}} (1 + a\hat{\tau}^2 N^2) \right] \frac{\partial \langle w^2 \rangle}{\partial z} \\ &+ \delta \hat{\tau} \langle w^2 \rangle \left[ -\frac{\partial^2 \langle w^3 \rangle}{\partial z^2} - \frac{\alpha}{\hat{\tau}} \frac{\partial \langle w^2 \rangle}{\partial z} (1 + a\hat{\tau}^2 N^2) \right]. \end{aligned}$$

This formula can be written in the form

$$\begin{aligned} \frac{\partial}{\partial t} \mathcal{H}^1 &= \kappa \hat{\tau} \frac{\partial \langle w^2 \rangle}{\partial z} \frac{\partial \langle w^3 \rangle}{\partial z} + \kappa \hat{\tau} \langle w^2 \rangle \frac{\partial^2 \langle w^3 \rangle}{\partial z^2} - 3 \langle w^2 \rangle \frac{\partial \langle w^2 \rangle}{\partial z} \\ &- \frac{\gamma}{\hat{\tau}} \mathcal{H}^1 (1 + a\hat{\tau}^2 N^2) + \gamma \delta (1 + a\hat{\tau}^2 N^2) \langle w^2 \rangle \frac{\partial \langle w^2 \rangle}{\partial z} + \delta \frac{\partial \hat{\tau}}{\partial t} \frac{\partial \langle w^2 \rangle}{\partial z} \langle w^2 \rangle \\ &- \delta \hat{\tau} \frac{\partial \langle w^2 \rangle}{\partial z} \frac{\partial \langle w^3 \rangle}{\partial z} - 2\delta \alpha (1 + a\hat{\tau}^2 N^2) \frac{\partial \langle w^2 \rangle}{\partial z} \langle w^2 \rangle - \delta \hat{\tau} \frac{\partial \langle w^2 \rangle}{\partial z} \frac{\partial^2 \langle w^3 \rangle}{\partial z^2}. \end{aligned}$$

As a result, we obtain

$$\begin{aligned} \frac{\partial}{\partial t} \mathcal{H}^1 &= (\kappa - \delta) \hat{\tau} \frac{\partial \langle w^2 \rangle}{\partial z} \frac{\partial \langle w^3 \rangle}{\partial z} + (\kappa - \delta) \hat{\tau} \langle w^2 \rangle \frac{\partial^2 \langle w^3 \rangle}{\partial z^2} + \langle w^2 \rangle \frac{\partial \langle w^2 \rangle}{\partial z} \left( \delta \frac{\partial \hat{\tau}}{\partial t} \right. \\ &\left. - 2\alpha \delta (1 + a\hat{\tau}^2 N^2) + \gamma \delta (1 + a\hat{\tau}^2 N^2) - 3 \right) - \frac{\gamma}{\hat{\tau}} (1 + a\hat{\tau}^2 N^2) \mathcal{H}^1. \quad (3.6) \end{aligned}$$

It follows from the equality  $\kappa = \delta$  and (3.4) that

$$\frac{\partial}{\partial t} \mathcal{H}^1 = \frac{\gamma}{\tau} (1 + a\hat{\tau}^2 N^2) \mathcal{H}^1.$$

Therefore

$$\mathcal{H}^1|_{t=t_1} = \mathcal{H}^1|_{t=t_0} \exp\left(-\gamma \int_{t_0}^{t_1} \left(\frac{1}{\hat{\tau}} + a\hat{\tau} N^2\right) ds\right).$$

This completes the proof of the theorem.  $\blacksquare$

**Theorem 2.** *Let  $a(2\alpha - \gamma) = \alpha(\varrho - a)$ ,  $\frac{3}{8} + 2\alpha - \gamma = \varrho - \alpha$  and  $\kappa = \delta$ . Then system (2.1)–(2.3) admits the invariant set  $D$  and its reduction on the set  $D$  is of the form:*

$$\langle w^2 \rangle = \hat{\tau} \epsilon, \quad (3.7)$$

$$\langle w^3 \rangle = -\delta \hat{\tau} \langle w^2 \rangle \frac{\partial \langle w^2 \rangle}{\partial z}, \quad (3.8)$$

$$\frac{\partial \epsilon}{\partial t} = \frac{\partial}{\partial z} \left[ \delta \hat{\tau} \langle w^2 \rangle \frac{\partial \epsilon}{\partial z} \right] - \varrho (1 + a\hat{\tau}^2 N^2) \frac{\epsilon}{\hat{\tau}}, \quad (3.9)$$

where the function  $\hat{\tau}(z, t) \equiv \hat{\tau}(t)$  solves the ordinary differential equation

$$\frac{d\hat{\tau}}{dt} = (\varrho - \alpha) (1 + a\hat{\tau}^2 N^2) \quad (\text{a version of equation (2.4) on the set } D). \quad (3.10)$$

**Proof.** Observe that, according to Theorem 1 the invariant set exists if the equation for  $\hat{\tau}$  admits a class of solutions which satisfy conditions (3.4). To find this class of solutions, we consider equation (2.4) for  $\hat{\tau}$ . As we have indicated above, this equation is a consequence of equations (2.1), (2.3). More precisely, calculating the time derivative for  $\hat{\tau}$ , we obtain

$$\begin{aligned} \frac{\partial \hat{\tau}}{\partial t} &= \frac{1}{\epsilon} \frac{\partial \langle w^2 \rangle}{\partial t} - \frac{\langle w^2 \rangle}{\epsilon^2} \frac{\partial \epsilon}{\partial t} = -\frac{1}{\epsilon} \frac{\partial \langle w^3 \rangle}{\partial z} - \alpha (1 + a\hat{\tau}^2 N^2) - \delta \hat{\tau}^2 \frac{\partial^2 \langle w^2 \rangle}{\partial z^2} \\ &\quad + \delta \langle w^2 \rangle \hat{\tau} \frac{\partial^2 \hat{\tau}}{\partial z^2} - \delta \frac{\hat{\tau}^2}{\langle w^2 \rangle} \left( \frac{\partial \langle w^2 \rangle}{\partial z} \right)^2 + \varrho (1 + a\hat{\tau}^2 N^2) + 2\delta \frac{\partial \hat{\tau}}{\partial z} \frac{\partial \langle w^2 \rangle}{\partial z} - \delta \frac{\partial \hat{\tau}}{\partial z} \langle w^2 \rangle. \end{aligned}$$

Thus the equation for  $\hat{\tau}$  is of the form

$$\begin{aligned} \frac{\partial \hat{\tau}}{\partial t} &= -\frac{\hat{\tau}}{\langle w^2 \rangle} \left[ \frac{\partial \langle w^3 \rangle}{\partial z} + \delta \hat{\tau} \langle w^2 \rangle \frac{\partial^2 \langle w^2 \rangle}{\partial z^2} + \delta \hat{\tau} \left( \frac{\partial \langle w^2 \rangle}{\partial z} \right)^2 \right] \\ &\quad + \delta \langle w^2 \rangle \hat{\tau} \frac{\partial^2 \hat{\tau}}{\partial z^2} + 2\delta \hat{\tau} \frac{\partial \hat{\tau}}{\partial z} \frac{\partial \langle w^2 \rangle}{\partial z} - \delta \left( \frac{\partial \hat{\tau}}{\partial z} \right)^2 \langle w^2 \rangle + (\varrho - \alpha) (1 + a\hat{\tau}^2 N^2). \end{aligned}$$

Obviously, it is sufficient to check the conditions of Theorem 1 only for  $(\langle w^2 \rangle, \langle w^3 \rangle, \hat{\tau}) \in D$ .

It is clear that equation (2.4) on the set  $D$  can be rewritten in the form

$$\frac{d\hat{\tau}}{dt} = (\varrho - \alpha) (1 + a\hat{\tau}^2 N^2)$$

for  $\hat{\tau}(z, t) \equiv \hat{\tau}(t)$  (the expression in square brackets equals zero). Using the equalities  $a(2\alpha - \gamma) = \alpha(\varrho - a)$ ,  $\frac{3}{8} + 2\alpha - \gamma = \varrho - \alpha$ , we conclude that the above-mentioned equation and equation defined by (3.4) coincide with each other. The proof is completed by simple checking that the function  $\langle w^2(z, t) \rangle = \hat{\tau}(t)\epsilon(z, t)$  satisfies identically (2.1) on the set  $D$ , where  $\hat{\tau}$  solves the equations (3.10).  $\blacksquare$

Theorem 1 is of a special interest in view of its application to Turbulent Models.

**Corollary 1.** *The equation  $\mathcal{H}^1(\langle w^2 \rangle, \langle w^3 \rangle, \hat{\tau}) = 0$  that defines an invariant set of (2.1)–(2.3) coincides with the algebraic triple correlation model or the Zeman–Lumley model [24].*

In other words, the algebraic expression represents the equation of an invariant set (manifold) generated by the differential equation for the triple correlation.

## 4 Solutions on invariant sets

Theorem 2 enables us to reduce (2.1)–(2.3) to the algebraic differential expressions (3.7)–(3.10) which can be easier analyzed. Using the obtained reduction, we construct explicit solutions to system (2.1)–(2.3) for  $N^2 \neq 0$ . In the case of nonstratified flow, i.e. for  $N^2 \equiv 0$ , it was proven in [4, 5] that system (2.1)–(2.3) admits a parametric group of scale transformation that enables us to find a selfsimilar solution of the form

$$\begin{aligned} \epsilon_a &= \frac{h(\xi)}{(t+t_0)^{3\mu+\nu}}, & \langle w_a^2 \rangle &= \frac{f(\xi)}{(t+t_0)^{2\mu}}, & \langle w_a^3 \rangle &= \frac{q(\xi)}{(t+t_0)^{3\mu}}, \\ \xi &= \frac{z-z_c}{L}, & L &= \lambda(t+t_0)^\nu, & z_c &= \lambda_0 L + \lambda_1, & t_0 &> 0, \end{aligned}$$

where  $\lambda, \lambda_i$  are model constants,  $t_0$  is a parameter and  $\nu = 1 - \mu$ . In [4, 5] we studied the existence of selfsimilar solutions and presented in detail the qualitative properties of the solution obtained.

A direct calculation shows that for stratified flows there are no selfsimilar solutions similar to  $\epsilon_a, \langle w_a^2 \rangle, \langle w_a^3 \rangle$ . Nevertheless we find a class of explicit solutions (2.1)–(2.3) by using the fact that system (2.1)–(2.3) is equivalent to algebraic differential expressions (3.7)–(3.10) under the above-mentioned hypotheses on the parameters of the model.

Let us rewrite (3.10) in the form

$$\frac{d\hat{\tau}}{dt} = BN^2\hat{\tau}^2(t) + D, \quad (4.1)$$

where  $B = a(2\alpha - \gamma)$ ,  $D = 3/\delta + 2\alpha - \gamma$ . We note that the form of solutions to (4.1) depends on the signs of quantities  $B, D$  and  $N^2$ . Positiveness (negativeness) of  $N^2$  corresponds to the case of stable (unstable) stratification. The signs  $B$  and  $D$  are determined by values of coefficients  $c_{**}$  which are experimentally known numbers. The calculated values of numbers  $B$  and  $D$  show that  $B < 0, D > 0$ .

### 4.1 Stable stratification

Integrating (4.1) and denoting by

$$A = \sqrt{\frac{D}{-BN^2}}$$

we can obtain the positive solution  $\hat{\tau}_s = \hat{\tau}_s(t)$  which is defined by the formula

$$\hat{\tau}_s(t) = A \tanh \left( 4\sqrt{(D/-B)BN}t + C_0 \right),$$



where  $C_0$  is a constant. Given the initial data  $\hat{\tau}_s(t_0)$ , the constant  $C_0$  can be easily determined. We note that function  $\hat{\tau}_s(t)$  has a horizontal asymptote; more exactly,  $\hat{\tau}_s(t) \rightarrow A$  as  $t \rightarrow \infty$ .

Our main aim is to find a solution to (3.9). The initial condition for equation (3.9) is determined by the physical model. We have

$$\epsilon_0(z) \equiv \epsilon(z, 0) = \begin{cases} \epsilon_-, & \text{if } z < 0, \\ \epsilon_+, & \text{if } z \geq 0, \end{cases}$$

where  $\epsilon_-, \epsilon_+$  are the positive numbers such that  $\epsilon_- \neq \epsilon_+$  and we assume that  $\epsilon_- < \epsilon_+$ . Set

$$\theta \equiv \theta(t) = \int_0^t \hat{\tau}_s^2(p) dp, \quad \varsigma(\theta) = \hat{\tau}_s^2(\theta^{-1}(t)), \quad \psi(\theta) = \frac{1 + a\varsigma^2(\theta)N^2}{\varsigma^3(\theta)}$$

and

$$\hat{\epsilon}(z, \theta) = u_s(z, \theta) \exp\left(-\int_0^\theta \psi(p) dp\right), \quad \text{where } \hat{\epsilon}(z, \theta) = \epsilon(z, t).$$

The function  $\theta(t)$  maps  $[0, +\infty)$  onto  $[0, +\infty)$  and for  $u_s$  we have:

$$\frac{\partial u_s}{\partial \hat{\theta}} = \frac{\partial}{\partial z} \left[ \delta u_s \frac{\partial u_s}{\partial z} \right], \quad \text{where } \hat{\theta} = \int_0^\theta \exp\left(-\int_0^\xi \psi(p) dp\right) d\xi, \quad (4.2)$$

$$u_s(z, 0) = \epsilon_- \quad \text{if } z < 0, \quad u_s(z, 0) = \epsilon_+ \quad \text{if } z \geq 0. \quad (4.3)$$

It is easy to check that  $\hat{\theta} : [0, +\infty) \rightarrow [0, \hat{\theta}_0)$ , where  $\hat{\theta}_0 < A/aN^2$  and

$$\exp\left(-\int_0^\theta \psi(p) dp\right) \rightarrow 0 \quad \text{as } \theta \rightarrow +\infty.$$

In studying the Cauchy problem for equation (3.9), we base our analysis on investigation of transformed problem (4.2), (4.3) for a finite time interval  $[0, \hat{\theta}_0)$ . Equation (4.2) is usually called the porous medium equation. It is well-known that solutions to (4.2), (4.3) are unique and invariant under a parametric group of the scale transformation (see [20]). Therefore  $u_s$  is a selfsimilar solution which can be represented in the form

$$u_s(z, \hat{\theta}) = u_s(\hat{\xi}), \quad \hat{\xi} = \frac{z - z_c}{\sqrt{2\hat{\theta}}}, \quad z_c(\hat{\theta}) = \lambda_0 \sqrt{2\hat{\theta}},$$

where  $\lambda_0$  is a model constant, and then (4.2), (4.3) is rewritten as

$$2\delta \frac{d^2 u_s}{d\hat{\xi}^2} + 2\delta \left( \frac{du_s}{d\hat{\xi}} \right)^2 + (\hat{\xi} + \lambda_0) \frac{du_s}{d\hat{\xi}} = 0, \quad (4.4)$$

$$u_s(-\infty) = \epsilon_-, \quad u_s(+\infty) = \epsilon_+. \quad (4.5)$$

Here  $z_c(\hat{\theta})$  is the so-called central line. Setting  $u_s(\hat{\xi}) = d\hat{\xi}/d\zeta$  where  $\zeta$  is a new variable, we obtain the following boundary value problem for the Blasius equation [15]

$$2\delta \frac{d^3 \hat{\xi}}{d\zeta^3} + (\hat{\xi} + \lambda_0) \frac{d^2 \hat{\xi}}{d\zeta^2} = 0, \quad (4.6)$$

$$\frac{d\hat{\xi}}{d\zeta} \Big|_{-\infty} = \epsilon_-, \quad \frac{d\hat{\xi}}{d\zeta} \Big|_{+\infty} = \epsilon_+. \quad (4.7)$$

Equation (4.6) arises in the context of the evolution of boundary layers in an incompressible fluid along a surface [15]. As a result, we have that there exists an one-parametric family of solutions to (4.6), (4.7). This implies existence of a solution to (4.4), (4.5). In [4] we proven this result directly for a problem of type (4.4), (4.5) and noted that this solution is essentially different from the well-known Barenblatt's solution [2].

**Remark 2.** For the first time, the class of positive solutions to the porous medium equation was introduced in [17].

Thus we arrive at the following

**Lemma 1.** *For any positive finite  $\epsilon_-$  and  $\epsilon_+$  ( $\epsilon_- < \epsilon_+$ ) there exists a unique positive solution  $u_s$  to (4.4), (4.5) (respectively (4.2), (4.3)) such that function  $u_s$  is increasing over  $(-\infty, +\infty)$ ; moreover  $u_s$  has convex and concave profiles for  $\hat{\xi} < 0$  and  $\hat{\xi} > 0$  respectively.*

**Remark 3.** The test configurations of profiles of the spectral flux  $\epsilon$  obtained by numerical and experimental methods (see, for example [1, 21]) coincides qualitatively with profiles of  $u_s$ .

Once we have determined  $u_s$ , we can find  $\langle w^2 \rangle$ . The function  $\langle w^2 \rangle$  is defined from the relation

$$\langle \hat{w}^2(z, \theta) \rangle = \hat{\tau}_s \hat{\epsilon}(z, \theta) = \hat{\tau}_s u_s(z, \theta) \exp\left(-\int_0^\theta \psi(p) dp\right), \quad \langle w^2(z, t) \rangle = \langle \hat{w}^2(z, \theta) \rangle.$$

For the triple correlation  $\langle w^3 \rangle$  we obtain

$$\langle \hat{w}^3(z, \theta) \rangle = -\delta \hat{\tau}_s \langle \hat{w}^2(z, \theta) \rangle \frac{\partial \langle \hat{w}^2(z, \theta) \rangle}{\partial z}, \quad \langle w^3(z, t) \rangle = \langle \hat{w}^3(z, \theta) \rangle. \quad (4.8)$$

## 4.2 Unstable stratification

Let us now find a solution to (4.1) for  $N^2 < 0$ . In this case we arrive at the following solution to equation (4.1)

$$\hat{\tau}_{ns} = A \tan\left(\sqrt{DBN^2}t + C_1\right),$$

where  $C_1$  is determined by the initial data  $\hat{\tau}(0)$ . As before, we introduce into consideration the new time variable

$$\theta = \theta(t) \equiv \int_0^t \hat{\tau}_{ns}^2(p) dp$$

and note that  $\theta \rightarrow +\infty$  as  $\sqrt{DBN^2}t + C_1 \rightarrow \pi/2$ . Then we obtain

$$u_{ns}(z, \theta) = \hat{\epsilon}(z, \theta) \exp\left(\int_0^\theta \psi(p) dp\right),$$

$$\psi(\theta) = \frac{1 + a\varsigma^2(\theta)N^2}{\varsigma^3(\theta)}, \quad \varsigma(\theta) = \hat{\tau}_{ns}^2(\theta^{-1}(t)),$$

and

$$\frac{\partial u_{ns}}{\partial \hat{\theta}} = \frac{\partial}{\partial z} \left[ \delta u_{ns} \frac{\partial u_{ns}}{\partial z} \right], \quad \text{where} \quad \hat{\theta} = \int_0^\theta \exp \left( - \int_0^\xi \psi(p) dp \right) d\xi.$$

Here  $\hat{\theta} \rightarrow +\infty$  as  $\sqrt{(DBN^2)t + C_1} \rightarrow \pi/2$  and  $\exp \left( - \int_0^\theta \psi(s) ds \right)$  is an increasing bounded function on the interval  $[0, +\infty)$  that coincides with experimental observations about increasing the spectral flux  $\epsilon$  in the case of unstable stratification of the flow. Further analysis goes along the same lines just as for  $N^2 > 0$ . Combining Lemma 1 with the above arguments, we claim the following

**Theorem 3.** *Let  $\kappa = \delta$  and*

$$a(2\alpha - \gamma) = \alpha(\varrho - a), \quad \frac{3}{\delta} + 2\alpha - \gamma = \varrho - \alpha.$$

*Then there exists a solution to system (2.1)–(2.3) of the following form*

$$\begin{aligned} \hat{\epsilon}(z, \theta) &= u_s(z, \theta) \exp \left( - \int_0^\theta \psi(s) ds \right), \quad \theta = \theta(t) \equiv \int_0^t \hat{\tau}_s^2(p) dp, \quad \epsilon(z, t) = \hat{\epsilon}(z, \theta), \\ \langle w^2(z, t) \rangle &= \hat{\tau}_s \epsilon(z, t), \quad \langle w^3(z, t) \rangle = -\delta \hat{\tau}_s \langle w^2(z, t) \rangle \frac{\partial \langle w^2(z, t) \rangle}{\partial z} \quad \text{for } N^2 > 0, \end{aligned}$$

and

$$\begin{aligned} \hat{\epsilon}(z, \theta) &= u_{ns}(z, \theta) \exp \left( - \int_0^\theta \psi(s) ds \right), \quad \theta = \theta(t) \equiv \int_0^t \hat{\tau}_{ns}^2(p) dp, \quad \epsilon(z, t) = \hat{\epsilon}(z, \theta), \\ \langle w^2(z, t) \rangle &= \hat{\tau}_{ns} \epsilon(z, t), \quad \langle w^3(z, t) \rangle = -\delta \hat{\tau}_{ns} \langle w^2(z, t) \rangle \frac{\partial \langle w^2(z, t) \rangle}{\partial z} \quad \text{for } N^2 < 0, \end{aligned}$$

where  $u_s(u_{ns})$  is a selfsimilar solution to (4.4), (4.5).

## 5 Conclusions

As a rule, the parametric models of turbulence represent the transport equations for the second-order moments which are completed by closure relations for the higher-order moments and dissipation tensor, and in many cases these closure relations are given in the so-called isotropic form. The assumption about the relaxation character of turbulence under its evolution to the equilibrium state (the homogeneous isotropic state with Gaussian distribution of turbulent fluctuations) is the base for such simplification. However, these models describe adequately certain statistical structures of investigated flows even if turbulence is characterized by anisotropic effects. In general, conditions (3.4) (which assume “uniformity” of the turbulent scales) can be used in the case of an equilibrium state of turbulent flows. Nevertheless, the results obtained provide a correct modelling for turbulent flows: the algebraic model for triple correlations (3.8) was used in [24] for studying turbulent structures in the convective boundary layer and moreover, it was testified for distinct turbulent flows (see [10]).

We conclude with some observations and comments. A version of formula (3.8) coincides with the well-known Hanjalic–Launder model [7] in the case of nonstratified flow [4]. It is

possible to get results similar to the results of this article for shear flows in the problem of plane turbulent wake. Maybe the most interesting and novel part of the criterion of invariance for the so-called locally equilibrium approximations (see [8]) in the problem of plane turbulent wake consists in showing that the Poisson bracket  $\{U, e\}$  equals zero. Here  $U$  is the velocity excess and  $e$  is the turbulent kinetic energy. An example of such an equality was obtained in [6] for a selfsimilar solution.

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