

Higher Order Terms in Multiscale Expansions: A Linearized KdV Hierarchy

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Abstract

We consider a wide class of model equations, able to describe wave propagation in dispersive nonlinear media. The Korteweg-de Vries (KdV) equation is derived in this general frame under some conditions, the physical meanings of which are clarified. It is obtained as usual at leading order in some multiscale expansion. The higher order terms in this expansion are studied making use of a multi-time formalism and imposing the condition that the main term satisfies the whole KdV hierarchy. The evolution of the higher order terms with respect to the higher order time variables can be described through the introduction of a linearized KdV hierarchy. This allows one to give an expression of the higher order time derivatives that appear in the right hand member of the perturbative expansion equations, to show that overall the higher order terms do not produce any secularity and to prove that the formal expansion contains only bounded terms.

1 Introduction

Soliton theory is based on the resolution of the so-called “completely integrable equations” by means of the inverse scattering transform (IST) method. When they describe particular physical situations, these equations arise as asymptotics of well-established models, and are derived from the latter by means of a multiscale expansion, or some equivalent formalism. The integrable nonlinear evolution equations appear as the leading order approximation in this perturbative approach. Corrections to this first approximation are often to be considered. At a given propagation time (or distance), more accuracy can obviously be obtained by retaining more than the main term only in the power series. Such additional terms are the “higher order terms”. On the other hand, for a very long propagation time, the physical solution goes away from the theoretical soliton because the nonlinear evolution equation gives only a first order approximation of this evolution. Corrections to the equation must be taken into account. The partial differential equations giving such corrections are what we call the “higher order” ones.

Consideration of such corrections have given rise to perturbation theories for solitons [6] that have found applications, e.g. in the frame of the physics of optical solitons in fibres,

which are related to optical telecommunications [4]. The effect of the higher order equations appears in these theories notably through the renormalization of the soliton velocity. From the purely theoretical point of view, it has been recently shown that the solvability of the higher order equations can in some sense be related to the problem of the complete integrability of the basic system [7, 10]. Furthermore, a physical interpretation of the equations of the Korteweg-de Vries (KdV) hierarchy has been found by Kraenkel, Manna and Pereira in the framework of the theory of water waves [11, 12, 13].

The Korteweg-de Vries (KdV) equation, which is known to describe the propagation of long waves in shallow water [9], is the first equation that was solved by the IST method [3, 14]. More physically speaking, the first observed soliton was a long wave in shallow water. The KdV equation appears as the evolution equation satisfied by the dominant term of the quantity describing the wave in some multiscale expansion. We investigate in this paper the higher orders terms of such an expansion. Our results apply *a priori* to the Maxwell–Landau model that describes electromagnetic wave propagation in ferromagnetic media, but, because they do not involve the explicit computations particular to this situation (written down elsewhere [16]), they are presented in a general abstract frame. The results are thus expected to apply to other physical situations. The conditions under which the KdV model can be derived in this general frame are discussed below from both the physical and the mathematical points of view. Regarding the higher order equations such conditions cannot be written down explicitly in a completely satisfactory way from the mathematical point of view, but their physical meaning is clarified. Further it has been proved [16] that these conditions are satisfied by the Maxwell–Landau model which describes wave propagation in ferromagnets.

We use, as did Kraenkel *et al*, a multiple time formalism. These authors have shown that the evolution of the dominant term relative to the higher order time variables is governed by the KdV hierarchy. Kodama and Taniuti showed that the evolution of the higher order terms relative to the first time variable is governed by the linearized KdV equation, but the way in which the higher order terms depend on the higher order time variables was not yet clarified and it was not proven that the higher order terms do not produce secular terms, while the coherence of the multiscale expansion necessitates the elimination of any secularities. We find that the evolution of these terms can be described by means of some linearized KdV hierarchy. This is the main result of the paper and enables us to prove that no unbounded term does appear in the expansion.

The paper is organized as follows. In Section 2 we describe the frame of the multiscale expansion, derive the KdV equation and discuss the physical hypotheses that allow this derivation. In Section 3 the problem of secularities is presented and the introduction of the KdV hierarchy allows us both to remove the secularity producing terms due to the main order and to determine the evolution of the latter relative to the higher order variables. Section 4 is devoted to the higher order time evolution of the higher order terms and to the linearized KdV hierarchy. Section 5 contains a conclusion. The linear part of the right hand member of the equations of the perturbative expansion plays an important role in the treatment of the secularity producing terms. Therefore the corresponding coefficients are computed explicitly in an appendix.

2 A multiscale expansion

2.1 The model and the scaling

We consider some set of partial differential equations that can be written as

$$(\partial_t + A\partial_x + E)u = B(u, u), \quad (2.1)$$

where the function u of the variables x and t is valued in \mathbb{R}^p , A and E are some $p \times p$ matrices, and $B : \mathbb{R}^p \times \mathbb{R}^p \rightarrow \mathbb{R}^p$ is bilinear. This system can describe the propagation of electromagnetic waves in a ferromagnetic medium, according to the Maxwell and Landau equations [16]. We consider the abstract system (2.1) rather than this latter particular situation for sake of simplicity. Indeed the study of a specific case would imply explicit computation of all the quantities involved by the multiscale expansion and increase considerably the size of the expressions, while the matter of the present paper does not depend upon the particular form of these quantities. It is in fact rather general, according to the fact that KdV solitons arise in many other physical frames. Considering an abstract frame thus avoids much computational detail, but in fact is not so easy because the derivation of the KdV model is not always achievable. It involves several assumptions, some of which are rather strong. These assumptions are introduced at the points where they have to be used. Then in subsection 2.3 we give their physical interpretation. In any case the required assumptions are satisfied by the Maxwell–Landau model, describing waves in ferromagnets. The proof of this is given in [16]. Thus the results of the paper are valid and completely justified for this latter model. The use of the abstract system (2.1) presents the two following advantages. Firstly it simplifies the algebra because it avoids considering many details peculiar to the physics of ferromagnets. Secondly it allows a discussion of the physical conditions required for the occurrence of KdV solitons, despite some feature concerning the higher order equations that cannot be completely solved from the mathematical point of view.

Regarding the system (2.1) the following assumptions are usually made: A is assumed either to be completely hyperbolic or symmetric and E is assumed to be skewsymmetric. The two latter hypotheses are closely related to the conservative character of the system and are satisfied by the Maxwell–Landau system under the scalar product

$$\left((\vec{E}, \vec{H}, \vec{M}) \mid (\vec{E}, \vec{H}, \vec{M}) \right) = \vec{E}^2 + \vec{H}^2 + \alpha \vec{M}^2$$

(with the notations of [16]). Note that this scalar product depends upon the zero-order term. Only weaker assumptions are necessary for the formal derivation of the KdV equation, as we will see below. However, some essential symmetry property of the higher order equations is satisfied only when the system is conservative. This is ensured, regarding the linear part of the system, by the symmetry and skewsymmetry hypotheses above. The corresponding assumption on the nonlinear part cannot be expressed in a simple way.

The variable u is expanded in a power series of some small parameter ε as

$$u = \varepsilon^2 u_2 + \varepsilon^3 u_3 + \dots, \quad (2.2)$$

and the slow variables

$$\xi = \varepsilon(x - Vt), \quad \tau = \varepsilon^3 t \quad (2.3)$$

are introduced. V is a speed to be determined. This multiscale expansion is well-known. It yields the Korteweg-de Vries (KdV) equation at leading order, ε^2 , in many physical cases. Starting the expansion at order ε^2 corresponds physically to specifying a low value for the order of magnitude of the amplitude. Setting a term of order ε in expansion (2.2) is *a priori* possible, but it is easily checked that it does not yield any evolution equation regarding the chosen time scales. Therefore we can omit it. It is seen below that, as is described in [8, 13], secularities, i.e. linear growth of the solutions with time, appear in the higher order terms. These secularities must be removed and this is achieved by imposing to the leading term some particular dependency with regard to higher order time variables (the equations of the KdV hierarchy). Therefore we introduce the variables τ_2, τ_3, \dots which are defined by

$$\tau_j = \varepsilon^{2j+1}t \quad (j \geq 1). \quad (2.4)$$

With this notation $\tau = \tau_1$.

2.2 The Korteweg-de Vries equation

At order ε^2 equation (2.1) yields

$$Eu_2 = 0. \quad (2.5)$$

Thus u_2 belongs to the kernel, $\ker(E)$, of E . We assume that the range, $\text{Rg}(E)$, of E is in direct sum with its kernel (hypothesis 1). This is satisfied if E is skewsymmetric. We also denote by Π_0 and Q_0 respectively the projectors onto $\ker(E)$ parallel to $\text{Rg}(E)$ and onto $\text{Rg}(E)$ parallel to $\ker(E)$. Equation (2.5) can be written as

$$u_2 = \Pi_0 u_2. \quad (2.6)$$

At the following order, ε^3 , we get the equation

$$Eu_3 + (A - V)\partial_\xi u_2 = 0. \quad (2.7)$$

The projection of equation (2.7) on $\ker(E)$ is

$$\Pi_0(A - V)\partial_\xi u_2 = 0. \quad (2.8)$$

Because of (2.6), equation (2.8) has a nonzero solution if V is an eigenvalue of $\Pi_0 A \Pi_0$. We assume that V is a simple nonzero eigenvalue of $\Pi_0 A \Pi_0$, and call a_0 an eigenvector, Π_1 and Q_1 the associated projectors: Π_1 the projector on $a_0 \mathbb{R}$, Q_1 the projector on $\text{Rg}(\Pi_0(A - V)\Pi_0)$, so that $\Pi_1 + Q_1 = \Pi_0$. This definition assumes that \mathbb{R}^p is the direct sum of the range and the kernel of $\Pi_0(A - V)\Pi_0$. This implies that not only the eigenspace but the characteristic space of the operator $\Pi_0 A \Pi_0$ relative to the eigenvalue V has dimension 1 (hypothesis 2). The assumption of the complete hyperbolicity of A ensures that this hypothesis is satisfied. When A is assumed to be symmetric, the dimensions of characteristic and eigenspace are the same.

According to equation (2.8) $\partial_\xi u_2 \in \ker(\Pi_0(A - V)\Pi_0)$. Thus $u_2 = a_0 \varphi_2$, where the function φ_2 has to be determined. The Q_0 -projection of equation (2.7) is

$$Q_0 u_3 = a_1 \partial_\xi \varphi_2. \quad (2.9)$$

The vector coefficient a_1 is defined by $a_1 = -E^{-1}(A - V)a_0$, where E^{-1} is a partial inverse of E . E^{-1} is precisely defined as follows. We call \hat{E} the restriction and corestriction of Q_0EQ_0 to $\text{Rg}(E)$: \hat{E} is invertible. We denote by $\hat{0}$ the zero operator defined on $\ker(E)$. Then E^{-1} is the direct sum of $\hat{0}$ and \hat{E}^{-1} . In other words, the part of the matrices of E and E^{-1} corresponding to $\text{Rg}(E)$ are inverses one of each other and the matrix of E^{-1} is completed by zeros to make a $p \times p$ matrix.

The equation of order ε^4 is

$$Eu_4 + (A - V)\partial_\xi u_3 = B(u_2, u_2). \quad (2.10)$$

It is divided into 3 parts by using the projectors Q_0, Q_1, Π_1 . The Π_1 -projection is a compatibility condition for φ_2 . Using $\Pi_1(A - V)\Pi_0 = 0$, and splitting u_3 into $u_3 = \Pi_0u_3 + Q_0u_3$, we find that the projection reduces to

$$q\partial_\xi^2\varphi_2 = r\varphi_2^2, \quad (2.11)$$

with $q = \Pi_1(A - V)a_1$ and $r = \Pi_1B(a_0, a_0)$. If $(q, r) \neq (0, 0)$, equation (2.11) has no bounded nonzero solution. The following conditions must thus be satisfied (hypotheses 3 and 4), *viz*

$$\Pi_1B(a_0, a_0) = 0, \quad \text{and} \quad \Pi_1(A - V)a_1 = 0. \quad (2.12)$$

The Q_1 - and then the Q_0 -projection of equation (2.10) yield expressions for Q_1u_3 and Q_0u_4 , respectively, as

$$Q_1u_3 = a'_1\partial_\xi\varphi_2 + \alpha_1 \int^\xi \varphi_2^2, \quad (2.13)$$

$$Q_0u_4 = a_1\partial_\xi\varphi_3 + a_2\partial_\xi^2\varphi_2 + \alpha_2\varphi_2^2. \quad (2.14)$$

Denoting by $(A - V)^{-1}$ a partial inverse of $\Pi_0(A - V)\Pi_0$ defined in an way analogous to E^{-1} , we define the vector coefficients by

$$a'_1 = -(A - V)^{-1}Q_1(A - V)a_1, \quad (2.15)$$

$$\alpha_1 = (A - V)^{-1}Q_1B(a_0, a_0), \quad (2.16)$$

$$a_2 = -E^{-1}Q_0(A - V)(a_1 + a'_1), \quad (2.17)$$

$$\alpha_2 = E^{-1}Q_0(B(a_0, a_0) - (A - V)\alpha_1). \quad (2.18)$$

With the use of all previous results the compatibility condition (Π_1 -projection) of the equation of order ε^5 yields the following evolution equation for φ_2 :

$$\partial_\tau\varphi_2 + \beta\varphi_2\partial_\xi\varphi_2 + \gamma\partial_\xi^3\varphi_2 = \delta\varphi_2 \int^\xi \varphi_2^2. \quad (2.19)$$

The scalar coefficients β, γ, δ are defined by

$$a_0\beta = 2\Pi_1(A - V)\alpha_2 - 2\Pi_1B(a_0, a_1 + a'_1), \quad (2.20)$$

$$a_0\gamma = \Pi_1(A - V)a_2, \quad (2.21)$$

$$a_0\delta = 2\Pi_1B(a_0, \alpha_1). \quad (2.22)$$

If the additional condition (hypothesis 5)

$$\Pi_1 B(a_0, \alpha_1) = 0 \quad (2.23)$$

is satisfied, equation (2.19) reduces to the KdV equation

$$\partial_\tau \varphi_2 + \beta \varphi_2 \partial_\xi \varphi_2 + \gamma \partial_\xi^3 \varphi_2 = 0. \quad (2.24)$$

2.3 Physical meaning of the hypotheses

The above formal derivation of the KdV equation necessitates 5 hypotheses that have been written down explicitly. Two of them, hypotheses 3 and 5, involve the nonlinearity, the three other the derivatives only. A physical interpretation of the latter can be found in the dispersion relation of the linearized system

$$(\partial_t + A \partial_x + E) u = 0. \quad (2.25)$$

The pulsation and polarization vector corresponding to a given wave vector, k , are denoted by $\omega(k)$ and $u(k)$ respectively. They satisfy

$$(-i\omega(k) + Aik + E) u(k) = 0. \quad (2.26)$$

The long wave approximation corresponds to $k = 0$ and $\omega(0) = 0$. Then equation (2.26) reduces to $E u(0) = 0$. It is equation (2.5). Taking the derivative of equation (2.26) with regard to k and then setting $k = 0$ we obtain

$$i[A - \omega'(0)]u(0) + E u'(0) = 0. \quad (2.27)$$

The prime denotes the derivative with regard to k . Together with the interpretation of the long wave approximation as a limit of oscillating waves for k close to 0, the analogy between equation (2.27) and equation (2.7) allows the identification between $u(0)$ and a_0 on one hand, and between $\omega'(0)$ and V on the other. V appears as a long-wave limit of the group velocity. Hypothesis 2 is thus that a unique polarization can propagate with this velocity. If this assumption is not satisfied, interactions between the various waves with same velocity must be taken into account.

Equation (2.27) yields also

$$Q_0 u'(0) = -E^{-1} i[A - V] a_0. \quad (2.28)$$

Thus $Q_0 u'(0) = i a_1$, with the definition above of a_1 . Taking once again the derivative of equation (2.26) with regard to k we obtain, for $k = 0$,

$$-i\omega''(0)a_0 + 2i[A - V]u'(0) + E u''(0) = 0. \quad (2.29)$$

Taking the Π_1 -projection of equation (2.29), decomposing $u'(0)$ into $u'(0) = i a_1 + \Pi_0 u'(0)$, and taking into account the relation $\Pi_1(A - V)\Pi_0 = 0$, we get

$$\omega''(0)a_0 = 2i\Pi_1[A - V]a_1. \quad (2.30)$$

Hypothesis 5, in (2.12), is thus

$$\frac{d^2\omega}{dk^2}(0) = 0.$$

The following feature can be observed: $\omega'''(0)$ can be computed by the same method, using the second derivative of equation (2.26). The expression obtained gives an interpretation of the coefficient γ of the third derivative in the KdV equation, (2.19), as

$$\gamma = \frac{-1}{6} \frac{d^3\omega}{dk^3}(0). \quad (2.31)$$

Hypothesis 1 appears mainly as a technical assumption without special physical meaning needed to solve equation (2.28). If E is skewsymmetric, this hypothesis is satisfied. Assume that E has a nonzero symmetrical part E_s . Taking the scalar product of equation (2.26) by $u(k)$, then comparing the result to its complex conjugate and making use of the symmetry assumptions (A being symmetric), we obtain the following expression for the imaginary part $\omega_i(k)$ of $\omega(k)$: $\omega_i(k) = -(u|E_s u)/(u|u)$. Thus the system cannot be conservative if E is not skew-symmetric, and hypothesis 1 is always satisfied by conservative systems.

The two hypotheses 3 and 5 concerning the nonlinearity obviously cannot be understood through the study of the linearized equation (2.25). Hypothesis 3 can be written as

$$\Pi_1 B(\Pi_1, \Pi_1) = 0, \quad (2.32)$$

while the assumption

$$\Pi_1 B(\Pi_1, Q_1) = 0 \quad (2.33)$$

implies hypothesis 5. Conditions (2.32) and (2.33) are particular expressions of a very general condition called the “transparency” condition in the rigorous mathematical theory of multiscale expansions by Joly, Métivier and Rausch [5]. Condition (2.32) excludes quadratic self-interaction for the chosen propagation mode, while condition (2.33) excludes interaction at the same order for different polarizations. If condition (2.32), that is hypothesis 3, is not satisfied, the nonlinear term appears sooner in the expansion. The nonlinear evolution of the wave will be described by a nonlinear evolution equation other than the KdV equation and for other space, amplitude and time scales. If condition (2.33), or more precisely hypothesis 5, is not satisfied, the above computation is valid, but the evolution equation is equation (2.19) instead of KdV. It is an integro-differential equation involving a cubic nonlinear term.

2.4 The linearized KdV equation

Regarding the higher order terms, the evolution equation relative to the first order time $\tau = \tau_1$ is found in the following way. Equation (2.1), written for some given order ε^n , yields a compatibility condition, its Π_1 -projection

$$\Pi_1(A - V)\partial_\xi Q_0 u_{n-1} + \sum_{j \geq 1} \partial_{\tau_j} \Pi_1 u_{n-2j-1} = \sum_{p=2}^{n-2} \Pi_1 B(u_p, u_{n-p}), \quad (2.34)$$

and recurrence formulas for the determined parts of u_n , its Q_0 -projection

$$Q_0 u_n = E^{-1} Q_0 \left[-(A - V) \partial_\xi u_{n-1} - \sum_{j \geq 1} \partial_{\tau_j} u_{n-2j-1} + \sum_{p=2}^{n-2} B(u_p, u_{n-p}) \right], \quad (2.35)$$

and its Q_1 -projection

$$Q_1 u_n = (A - V)^{-1} Q_1 \times \left[-(A - V) Q_0 u_n - \sum_{j \geq 1} \int^{\xi} \partial_{\tau_j} u_{n-2j} + \sum_{p=2}^{n-1} \int^{\xi} B(u_p, u_{n+1-p}) \right]. \quad (2.36)$$

The component of u_n belonging to $\text{Rg}(\Pi_1)$ is proportional to some unknown real function φ_n , and u_n is

$$u_n = a_0 \varphi_n + Q_0 u_n + Q_1 u_n. \quad (2.37)$$

We call \mathcal{S}_p the set of expressions involving u_2, u_3, \dots, u_p , but not u_{p+1} and subsequent terms. The same notation \mathcal{S}_p will hold below for any function belonging to this set. Due to equations (2.35) and (2.36) u_n can be written as

$$u_n = a_0 \varphi_n + \mathcal{S}_{n-1}. \quad (2.38)$$

The functions φ_n are the unknowns of the problem. Note that, with these notations, \mathcal{S}_p is the set of expressions involving $\varphi_2, \varphi_3, \dots, \varphi_p$ and their derivatives. Applying two times the recurrence formulas (2.35) and (2.36) onto the expression (2.38) of u_n we obtain

$$u_n = a_0 \varphi_n + Q_0 u_n + Q_1 u_n, \quad (2.39)$$

with

$$Q_0 u_n = a_1 \partial_\xi \varphi_{n-1} + a_2 \partial_\xi^2 \varphi_{n-2} + 2\alpha_2 \varphi_2 \varphi_{n-2} + \mathcal{S}_{n-3}, \quad (2.40)$$

$$Q_1 u_n = a'_1 \partial_\xi \varphi_{n-1} + 2\alpha_1 \int^{\xi} \varphi_2 \varphi_{n-1} + \mathcal{S}_{n-2}. \quad (2.41)$$

Equations (2.39)–(2.41) are correct if $n - 2 > 2$, i.e. $n \geq 5$. For $n = 4$, the coefficient 2 before α_2 vanishes, as seen above. The dependency of $Q_1 u_n$ with regard to φ_{n-2} is not needed in the computation of the higher order equations below. This is fortunate because of several integral nonlinear terms arising in it that would greatly hinder the computation. When use is made of the expressions (2.39) to (2.41) of u_n in the compatibility condition (2.34), it gives an evolution equation for φ_{n-3} . Indeed, u_n does not appear in the equation. The dependency with regard to φ_{n-1} , φ_{n-2} and φ_{n-3} is explicitly written. Because of the condition $\Pi_1(A - V)a = 0$, which determines the velocity V , φ_{n-1} vanishes from the equation. Because of the relations (2.12) and (2.23) that are satisfied under the present hypotheses, φ_{n-2} and a term $\delta\varphi_2 \left(\int^{\xi} \varphi_2 \varphi_{n-3} + \varphi_{n-3} \int^{\xi} \varphi_2^2 \right)$ also vanish and the equation obtained is, with $n - 3 = l$:

$$\partial_{\tau_1} \varphi_l + \beta \partial_\xi (\varphi_2 \varphi_l) + \gamma \partial_\xi^3 \varphi_l = \Xi_l(\varphi_2, \dots, \varphi_{l-1}). \quad (2.42)$$

Equation (2.42) is the linearized KdV equation obtained by the linearization of the KdV equation (2.24) about its solution φ_2 with an additional right hand member $\Xi_l(\varphi_2, \dots, \varphi_{l-1})$ depending on the previously determined terms.

This is true with no further hypothesis than the existence of some simple eigenvalue V of $\Pi_0(A - V)\Pi_0$ and conditions (2.12) and (2.23). However, in the general case, due to the term $\int^\xi B(u_p, u_{n+1-p})$ in the recurrence formula (2.36), the right hand member Ξ_l involves *a priori* many integrations relative to the variable ξ . The property

$$Q_1 B(\cdot, \cdot) \equiv 0, \quad (2.43)$$

satisfied in ferromagnets, ensures their vanishing.

3 The second order time evolution and the KdV hierarchy

3.1 The problem of the secularities

It is well known that the KdV equation (2.24) is completely integrable, i.e. that the Cauchy problem for it can be solved by use of the Inverse Scattering Transform (IST) method [1, 3, 14]. The IST method gives also explicit formulas for the resolution of the linearized KdV equation (2.42) [8, 15]. This latter equation is linear, but it is necessary to solve it by this method, because the solution φ_2 of the KdV equation intervenes in it as an essential parameter. In the general case φ_2 can be only expressed in terms of its inverse transform, thus also the solution of the linearized KdV equation (2.42). Unfortunately the solution computed this way is not always bounded (I mean uniformly bounded in ξ as $\tau \rightarrow +\infty$). As an example, assume that at $\tau = 0$, $\varphi_l \equiv 0$, and replace the right hand member $\Xi_l(\varphi_2, \dots, \varphi_{l-1})$ by $\partial_\xi \varphi_2$. Then the solution φ_l of equation (2.42) is [8, 15]

$$\varphi_l = \tau \partial_\xi \varphi_2. \quad (3.1)$$

A solution like (3.1) is called “secular”. This phenomenon occurs when a term in the right hand member “resonates”, that is, from the mathematical point of view, is a solution of the homogeneous equation ((2.42) with $\Xi_l(\varphi_2, \dots, \varphi_{l-1}) = 0$). Physically, this resonance phenomenon occurs when each component of the inverse scattering transform of the source term (some part of the right hand member $\Xi_l(\varphi_2, \varphi_3, \dots, \varphi_{l-1})$) evolves in time in the same way as the corresponding component in the transform of the main term. In order to remove the secular terms, we use the method of Kraenkel, Manna, and Pereira [12, 13] that consists in the introduction of additional evolution equations, which describe the evolution of the main term relative to the higher order time variables τ_2, τ_3, \dots . We check that, according to the papers cited, these equations must be those of the so-called KdV hierarchy, normalized in order to cancel the linear terms in the right hand member $\Xi_l(\varphi_2, \varphi_3, \dots, \varphi_{l-1})$. Indeed references [8, 13] state that the secularity producing terms are the linear ones. We showed in [15] that they are rather the derivatives of the conserved densities of the KdV equation, and that the procedure of Kraenkel et al. in fact removes completely these latter terms from the right hand member.

3.2 The next order equation

The standard expansion used to derive the KdV equation in hydrodynamics has some parity relative to ε . As an example, Su [17] uses series expansions in integer powers of

some perturbative parameter ϵ , defined in such a way that $\epsilon = \varepsilon^2$, ε being the perturbative parameter of the present paper. This corresponds to a vanishing of all odd terms in the present expansion. The equation (2.42) governing the τ_1 evolution of φ_n is obtained at order ε^{n+3} , which is even when n is odd. Due to the homogeneity properties of the expansion, the right hand member $\Xi_n(\varphi_2, \dots, \varphi_{n-1})$ contains only even order derivatives of φ_2 , which describe losses. Physically, if the initial system (2.1) is conservative, the same feature can be expected for any equation in the expansion. Therefore the even order derivatives of φ_2 must vanish. We assume that the present model has the same property in the following sense: $\Xi_3(\varphi_2) = 0$. Thus, with a zero initial condition, $\varphi_3 \equiv 0$, and so on: if all φ_p with odd p are zero up to $(n-2)$, n being odd, then $\Xi_n(\varphi_2, \dots, \varphi_{n-1})$ is zero and, with a zero initial condition, φ_n is also zero. Thus, if the corresponding initial conditions are zero, all the Ξ_n and all the φ_n with odd values of the integer n are zero. This occurs in the particular case in which the system (2.1) describes wave propagation in a ferromagnetic medium. This last feature is proved, using homogeneity properties, in [16]. Few results can be proved on this point in the general case. The right hand member $\Xi_3(\varphi_2)$ contains a term $K\partial_\xi^4\varphi_2$, where the coefficient K is given by

$$a_0K = \Pi_1(A - V)E^{-1}Q_0(A - V)(1 - (A - V)^{-1}Q_1(A - V))a_2. \quad (3.2)$$

In the same way as the expression (2.30) for ω'' has been derived, we can give an expression for $\omega^{(4)} = d^4\omega/dk^4|_{k=0}$. After comparison with (3.2), it is seen that $\omega^{(4)} = 24iK$. Observe that the right hand member of (3.2) contains only real terms. Thus K is real and $\omega^{(4)}$ is purely imaginary. The requirement that the linear term $K\partial_\xi^4\varphi_2$ in $\Xi_3(\varphi_2)$ vanish is equivalent to the requirement that $\omega^{(4)}$ be real, which must be satisfied if the system is conservative. This generalizes to the higher even order derivatives and ensures the vanishing of the linear terms, but we omit the proof because we must admit the generalization of this feature to the nonlinear terms. Note that only the linear terms are suspected to be secularity producing. Thus the vanishing of the linear term is expected to ensure that all terms in the expansion are bounded. Nonetheless, we make the assumption above, because it is satisfied in all the examples of which we know, even if we are not able to prove it in the general case.

With this hypothesis the second nontrivial equation of the perturbative expansion is the equation (2.42) for φ_4 . Its right hand member $\Xi_4(\varphi_2)$ is polynomial with regard to φ_2 and its derivatives, with the homogeneity of the terms of order ε^7 in the expansion and can thus be expanded according to formula (A.16) as

$$\Xi_4(\varphi_2) = \sum_{\substack{(m_j)_{j \geq 1}, k \geq 0 \\ \sum_{j \geq 1} 2jm_j + k = 5}} \Xi((m_j)_{j \geq 1}, k) \prod_{j \geq 1} \left(\int_{-\infty}^{\xi} \partial_{\tau_j} \right)^{m_j} \partial_\xi^k \varphi_2 + \mathcal{O}_2. \quad (3.3)$$

(\mathcal{O}_2 designates here an expression in φ_2 without linear terms.) The indices $(m_j)_{j \geq 1}$, k in this sum can take the values

$$((m_j)_{j \geq 1}, k) = ((0, \dots), 5); ((1, 0, \dots), 3); ((2, 0, \dots), 1); ((0, 1, 0, \dots), 1) \quad (3.4)$$

only. Furthermore, the coefficient $\Xi((0, 1, 0, \dots), 1)$ of $\partial_{\tau_2}\varphi_2$ is -1 . Thus expression (3.3)

can be expanded in the following way:

$$\begin{aligned} \Xi_4(\varphi_2) = & -\partial_{\tau_2}\varphi_2 + \Xi((2, 0, \dots), 1) \left(\int_{-\infty}^{\xi} \partial_{\tau_1} \right)^2 \partial_{\xi}\varphi_2 \\ & + \Xi((1, 0, \dots), 3) \left(\int_{-\infty}^{\xi} \partial_{\tau_1} \right) \partial_{\xi}^3\varphi_2 + \Xi((0, \dots), 5) \partial_{\xi}^5\varphi_2 + \mathcal{O}_2. \end{aligned} \quad (3.5)$$

As φ_2 satisfies the KdV equation (2.24), we have

$$\int_{-\infty}^{\xi} \partial_{\tau_1}\varphi_2 = -\gamma\partial_{\xi}^2\varphi_2 + \mathcal{O}_2. \quad (3.6)$$

Thus

$$\Xi_4(\varphi_2) = -\partial_{\tau_2}\varphi_2 - \gamma_2\partial_{\xi}^5\varphi_2 + \mathcal{O}_2, \quad (3.7)$$

with

$$-\gamma_2 = \Xi((2, 0, \dots), 1)(-\gamma)^2 + \Xi((1, 0, \dots), 3)(-\gamma) + \Xi((0, \dots), 5). \quad (3.8)$$

The coefficient γ_2 is explicitly computed in the Appendix (equation (A.19)).

3.3 The KdV hierarchy

The linear terms computed above are secularity producing. Because the solution φ_4 of the linearized KdV equation must be bounded, they must vanish. This imposes some evolution equation for φ_2 relative to the second-order time variable τ_2 , such that

$$\partial_{\tau_2}\varphi_2 = -\gamma_2\partial_{\xi}^5\varphi_2 + \mathcal{O}_2. \quad (3.9)$$

The nonlinear terms in equation (3.9) are not free, but imposed by the compatibility conditions between the KdV equation (2.24) and (3.9): the Schwartz conditions $\partial_{\tau_1}\partial_{\tau_2}\varphi_2 = \partial_{\tau_2}\partial_{\tau_1}\varphi_2$. Kraenkel, Manna, and Pereira [13] have shown that the only equation that has the same homogeneity properties as Ξ_4 and that satisfies this condition is the second equation of the so-called KdV hierarchy. The same requirements are found at higher orders. The KdV hierarchy is the following set of equations [2]:

$$\partial_{T_n}v = \partial_X\mathcal{L}^n v \quad (n \text{ integer}), \quad (3.10)$$

where

$$\mathcal{L} = \frac{-1}{4}\partial_X^2 - v + \frac{1}{2}\int^X dX(\partial_X v). \quad (3.11)$$

For $n = 1$ it is the KdV equation, but with values of the coefficients $\beta = 3/2$, $\gamma = 1/4$. The coefficients become equal to those of equation (2.24) by setting

$$v = \frac{\beta}{6\gamma}\varphi_2, \quad X = \xi \quad \text{and} \quad T_1 = 4\gamma\tau_1. \quad (3.12)$$

For $n = 2$ the equation (3.10) of the hierarchy is

$$\partial_{T_2} v = \frac{1}{16} \partial_X^5 v + \frac{5}{4} (\partial_X v) \partial_X^2 v + \frac{5}{8} v \partial_X^3 v + \frac{15}{8} v^2 \partial_X v. \quad (3.13)$$

The important feature is the existence of the Hirota τ -function [2], that is a function of all the variables (X, T_1, T_2, \dots) , related to v by

$$v(X, T_1, T_2, \dots) = 2 \partial_X^2 \ln \tau(X, T_1, T_2, \dots) \quad (3.14)$$

(take care to avoid confusion between the Hirota τ -function and the time variables τ_j). The existence of τ insures that a solution v of the whole system yielded by all equations of the hierarchy exists. As the system admits a solution, the Schwartz conditions are satisfied at any order, i.e. $\partial_{T_j} \partial_{T_p} v = \partial_{T_p} \partial_{T_j} v$ for any $j, p \geq 1$. Furthermore these conditions are satisfied formally and identically by the equations themselves and not only for some particular solution. The authors cited have checked by formal computation on the first orders that, apart from the symmetries of the KdV hierarchy, it is the only compatible system which has the required homogeneity properties and for which the first equation is the KdV one. The symmetries of the hierarchy are, after those of the KdV equation, a free scaling coefficient for each time variable. We must identify

$$T_2 = -16\gamma_2\tau_2 \quad (3.15)$$

with γ_2 given by equation (3.8) and impose that φ_2 satisfy the evolution equation

$$\frac{-1}{16\gamma_2} \partial_{\tau_2} \varphi_2 = \partial_\xi \mathcal{L}^2 \varphi_2 \quad (3.16)$$

with

$$\mathcal{L} = \frac{-1}{4} \partial_\xi^2 - \frac{\beta}{6\gamma} \varphi_2 + \frac{\beta}{12\gamma} \int_{-\infty}^{\xi} d\xi (\partial_\xi \varphi_2). \quad (3.17)$$

Then $\Xi_4(\varphi_2)$ no longer contains any linear term. This implies, due to the procedure, that it no longer contains any secularity producing term [15].

The linear terms in φ_2 are removed in the same way at each order. We impose for each $p \geq 2$ that

$$\frac{-1}{(-4)^p \gamma_p} \partial_{\tau_p} \varphi_2 = \partial_\xi \mathcal{L}^p \varphi_2, \quad (3.18)$$

(\mathcal{L} as above) with γ_p defined by $\gamma_1 = \gamma$ and

$$\gamma_{p+1} = \sum_{\substack{(m_j)_{1 \leq j \leq p-1}, k \geq 0 \\ \sum_{j=1}^{p-1} 2jm_j + k = 2p+3}} \Xi((m_j)_{1 \leq j \leq p-1}, k) \prod_{j=1}^{p-1} (-\gamma_j)^{m_j}. \quad (3.19)$$

For $p = 1$ equation (3.18) coincides with equation (2.24) with $\beta_1 = \beta$. We use here the same scheme as in the first case $p = 2$. We have

$$\mathcal{L}^p = \left(\frac{-1}{4} \partial_\xi^2 \right)^p + \mathcal{O}_1. \quad (3.20)$$

Equation (3.18) gives

$$\partial_{\tau_p} \varphi_2 = -\gamma_p \partial_\xi^{2p+1} \varphi_2 + \mathcal{O}_2. \quad (3.21)$$

Then the linear term in φ_2 vanishes from $\Xi_p(\varphi_2, \varphi_4, \dots, \varphi_{p-2})$. In this way all secularity producing terms due to φ_2 vanish at any order.

4 The third order and the linearized KdV hierarchy

4.1 The higher order time evolution of the higher order terms

The third equation of our perturbation expansion is equation (2.42), written for $l = 6$, with its right hand member $\Xi_6(\varphi_2, \varphi_4)$ given by formula (A.16). It is of order ε^9 . It involves not only φ_2, φ_4 and their space derivatives but also $\partial_{\tau_1} \varphi_2, \partial_{\tau_2} \varphi_2, \partial_{\tau_3} \varphi_2, \partial_{\tau_1} \varphi_4$ and $\partial_{\tau_2} \varphi_4$ of orders $\varepsilon^5, \varepsilon^7, \varepsilon^9, \varepsilon^7$ and ε^9 respectively. $\partial_{\tau_1} \varphi_2$ is replaced by space derivatives and nonlinear terms using the KdV equation (2.24). We make use of the second and third equations of the hierarchy for φ_2 so that $\partial_{\tau_2} \varphi_2$ and $\partial_{\tau_3} \varphi_2$ also vanish, replaced by functions of the space derivatives. The right hand member becomes

$$\begin{aligned} \Xi_6(\varphi_2, \varphi_4) &= \partial_{\tau_2} \varphi_4 + \Xi((2, 0, \dots), 1) \left(\int_{-\infty}^{\xi} \partial_{\tau_1} \right)^2 \partial_\xi \varphi_4 \\ &\quad + \Xi((1, 0, \dots), 3) \left(\int_{-\infty}^{\xi} \partial_{\tau_1} \right) \partial_\xi^3 \varphi_4 + \Xi((0, \dots), 5) \partial_\xi^5 \varphi_4 + \mathcal{O}_2. \end{aligned} \quad (4.1)$$

(Here \mathcal{O}_2 represents some expression in φ_2, φ_4 , their derivatives and primitives, without linear terms). The τ_1 -evolution of φ_4 is also known, and described by the linearized KdV equation (2.42), with the right hand member $\Xi_4(\varphi_2)$. Using this relation removes the explicit dependency on $\partial_{\tau_1} \varphi_4$, and the right hand member $\Xi_6(\varphi_2, \varphi_4)$ reduces to

$$\Xi_6(\varphi_2, \varphi_4) = \partial_{\tau_2} \varphi_4 + \gamma_2 \partial_\xi^5 \varphi_4 + \mathcal{O}_2. \quad (4.2)$$

Two questions arise at this point:

- How is the τ_2 -dependency of φ_4 defined?
- Is the right hand member $\Xi_6(\varphi_2, \varphi_4)$ secularity producing due to the terms in φ_4 , in particular the linear ones?

φ_4 is the solution of the linearized KdV equation (2.42), with the right hand member $\Xi_4(\varphi_2)$ and some given initial data $\varphi_4(\xi, \tau_1 = 0)$. It is expressed as an integral and linear combination of the squared Jost functions, defined in the solution of the KdV equation through the IST method [8, 15]. We denote by $\varphi_4^{(1)}$ the solution of the homogeneous linearized KdV equation with the same initial data $\varphi_4^{(1)}(\xi, \tau_1 = 0) \equiv \varphi_4(\xi, \tau_1 = 0)$, and by $\varphi_4^{(2)}$ the solution of the linearized KdV equation with the right hand member $\Xi_4(\varphi_2)$ and vanishing initial data $\varphi_4^{(2)}(\xi, \tau_1 = 0) \equiv 0$, so that $\varphi_4 = \varphi_4^{(1)} + \varphi_4^{(2)}$. It is shown in [15] that $\varphi_4^{(1)}$ is secularity producing while $\varphi_4^{(2)}$ is not.

Since the τ_2 -dependency of the solution φ_2 of KdV is determined, the squared Jost functions are also known for all (ξ, τ_1, τ_2) , and so are the spectral components of the right

hand member $\Xi_4(\varphi_2)$. Thus $\varphi_4^{(2)}$ is completely determined and its τ_2 -dependency is well defined without additional condition. Furthermore the evolution equation that makes explicit this dependency is not needed for the computation of the function $\varphi_4(\xi, \tau_1, \tau_2)$. The situation is different with $\varphi_4^{(1)}$, while the initial condition is *a priori* free. In the previous subsection we saw that the compatibility conditions fixed the τ_2 -dependency of φ_2 , apart from a scaling coefficient for this time variable. The τ_2 -dependency of $\varphi_4^{(1)}$ is determined below in the same way, but the scaling coefficients are no longer free.

Consider some solution v_0 of the KdV equation, and v_1 of the homogeneous linearized KdV equation. We use here the normalization of formulas (3.10)–(3.11). The latter equation is

$$\partial_{T_1} v_1 + \frac{3}{2} \partial_X (v_0 v_1) + \frac{1}{4} \partial_X^3 v_1 = 0. \quad (4.3)$$

We assume also that v_0 and v_1 are smooth functions of all the variables X, T_1, T_2, \dots , and that their dependency relative to each time variable satisfies the homogeneity properties of the KdV hierarchy. We saw above that under these conditions the function v_0 , solution of the KdV equation, satisfies the complete hierarchy, with some scaling constant for each time variable. Consider some small parameter η . The function

$$v = v_0 + \eta v_1 \quad (4.4)$$

satisfies the KdV equation apart from a term of order η^2 . Thus it satisfies the whole hierarchy (still apart from terms of order η^2). By linearization of the n^{th} equation of the hierarchy (3.10), we find that

$$\partial_{T_n} v_1 = \partial_X \mathcal{D}_n v_1, \quad (4.5)$$

with

$$\mathcal{D}_n v_1 = (d_1 \mathcal{L}^{n-1} + \mathcal{L} d_1 \mathcal{L}^{n-2} + \dots + \mathcal{L}^{n-1} d_1) v_0 + \mathcal{L}^n v_1 \quad (4.6)$$

and

$$d_1 = \frac{d\mathcal{L}}{dv}(v_1) = -v_1 + \frac{1}{2} \int^X dX (\partial_X v_1). \quad (4.7)$$

This result is in particular valid for $v_1 = \varphi_4^{(1)}$, but also for the following orders $v_1 = \varphi_p^{(1)}$, with analogous notations, for any even $p \geq 4$. While the higher order time evolution of the part $\varphi_4^{(2)}$ of the term of order ε^4 coming from the right hand member $\Xi_4(\varphi_2)$ of the linearized KdV equation is defined by the main order itself, the evolution of the part $\varphi_4^{(1)}$ coming from the initial data is determined in an analogous way as the main term: $\varphi_4^{(1)}$ satisfies the linearized KdV hierarchy (4.5).

Formal identities are related to this feature. They are found by the following reasoning. We assume as previously that v_1 satisfies the homogeneous KdV equation. We have seen that v_1 must satisfy equations (4.5). The existence of v_1 as a function of all the variables X, T_1, T_2, \dots is not in doubt, because it is ensured by the existence of the Hirota τ -function [2]. Thus we have

$$\partial_{T_n} \partial_{T_j} v_1 = \partial_{T_j} \partial_{T_n} v_1 \quad (4.8)$$

for any integers n and j . We denote by $(\partial_{T_n} \mathcal{D}_j)$ the T_n -partial derivative of \mathcal{D}_j which depends on T_n through v , and by $\partial_{T_n} \mathcal{D}_j$ the operator that applies successively \mathcal{D}_j and ∂_{T_n} . With analogous notation, $\partial_X \mathcal{D}_j \equiv (\partial_X \mathcal{D}_j) + \mathcal{D}_j \partial_X$. With the use of equation (4.5) equation (4.8) becomes

$$((\partial_{T_n} \mathcal{D}_j) + \mathcal{D}_j \partial_X \mathcal{D}_n) v_1 = ((\partial_{T_j} \mathcal{D}_n) + \mathcal{D}_n \partial_X \mathcal{D}_j) v_1. \quad (4.9)$$

As in the case of the KdV hierarchy itself (nonlinearized), equation (4.9) is valid for any function v_1 . Thus the following identity holds formally:

$$(\partial_{T_n} \mathcal{D}_j) + \mathcal{D}_j \partial_X \mathcal{D}_n = (\partial_{T_j} \mathcal{D}_n) + \mathcal{D}_n \partial_X \mathcal{D}_j. \quad (4.10)$$

This identity can also be written as

$$[\partial_{T_j} - \partial_X \mathcal{D}_j, \partial_{T_n} - \partial_X \mathcal{D}_n] = 0, \quad (4.11)$$

where $[M, N] = MN - NM$ denotes the commutator of the operators M and N , and the Schwartz conditions $[\partial_{T_j}, \partial_{T_n}] = 0$ are assumed. The commutator in equation (4.11) is easily computed and vanishes due to the Schwartz condition and to the formal identity (4.10).

Using the scaling definition for v_0 and T_1, T_2, \dots , we can write down the equations of the linearized KdV hierarchy for the functions $\varphi_p^{(1)}$ as

$$\partial_{\tau_n} \varphi_p^{(1)} + (-4)^n \gamma_n \partial_\xi \mathcal{D}_n \varphi_p^{(1)} = 0, \quad (4.12)$$

with \mathcal{D}_n defined by (4.6)–(4.7). In the case that $n = 2$ and $p = 4$ this equation has the explicit form

$$\begin{aligned} \partial_{\tau_2} \varphi_4^{(1)} &= -\gamma_2 \partial_\xi^5 \varphi_4^{(1)} \\ &\quad - \frac{5}{3} \frac{\beta \gamma_2}{\gamma} \left[\varphi_2 \partial_\xi^3 \varphi_4^{(1)} + 2 (\partial_\xi \varphi_2) \partial_\xi^2 \varphi_4^{(1)} + 2 (\partial_\xi^2 \varphi_2) \partial_\xi \varphi_4^{(1)} + (\partial_\xi^3 \varphi_2) \varphi_4^{(1)} \right] \\ &\quad - \frac{5}{6} \frac{\beta^2 \gamma_2}{\gamma^2} \left[\varphi_2^2 \partial_\xi \varphi_4^{(1)} + 2 \varphi_2 (\partial_\xi \varphi_2) \varphi_4^{(1)} \right]. \end{aligned} \quad (4.13)$$

4.2 All secular terms vanish

The function $\varphi_4^{(1)}$ is secularity producing [15]. The following term φ_6 would thus be secular if $\varphi_4^{(1)}$ appears in $\Xi_6(\varphi_4, \varphi_2)$. Equation (4.13) yields

$$\partial_{\tau_2} \varphi_4^{(1)} = -\gamma_2 \partial_\xi^5 \varphi_4^{(1)} + \mathcal{O}_2. \quad (4.14)$$

Together with equation (4.2), this allows one to compute the linear part of the right hand member to obtain

$$\Xi_6(\varphi_4, \varphi_2) = -\partial_{\tau_2} \varphi_4^{(2)} - \gamma_2 \partial_\xi^5 \varphi_4^{(2)} + \mathcal{O}_2. \quad (4.15)$$

We admit that the only secularity producing terms due to $\varphi_4^{(1)}$ are the linear ones and recall that $\varphi_4^{(2)}$ is not secularity producing. Thus, due to the equation of the linearized

KdV hierarchy (4.13), all secularity producing terms vanish from the right hand member $\Xi_6(\varphi_4, \varphi_2)$ and φ_6 is bounded.

In order to justify the same property at any order, we have to compute the linear terms in φ_p that appear in the expression of $\Xi_n(\varphi_2, \dots, \varphi_{n-2})$. The computation of all linear terms in these right hand members is detailed in the Appendix. The general expression of the linear part of Ξ_n is

$$\begin{aligned} & \Xi_n(\varphi_2, \dots, \varphi_{n-2}) \\ &= \sum_{\substack{(m_j)_{j \geq 1}, k \geq 0, p \geq 2 \\ \sum_{j \geq 1} 2jm_j + k + p = n + 3}} \Xi((m_j)_{j \geq 1}, k) \prod_{j \geq 1} \left(\int_{-\infty}^{\xi} \partial_{\tau_j} \right)^{m_j} \partial_{\xi}^k \varphi_p + \mathcal{R}. \end{aligned} \quad (4.16)$$

The remainder \mathcal{R} contains nonlinear terms in φ_p , as well as terms in φ_j , $j \neq p$. From equation (4.12) of the linearized KdV hierarchy we see that

$$\partial_{\tau_j} \varphi_p^{(1)} = -\gamma_j \partial_{\xi}^{2j+1} \varphi_p^{(1)} + \mathcal{O}_2. \quad (4.17)$$

Thus

$$\begin{aligned} & \Xi_n(\varphi_2, \dots, \varphi_{n-2}) \\ &= \sum_{\substack{(m_j)_{j \geq 1}, k \geq 0 \\ \sum_{j \geq 1} 2jm_j + k + p = n + 3}} \Xi((m_j)_{j \geq 1}, k) \prod_{j \geq 1} (-\gamma_j)^{m_j} \partial_{\xi}^{(\sum_{j \geq 1} 2jm_j + k)} \varphi_p^{(1)} + \mathcal{R}'. \end{aligned} \quad (4.18)$$

The remainder \mathcal{R}' contains nonlinear terms, terms in φ_j , $j \neq p$, and in the part $\varphi_p^{(2)}$ of φ_p that comes from the right hand member $\Xi_p(\varphi_2, \dots, \varphi_{p-2})$. These latter terms are not secularity producing. Using the fact that, in the sum in (4.18), $m_j = 0$ for $j > s = (n + 3 - p - 1)/2$, and expression (3.19) of γ_j , we find that $\Xi_n(\varphi_2, \dots, \varphi_{n-2})$ reduces to \mathcal{R}' . This result is valid for each p , thus $\Xi_n(\varphi_2, \dots, \varphi_{n-2})$ does not contain any secularity producing term and φ_n is not secular.

4.3 The higher order time derivatives

On the other hand the expression of the solution φ_4 of the linearized KdV equation involves its spectral transform and an explicit computation of its τ_2 -derivative seems quite impossible. However, it is needed for the explicit computation of the right hand member $\Xi_6(\varphi_2, \varphi_4)$. $\partial_{\tau_2} \varphi_4^{(1)}$ is directly given by equation (4.13). We now seek an analogous expression of $\partial_{\tau_2} \varphi_4^{(2)}$. With the use of the definitions of this section, the linearized KdV equation (2.42) can be written as

$$(\partial_{\tau_1} + 4\gamma \partial_{\xi} \mathcal{D}_1) \varphi_4^{(2)} = \Xi_4(\varphi_2). \quad (4.19)$$

We apply the operator $(\partial_{\tau_2} - 16\gamma_2 \partial_{\xi} \mathcal{D}_2)$ to both sides of this equation and make use of identity (4.11), to obtain

$$(\partial_{\tau_1} + 4\gamma \partial_{\xi} \mathcal{D}_1) (\partial_{\tau_2} - 16\gamma_2 \partial_{\xi} \mathcal{D}_2) \varphi_4^{(2)} = (\partial_{\tau_2} - 16\gamma_2 \partial_{\xi} \mathcal{D}_2) \Xi_4(\varphi_2). \quad (4.20)$$

$(\partial_{\tau_1} + 4\gamma\partial_\xi\mathcal{D}_1)$ is the operator on the left hand side of the linearized KdV equation that admits a unique solution for a given initial data. $\varphi_4^{(2)}$ is defined in such a way that it vanishes at $\tau_1 = 0$ for any ξ and τ_2 . Thus its ξ and τ_2 derivatives also vanish and

$$(\partial_{\tau_2} - 16\gamma_2\partial_\xi\mathcal{D}_2)\varphi_4^{(2)}\Big|_{\tau_1=0} \equiv 0. \quad (4.21)$$

We use the following notation to denote this solution:

$$(\partial_{\tau_2} - 16\gamma_2\partial_\xi\mathcal{D}_2)\varphi_4^{(2)} = (\partial_{\tau_1} + 4\gamma\partial_\xi\mathcal{D}_1)^{-1}[(\partial_{\tau_2} - 16\gamma_2\partial_\xi\mathcal{D}_2)\Xi_4(\varphi_2)]. \quad (4.22)$$

$\partial_{\tau_2}\Xi_4(\varphi_2)$ is easily computed while the expression of Ξ_4 is explicitly known through the perturbative expansion and $\partial_{\tau_2}\varphi_2$ is determined by the second equation of the hierarchy (3.16). It is

$$\partial_{\tau_2}\Xi_4(\varphi_2) = -16\gamma_2\frac{d\Xi_4(\varphi_2)}{d\varphi_2}\partial_\xi\mathcal{L}^2\varphi_2, \quad (4.23)$$

where $d\Xi_4(\varphi_2)/d\varphi_2$ is the differential operator obtained by linearization of $\Xi_4(\varphi_2)$ about φ_2 . $\partial_{\tau_2}\varphi_4$ is thus given by the formula

$$\begin{aligned} \partial_{\tau_2}\varphi_4^{(2)} &= 16\gamma_2\left(\partial_\xi\mathcal{D}_2\varphi_4^{(2)} - (\partial_{\tau_1} + 4\gamma\partial_\xi\mathcal{D}_1)^{-1}\right. \\ &\quad \left.\times\left[\frac{d\Xi_4(\varphi_2)}{d\varphi_2}\partial_\xi\mathcal{L}^2\varphi_2 + \partial_\xi\mathcal{D}_2\Xi_4(\varphi_2)\right]\right). \end{aligned} \quad (4.24)$$

More generally $\varphi_p^{(2)}$ is the solution of the linearized KdV equation with the right hand member $\Xi_p(\varphi_2, \dots, \varphi_{p-2})$ and zero initial data and

$$\begin{aligned} (\partial_{\tau_1} + 4\gamma\partial_\xi\mathcal{D}_1)(\partial_{\tau_n} - (-4)^n\gamma_n\partial_\xi\mathcal{D}_n)\varphi_p^{(2)} \\ = (\partial_{\tau_n} - (-4)^n\gamma_n\partial_\xi\mathcal{D}_n)\Xi_p(\varphi_2, \dots, \varphi_{p-2}). \end{aligned} \quad (4.25)$$

Because, for $n \neq 1$, $(\partial_{\tau_n} - (-4)^n\gamma_n\partial_\xi\mathcal{D}_n)$ does not contain the partial derivative ∂_{τ_1} , $(\partial_{\tau_n} - (-4)^n\gamma_n\partial_\xi\mathcal{D}_n)\varphi_p^{(2)}\Big|_{\tau_1=0} \equiv 0$. Using the above notations we obtain the following expression of the higher order time derivative:

$$\begin{aligned} \partial_{\tau_n}\varphi_p^{(2)} &= (-4)^n\gamma_n\partial_\xi\mathcal{D}_n\varphi_p^{(2)} + (\partial_{\tau_1} + 4\gamma\partial_\xi\mathcal{D}_1)^{-1} \\ &\quad \times\left[\sum_{2l=2}^{p-2}\frac{d\Xi_p(\varphi_2, \dots, \varphi_{p-2})}{d\varphi_{2l}}\partial_{\tau_n}\varphi_{2l} - (-4)^n\gamma_n\partial_\xi\mathcal{D}_n\Xi_p(\varphi_2, \dots, \varphi_{p-2})\right]. \end{aligned} \quad (4.26)$$

For $p = 4$ the time derivatives in the right hand member of equation (4.26) reduce to $\partial_{\tau_n}\varphi_2 = -(-4)^n\gamma_n\partial_\xi\mathcal{L}^n\varphi_2$, according to equation (3.18). For larger values of p , it involves $\partial_{\tau_n}\varphi_{2l}$ with $2l \geq 4$ that divides into $\partial_{\tau_n}\varphi_{2l} = \partial_{\tau_n}\varphi_{2l}^{(1)} + \partial_{\tau_n}\varphi_{2l}^{(2)}$, where $\partial_{\tau_n}\varphi_{2l}^{(1)}$ is given by the linearized KdV hierarchy (4.12), and $\partial_{\tau_n}\varphi_{2l}^{(2)}$ by the same equation (4.26), in a recurrent way.

We have proved that the introduction of the KdV hierarchy removes all unbounded or secular solutions from the perturbative expansion and given the expression of the higher order time derivatives of all terms in the perturbative expansion.

5 Conclusion

We have studied the higher order terms in the perturbative expansion that describes KdV solitons in a rather general frame, including electromagnetic wave propagation in ferromagnetic media. Using various mathematical techniques, we have been able to write down the equations satisfied by the quantities of any order in this expansion. In every case it is the linearized KdV equation, with some right hand member. We have summarized the known results about this equation. Unbounded or secular solutions can be removed by suppressing linear terms in the right hand member of the equations. This is done by imposing that the main term satisfy all equations of the KdV hierarchy. There exist scaling coefficients for the higher order time variables in the hierarchy, which are determined by the requirement that the linear terms in the right hand member of the linearized KdV equation vanish. They are computed using explicit recurrence formulas. Furthermore the behaviour of the higher order terms relative to the higher order time variables was not known. These terms are divided into two parts: one part comes from the initial data, and its higher order time evolution is described by the linearized KdV hierarchy. The other part comes from the right hand member of the linearized KdV equation that describes its evolution relative to the main time scale. It is completely determined, including its higher order time evolution, by this latter equation.

A quasi-explicit formula for the higher order time derivatives of this second part of the corrective terms has also been given. It involves also the expression of the linearized KdV hierarchy. The main result is that none of the higher order terms produces any secularity. Thus the existence of the expansion is established up to any order.

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A Appendix. Computation of the linear terms in the right hand member of the equations of the perturbative expansion

The right hand member $\Xi_l(\varphi_2, \varphi_3, \dots, \varphi_{l-1})$ of the linearized KdV equation (2.42) is defined by

$$a_0 \Xi_l(\varphi_2, \varphi_3, \dots, \varphi_{l-1}) = \Pi_1 \mathcal{E}_{q(l+3)} + a_0 (\partial_{\tau_1} \varphi_l + \beta \partial_{\xi}(\varphi_2 \varphi_l) + \gamma \partial_{\xi}^3 \varphi_l) \quad (\text{A.1})$$

for each $l \geq 3$, with

$$\mathcal{E}q_{(n)} = -Eu_n - (A - V)\partial_\xi u_{n-1} - \sum_{j \geq 1} \partial_{\tau_j} u_{n-2j-1} + \sum_{p=2}^{n-2} B(u_p, u_{n-p}), \quad (\text{A.2})$$

so that the n^{th} equation (2.34)–(2.36) of the perturbative scheme is $\mathcal{E}q_{(n)} = 0$ ($\mathcal{E}q_{(n)}$ is the difference of the right hand member minus the left hand member). We write \mathcal{O}_2 for any polynomial expression in the φ_j and their derivatives and primitives that does not contain any linear term. This is analogous to the usual Landau notation $O(\varphi_j^2)$, except that we do not assume that the φ_j are small in any way. Note that the use of the KdV equation satisfied by φ_2 , and so on could change the linear terms in $\Xi_l(\varphi_2, \varphi_3, \dots, \varphi_{l-1})$. Thus in order to define the linear part of $\Xi_l(\varphi_2, \varphi_3, \dots, \varphi_{l-1})$ in a unique way, we assume that no use of these properties has been done. We made the standard hypothesis that, for zero initial conditions at these orders, all φ_j with an odd value of j are zero, but it is more convenient for the present formal computation to forget this feature. Formal first order terms, vector u_1 and function φ_1 , satisfying equations (2.34) and (2.35) are introduced, although it can be shown that they are necessary zero. This is consistent with the requirement that no use of the equations has been made: we compute formally the right hand member without solving the equations in any way.

We write an *a priori* formula for the linear part of u_n :

$$u_n = \sum_{\substack{\bar{m} = ((m_j)_{j \geq 1}, k, l) \\ m_j, k \geq 0, l \geq 1, d(\bar{m}) = n}} \tilde{u}(\bar{m}) \prod_{j \geq 1} \left(\int^{\xi} \partial_{\tau_j} \right)^{m_j} \partial_\xi^k \varphi_l + \mathcal{O}_2, \quad (\text{A.3})$$

where the components of the vector coefficients $\tilde{u}(\bar{m})$ have to be determined. The homogeneity properties of u_n are described by the “degree” $d(\bar{m})$, defined by

$$d((m_j)_{j \geq 1}, k, l) = \left(\sum_{j \geq 1} 2j m_j \right) + k + l. \quad (\text{A.4})$$

No explicit dependency on n has to be written down because $n = d(\bar{m})$. The substitution of formula (A.3) into the recurrence formulas (2.35) and (2.36) for u_n yields a new recurrence formula that allows one to compute these coefficients. This shows by induction that formula (A.3) is valid for any value of the integer n and that the values found for the coefficients are valid. These recurrence formulas are

$$\text{For all } l \geq 1: \quad \tilde{u}((0), 0, l) = a_0 \quad (\text{A.5})$$

$$\text{For all } k \text{ and } l \geq 1: \quad \tilde{u}((0), k, l) = F_1(\tilde{u}((0), k-1, l)). \quad (\text{A.6})$$

For all $(m_j)_{j \geq 1} \neq (0)$ and $l \geq 1$,

$$\tilde{u}((m_j)_{j \geq 1}, 0, l) = \sum_{i \geq 1} F_3(\tilde{u}((m_j - \delta_{i,j})_{j \geq 1}, 0, l)). \quad (\text{A.7})$$

For all $(m_j)_{j \geq 1} \neq (0)$, $k, l \geq 0$,

$$\begin{aligned} \tilde{u}((m_j)_{j \geq 1}, k, l) &= F_1(\tilde{u}((m_j)_{j \geq 1}, k-1, l)) \\ &+ \sum_{i \geq 1} [F_2(\tilde{u}((m_j - \delta_{i,j})_{j \geq 1}, k-1, l)) + F_3(\tilde{u}((m_j - \delta_{i,j})_{j \geq 1}, k, l))]. \end{aligned} \quad (\text{A.8})$$

$\delta_{i,j}$ is the Kronecker symbol and the operators F_j are defined by

$$F_1 = F_2(A - V), \quad (\text{A.9})$$

$$F_2 = -[1 - (A - V)^{-1}Q_1(A - V)]E^{-1}Q_0, \quad (\text{A.10})$$

$$F_3 = -(A - V)^{-1}Q_1. \quad (\text{A.11})$$

Formulas (A.5) to (A.8) define $\tilde{u}((m_j)_{j \geq 1}, k, l)$. We see that this quantity does not depend upon l . We write more simply $\tilde{u}((m_j)_{j \geq 1}, k)$ in the following. For some particular terms explicit expressions can be given. It is straightforwardly seen from equations (A.6) and (A.5) that, for all k ,

$$\tilde{u}((0), k) = F_1^k a_0. \quad (\text{A.12})$$

In the particular case that $k = 1$ this is

$$\tilde{u}((0), 1) = a_1 + a'_1. \quad (\text{A.13})$$

Equations (A.7) and (A.5) yield in a similar way

$$\tilde{u}((m_j)_{j \geq 1}, 0) = F_3^{\sum_{j \geq 1} m_j} a_0. \quad (\text{A.14})$$

The definition of a_0 shows that $Q_1 a_0 = 0$. Thus $F_3 a_0$ is zero and $\tilde{u}((m_j)_{j \geq 1}, 0) = 0$ for all $(m_j)_{j \geq 1}$. In the same way, due to $Q_0 a_0 = 0$, $F_2 a_0 = 0$.

The formulas obtained are substituted into the definition (A.2) of Ξ_l to give

$$\begin{aligned} \Pi_1 \mathcal{E}q(n) &= - \sum_{\substack{(m_j)_{j \geq 1}, k \geq 0, l \geq 1 \\ d(\bar{m})=n-1}} \Pi_1(A - V)Q_0 \tilde{u}(\bar{m}) \prod_{j \geq 1} \left(\int_{-\infty}^{\xi} \partial_{\tau_j} \right)^{m_j} \partial_{\xi}^{k+1} \varphi_l \\ &- \sum_{i \geq 1} \sum_{\substack{(m_j)_{j \geq 1}, k \geq 0, l \geq 1 \\ d(\bar{m})=n-2i-1}} \Pi_1 \tilde{u}(\bar{m}) \prod_{j \geq 1} \left(\int_{-\infty}^{\xi} \partial_{\tau_j} \right)^{m_j + \delta_{i,j}} \partial_{\xi}^{k+1} \varphi_l + \mathcal{O}_2. \end{aligned} \quad (\text{A.15})$$

Thus the right hand member Ξ_n is

$$\begin{aligned} \Xi_n(\varphi_2, \varphi_3, \dots, \varphi_{n-1}) &= \sum_{\substack{(m_j)_{j \geq 1}, k \geq 0 \\ 1 \leq l \leq n-1 \\ \sum_{j \geq 1} 2jm_j + k + l = n+3}} \Xi((m_j)_{j \geq 1}, k) \prod_{j \geq 1} \left(\int_{-\infty}^{\xi} \partial_{\tau_j} \right)^{m_j} \partial_{\xi}^k \varphi_l + \mathcal{O}_2, \end{aligned} \quad (\text{A.16})$$

with

$$a_0 \Xi((m_j)_{j \geq 1}, k) = -\Pi_1(A - V)Q_0 \tilde{u}((m_j)_{j \geq 1}, k - 1) - \sum_{i \geq 1} \Pi_1 \tilde{u}((m_j - \delta_{i,j})_{j \geq 1}, k - 1), \quad (\text{A.17})$$

where $\tilde{u}((m_j)_{j \geq 1}, k)$ is defined by the recurrence formulas (A.5) to (A.8). Note the remarkable feature that the coefficient $\Xi((m_j)_{j \geq 1}, k)$ does not depend on n .

The term in which $\partial_{\tau_j} \varphi_l$ appears with the largest value of the index j , in a given Ξ_n , can be computed explicitly. Because n is necessary even, we write $n = 2p$. Due to the homogeneity properties of $\Xi_n(\varphi_1, \dots)$ we see that the term sought is proportional to $\partial_{\tau_p} \varphi_2$. The corresponding coefficient is $\Xi((\delta_{j,p}), 1)$. By direct application of the previous formulas, we find that

$$\Xi((\delta_{j,p}), 1) = -1. \quad (\text{A.18})$$

The coefficient γ_2 that defines the second order time scale τ_2 is computed using the expansion (3.8) and the above recurrence formulas. It is

$$\gamma_2 a_0 = \Pi_1 [-\gamma F_1 + (A - V)Q_0 (F_1^3 - \gamma [F_1 F_3 + F_3 F_1 + F_2])] F_1 a_0. \quad (\text{A.19})$$

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