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Group Invariant Solution and Conservation Law for a Free Laminar Two-Dimensional Jet

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Abstract

A group invariant solution for a steady two-dimensional jet is derived by considering a linear combination of the Lie point symmetries of Prandtl's boundary layer equations for the jet. Only two Lie point symmetries contribute to the solution and the ratio of the constants in the linear combination is determined from conservation of total momentum flux in the downstream direction. A conservation law for the differential equation for the stream function is derived and it is shown that the Lie point symmetry associated with the conservation law is the same as that which generates the group invariant solution. This establishes a connection between the conservation law and conservation of total momentum flux.

1 Introduction

The theory of laminar jets has many applications in science and engineering. In this paper we will consider the two-dimensional steady laminar flow of a thin jet from a long narrow orifice into a fluid at rest. Since the jet is thin, the velocity in the direction of the jet varies more rapidly across the jet than along the jet. Prandtl's boundary layer theory therefore applies. There is no bounding wall. It is a free boundary layer and it is an example of a flow without an outer flow. Since there is no solid boundary present and the pressure gradient in the direction of the jet vanishes, the total flux of momentum in the downstream direction is constant and independent of the distance from the orifice. This conserved quantity plays an important part in the solution of the problem.

Schlichting [7] was the first to apply laminar boundary layer theory to the steady flow produced by a free two-dimensional jet emerging into a fluid at rest. He solved the resulting ordinary differential equation numerically. Later, Bickley [1] solved the differential equation analytically. The application of the boundary layer approximation to laminar jets is discussed fully in standard texts on boundary layer theory such as by Schlichting [8], Schlichting and Gersten [9] and Rosenhead [6]. The standard procedure is to obtain a similarity solution by assuming a certain form for the stream function.

We will derive a group invariant solution for the two-dimensional steady laminar jet by considering a linear combination of the Lie point symmetries of the partial differential equation for the stream function. It is not necessary to assume a specific form for the stream function. This method was introduced by Momoniat *et. al.* [5] who considered the axisymmetric spreading of a thin liquid drop. It is found that the similarity solution of Schlichting [7] and Bickley [1] is the group invariant solution.

There is a close connection between conservation laws for a differential equation and the Lie point symmetries of the differential equation. By using a result of Kara and Mahomed [3, 4] connecting conservation laws and Lie point symmetries, we will establish a connection between a simple conservation law for the partial differential equation for the stream function and the condition that the total flux of momentum in the direction of the jet is independent of the distance from the orifice.

2 Mathematical formulation

Consider a steady two-dimensional thin jet which emerges from a long narrow orifice in a wall into a fluid which is at rest. The surrounding fluid consists of the same fluid as the jet itself and is viscous and incompressible. Choose the x-axis along the jet with $x = 0$ at the wall and the y-axis perpendicular to the jet with $y = 0$ at the orifice. Since the jet is thin, $v_x(x, y)$ varies much more rapidly with y than with x. The boundary layer approximation therefore applies to the two-dimensional flow produced by the jet. Since the fluid velocity vanishes outside the jet, it follows from Euler's equation that $\partial p/\partial x$ vanishes outside the jet. Since in the boundary layer approximation, $p = p(x)$, it follows that $\mathrm{d}p/\mathrm{d}x$ vanishes in the jet. Hence, Prandtl's two-dimensional boundary layer equations in the jet are

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \nu \frac{\partial^2 v_x}{\partial y^2}, \tag{2.1}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0, \tag{2.2}$$

where ν is the kinematic viscosity of the fluid. The boundary conditions are

$$y = 0: \qquad v_y = 0, \qquad \frac{\partial v_x}{\partial y} = 0, \tag{2.3}$$

$$y = \pm\infty: \qquad v_x = 0. \tag{2.4}$$

By integrating (2.1) with respect to y from $y = -\infty$ to $y = \infty$ and using (2.3) it can be verified that J is a constant independent of x where

$$J = \rho \int_{-\infty}^{\infty} v_x^2(x, y)\, \mathrm{d}y = 2\rho \int_0^{\infty} v_x^2(x, y)\, \mathrm{d}y. \tag{2.5}$$

J is the total flux in the x-direction of the x-component of momentum and it is constant because there is no solid boundary present and $\mathrm{d}p/\mathrm{d}x$ is neglected.

A stream function $\psi(x, y)$ is introduced which is defined by

$$v_x(x, y) = \frac{\partial \psi}{\partial y}, \qquad v_y(x, y) = -\frac{\partial \psi}{\partial x}. \tag{2.6}$$

Equation (2.2) is identically satisfied. Equation (2.1) becomes

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} = \nu\frac{\partial^3\psi}{\partial y^3}. \tag{2.7}$$

The problem can be stated as follows. Solve the partial differential equation (2.7) for $\psi(x, y)$ subject to the boundary conditions

$$y = 0: \qquad \frac{\partial\psi}{\partial x} = 0, \qquad \frac{\partial^2\psi}{\partial y^2} = 0, \tag{2.8}$$

$$y = \pm\infty: \qquad \frac{\partial\psi}{\partial y} = 0. \tag{2.9}$$

and subject to the condition that J is a given constant independent of x where

$$J = 2\rho\int_0^\infty \left(\frac{\partial\psi}{\partial y}(x, y)\right)^2 \mathrm{d}y. \tag{2.10}$$

The approach followed by Schlichting [7, 8] was to assume a similarity solution for the stream function of the form

$$\psi(x, y) = x^p f\left(\frac{y}{x^q}\right). \tag{2.11}$$

The exponents p and q were determined by imposing the condition that (2.7) reduce to an ordinary differential equation for f and that J is independent of x. We will look for a group invariant solution and it will not be necessary to assume a specific form for the stream function $\psi(x, y)$.

3 Lie point symmetry generators

Equation (2.7) can be written as

$$F(\psi_x, \psi_y, \psi_{xy}, \psi_{yy}, \psi_{yyy}) = 0, \tag{3.1}$$

where

$$F = \psi_y\psi_{xy} - \psi_x\psi_{yy} - \nu\psi_{yyy} \tag{3.2}$$

and a subscript denotes partial differentiation. The Lie point symmetry generators

$$X = \xi^1(x, y, \psi)\frac{\partial}{\partial x} + \xi^2(x, y, \psi)\frac{\partial}{\partial y} + \eta(x, y, \psi)\frac{\partial}{\partial \psi} \tag{3.3}$$

of the partial differential equation (3.1) are obtained by solving the determining equation [2]

$$X^{[3]}F\Big|_{F=0} = 0, \tag{3.4}$$

where

$$X^{[3]} = X + \zeta_1 \frac{\partial}{\partial \psi_x} + \zeta_2 \frac{\partial}{\partial \psi_y} + \zeta_{11} \frac{\partial}{\partial \psi_{xx}} + \zeta_{12} \frac{\partial}{\partial \psi_{xy}} + \zeta_{22} \frac{\partial}{\partial \psi_{yy}}$$
$$+ \zeta_{111} \frac{\partial}{\partial \psi_{xxx}} + \zeta_{112} \frac{\partial}{\partial \psi_{xxy}} + \zeta_{122} \frac{\partial}{\partial \psi_{xyy}} + \zeta_{222} \frac{\partial}{\partial \psi_{yyy}} \tag{3.5}$$

and

$$\zeta_i = D_i(\eta) - \psi_s D_i(\xi^s), \tag{3.6}$$
$$\zeta_{ij} = D_j(\eta_i) - \psi_{is} D_j(\xi^s), \tag{3.7}$$
$$\zeta_{ijk} = D_k(\eta_{ij}) - \psi_{ijs} D_k(\xi^s), \tag{3.8}$$

with summation over repeated indices. In (3.6) to (3.8), D_1 and D_2 are the operators of total differentiation with respect to x and y respectively:

$$D_1 = \frac{\partial}{\partial x} + \psi_x \frac{\partial}{\partial \psi} + \psi_{xx} \frac{\partial}{\partial \psi_x} + \psi_{yx} \frac{\partial}{\partial \psi_y} + \cdots , \tag{3.9}$$
$$D_2 = \frac{\partial}{\partial y} + \psi_y \frac{\partial}{\partial \psi} + \psi_{xy} \frac{\partial}{\partial \psi_x} + \psi_{yy} \frac{\partial}{\partial \psi_y} + \cdots . \tag{3.10}$$

Since F depends only on ψ_x, ψ_y, ψ_{xy}, ψ_{yy} and ψ_{yyy}, the coefficients ζ_{11}, ζ_{111}, ζ_{112} and ζ_{122} do not need to be calculated. The coefficient ζ_{222} depends on ψ_{yyy} which is eliminated from (3.4) using the partial differential equation (3.1). Equation (3.4) is separated according to the derivatives of ψ. It is found that

$$X = c_1 X_1 + c_2 X_2 + c_3 X_3 + c_4 X_4 + X_g, \tag{3.11}$$

where c_1, c_2, c_3 and c_4 are constants and

$$X_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad X_2 = x\frac{\partial}{\partial x} + \psi\frac{\partial}{\partial \psi},$$
$$X_3 = \frac{\partial}{\partial x}, \qquad X_4 = \frac{\partial}{\partial \psi}, \qquad X_g = g(x)\frac{\partial}{\partial y}, \tag{3.12}$$

where $g(x)$ is an arbitrary function. The Lie point symmetry generators of the partial differential equation (2.7) are given by (3.12).

4 Group invariant solution

In order to derive a group invariant solution of (2.7) we consider the linear combination (3.11) of the Lie point symmetry generators. Since X is determined up to an arbitrary multiplicative constant we divide (3.11) by c_1 which we can expect to be non-zero since X_1 is a non-trivial symmetry generator. We incorporate c_1 into the remaining constants and into $g(x)$ which is equivalent to taking $c_1 = 1$.

Now, $\psi = \Phi(x, y)$ is a group invariant solution of (2.7) provided

$$\left[\big((1 + c_2)x + c_3\big)\frac{\partial}{\partial x} + \big(y + g(x)\big)\frac{\partial}{\partial \psi} + (c_2\psi + c_4)\frac{\partial}{\partial \psi}\right]\big(\psi - \Phi(x, y)\big)\Big|_{\psi=\Phi} = 0, \tag{4.1}$$

which may be rewritten as

$$\big((1+c_2)x+c_3\big)\frac{\partial\Phi}{\partial x}+\big(y+g(x)\big)\frac{\partial\Phi}{\partial y}=c_2\Phi+c_4. \tag{4.2}$$

Equation (4.2) is a quasi-linear first order partial differential equation for $\Phi(x,y)$. Two independent solutions of the differential equations of the characteristic curves are

$$\frac{y}{\left(x+\frac{c_3}{1+c_2}\right)^{\frac{1}{1+c_2}}}-G(x)=a_1, \tag{4.3}$$

$$\frac{\Phi+\frac{c_4}{c_2}}{\left(x+\frac{c_3}{1+c_2}\right)^{\frac{c_2}{1+c_2}}}=a_2, \tag{4.4}$$

where

$$G(x)=(1+c_2)^{1/(1+c_2)}\int^x\frac{g(x)\,\mathrm{d}x}{\big[(1+c_2)x+c_3\big]^{\frac{2+c_2}{1+c_2}}} \tag{4.5}$$

and a_1 and a_2 are constants. Hence, since $\psi=\Phi(x,y)$, the group invariant solution of (2.8) is of the form

$$\psi(x,y)=\left(x+\frac{c_3}{1+c_2}\right)^{\frac{c_2}{1+c_2}}f(\xi)-\frac{c_4}{c_2}, \tag{4.6}$$

where $f(\xi)$ is an arbitrary function of ξ and

$$\xi=\frac{y}{\left(x+\frac{c_3}{1+c_2}\right)^{\frac{1}{1+c_2}}}-G(x). \tag{4.7}$$

We now substitute (4.6) into (2.7). This yields an ordinary differential equation for $f(\xi)$:

$$\nu\frac{\mathrm{d}^3f}{\mathrm{d}\xi^3}+\frac{c_2}{(1+c_2)}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(f(\xi)\frac{\mathrm{d}f}{\mathrm{d}\xi}\right)+2\left(\frac{\frac{1}{2}-c_2}{1+c_2}\right)\left(\frac{\mathrm{d}f}{\mathrm{d}\xi}\right)^2=0. \tag{4.8}$$

Consider next the condition that J, defined by (2.10), is constant independent of x. By making the change of variable from y to ξ at any given position x, (2.10) becomes

$$J=2\rho\left(x+\frac{c_2}{1+c_2}\right)^{\frac{2c_2-1}{c_2+1}}\int_0^\infty\left(\frac{\mathrm{d}f}{\mathrm{d}\xi}\right)^2\mathrm{d}\xi. \tag{4.9}$$

Thus J is independent of x provided

$$c_2=\frac{1}{2}. \tag{4.10}$$

When $c_2=\frac{1}{2}$, (4.9) becomes

$$J=2\rho\int_0^\infty\left(\frac{\mathrm{d}f}{\mathrm{d}\xi}\right)^2\mathrm{d}\xi \tag{4.11}$$

and (4.6) and (4.7) reduce to

$$\psi(x,y) = \left(x + \frac{2}{3}c_3\right)^{1/3} f(\xi) - 2c_4, \tag{4.12}$$

$$\xi = \frac{y}{\left(x + \frac{2}{3}c_2\right)^{2/3}} - G(x). \tag{4.13}$$

The differential equation (4.8) becomes

$$\frac{\mathrm{d}^3 f}{\mathrm{d}\xi^3} + \frac{1}{3\nu}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(f(\xi)\frac{\mathrm{d}f}{\mathrm{d}\xi}\right) = 0. \tag{4.14}$$

We see that the result $c_2 = 1/2$ puts the differential equation (4.8) in a form that can be integrated analytically.

Finally, consider the boundary conditions on $f(\xi)$. The function $g(x)$ in (4.5) is arbitrary. To make $\xi = 0$ correspond to $y = 0$ we choose $g(x) = 0$ and therefore $G(x) = 0$. Since

$$\frac{\partial\psi}{\partial x} = \frac{1}{3\left(x + \frac{2}{3}c_3\right)^{2/3}}\left(f(\xi) - 2\xi\frac{\mathrm{d}f}{\mathrm{d}\xi}\right), \tag{4.15}$$

$$\frac{\partial\psi}{\partial y} = \frac{1}{\left(x + \frac{2}{3}c_3\right)^{1/3}}\frac{\mathrm{d}f}{\mathrm{d}\xi}, \qquad \frac{\partial^2\psi}{\partial y^2} = \frac{1}{\left(x + \frac{2}{3}c_3\right)}\frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2}, \tag{4.16}$$

the boundary conditions (2.8) and (2.9) on $\psi(x,y)$ yield the following boundary conditions on $f(\xi)$:

$$f(0) = 0, \qquad f''(0) = 0, \qquad f'(\pm\infty) = 0. \tag{4.17}$$

For completeness we outline briefly the solution of (4.14) subject to the boundary conditions (4.17). Integration of (4.14) once with respect to ξ gives

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2} + \frac{1}{3\nu}f(\xi)\frac{\mathrm{d}f}{\mathrm{d}\xi} = k, \tag{4.18}$$

where k is a constant. The boundary conditions (4.17) give $k = 0$ provided

$$f(0)\frac{\mathrm{d}f(0)}{\mathrm{d}\xi} = 0. \tag{4.19}$$

We will take $k = 0$ and check that the solution obtained satisfies (4.19). Equation (4.18) becomes

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2} + \frac{1}{6\nu}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(f^2(\xi)\right) = 0 \tag{4.20}$$

and integration with respect to ξ gives

$$\frac{\mathrm{d}f}{\mathrm{d}\xi} + \frac{1}{6\nu}f^2(\xi) = \frac{\alpha^2}{6\nu}, \tag{4.21}$$

where α is a constant. Since v_x is positive, it follows from (2.6) and (4.16) that $\mathrm{d}f/\mathrm{d}\xi$ is positive. Hence the left hand side of (4.21) is positive and we therefore wrote α^2 instead of α on the right hand side. Equation (4.21) is a variables separable differential equation. Its solution subject to the boundary condition $f(0) = 0$ is

$$f(\xi) = \alpha \tanh\left(\frac{\alpha}{6\nu}\xi\right). \tag{4.22}$$

It is readily verified that (4.19) is satisfied by (4.22). The constant α is obtained in terms of the given flux of momentum, J, by substituting (4.22) into (4.11):

$$\alpha = \left(\frac{9\nu J}{2\rho}\right)^{1/3}. \tag{4.23}$$

The boundary condition $f'(\pm\infty) = 0$ was not used. It is satisfied by the solution (4.22).

It remains to determine the constants c_3 and c_4. The long narrow orifice in the wall is assumed to be infinitely small. In order to have a finite volume of flow and a finite momentum it is necessary to assume an infinite fluid velocity at the orifice [8]. Now

$$v_x(x,0) = \frac{\partial\psi}{\partial y}(x,0) = \frac{\alpha^2}{6\nu\left(x + \frac{2}{3}c_3\right)^{1/3}}. \tag{4.24}$$

We therefore take $c_3 = 0$ to ensure that $v_x(x,0) = \infty$ at $x = 0$. Also, a stream function $\psi(x,y)$ is determined up to an arbitrary additive constant. Hence from (4.12) we choose $c_4 = 0$.

From (4.12), (4.13) and (4.22) we obtain the group invariant solution

$$\psi(x,y) = \alpha x^{1/3} \tanh\left(\frac{\alpha}{6\nu}\xi\right), \tag{4.25}$$

where α is given by (4.23) and

$$\xi = \frac{y}{x^{2/3}}. \tag{4.26}$$

This result agrees with the solution obtained by the combined work of Schlichting [7] and Bickley [1] and given in standard texts [6, 8]. The solution derived by Schlichting and Bickley is therefore the group invariant solution.

Since $c_1 = 1$, $c_2 = 1/2$, $c_3 = 0$, $c_4 = 0$ and $g(x) = 0$, the Lie point symmetry which generates the group invariant solution is

$$X = \frac{3}{2}x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{1}{2}\psi\frac{\partial}{\partial \psi}. \tag{4.27}$$

5 Conservation law

We will establish a connection between a conservation law for the differential equation (2.7), the Lie point symmetry (4.27) which generated the group invariant solution of the differential equation and conservation of the total flux of momentum in the direction of the jet.

The equation

$$D_1T^1 + D_2T^2 = 0 \tag{5.1}$$

is a conservation law for the differential equation (2.7) if (5.1) is satisfied for all solutions $\psi(x, y)$ of (2.7) [2]. In (5.1) D_1 and D_2 are the operators of total differentiation defined by (3.9) and (3.10). The quantities $T^i(x, y, \psi, \psi_x, \psi_y, \ldots)$, where $i = 1$ and 2, are the components of the conserved vector $T = \left(T^1, T^2\right)$.

It is readily verified that

$$D_1\left(\psi_y^2\right) + D_2(-\psi_x\psi_y - \nu\psi_{yy}) = 0 \tag{5.2}$$

for all solutions $\psi(x, y)$ of the partial differential equation (2.7). Equation (5.2) is a conservation law for (2.7) and the components of the conserved vector are

$$T^1 = \psi_y^2, \qquad T^2 = -\psi_x\psi_y - \nu\psi_{yy}. \tag{5.3}$$

Now, from a result due to Kara and Mahomed [3, 4], if

$$X = \xi^1\frac{\partial}{\partial x} + \xi^2\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial \psi} \tag{5.4}$$

is a Lie point symmetry of (2.7) and T^1 and T^2 are the components of the conserved vector of (2.7) associated with X then

$$X^{[l]}T^i + T^iD_k\xi^k - T^kD_k\xi^i = 0, \qquad i = 1, 2, \tag{5.5}$$

where $X^{[l]}$ is the l^{th} prolongation of X and the repeated index is summed form $k = 1$ to $k = 2$. We will use this result to determine the Lie point symmetry associated with the conserved vector (5.3).

For the conserved vector (5.3), $l = 2$. When expanded, (5.5) consists of the two equations

$$i = 1: \qquad X^{[2]}T^1 + T^1D_2\xi^2 - T^2D_2\xi^1 = 0, \tag{5.6}$$

$$i = 2: \qquad X^{[2]}T^2 + T^2D_1\xi^1 - T^1D_1\xi^2 = 0. \tag{5.7}$$

Consider the Lie point symmetry (3.11) with $c_1 = 1$. Using (3.6) and (3.7) it follows that

$$\begin{aligned} X^{[2]} = {} & \left[(1 + c_2)x + c_3\right]\frac{\partial}{\partial x} + \left(y + g(x)\right)\frac{\partial}{\partial y} + (c_2\psi + c_4)\frac{\partial}{\partial \psi} + \left(-\psi_x - \frac{\mathrm{d}g}{\mathrm{d}x}\psi_y\right)\frac{\partial}{\partial \psi_x} \\ & + (c_2 - 1)\psi_y\frac{\partial}{\partial \psi_y} + \zeta_{11}\frac{\partial}{\partial \psi_{xx}} + \zeta_{12}\frac{\partial}{\partial \psi_{xy}} + (c_2 - 2)\psi_{yy}\frac{\partial}{\partial \psi_{yy}}. \end{aligned} \tag{5.8}$$

By using (5.3) for T^1 and T^2, equations (5.6) and (5.7) become

$$(2c_2 - 1)T^1 = 0, \qquad (2c_2 - 1)T^2 = 0. \tag{5.9}$$

Thus (5.6) and (5.7) are satisfied provided $c_2 = 1/2$.

Hence X given by (3.11) is the Lie point symmetry generator associated with the conserved vector (5.3) provided $c_2 = 1/2$. This is the same condition on c_2 as obtained from (4.9) by insisting that J is independent of x. Hence c_2 as determined by (5.5) has the same value as in the Lie point symmetry (4.27) which generates the group invariant solution. This establishes a connection between the conserved vector T and the total momentum flux J through the Lie point symmetry generator. No condition is placed on c_3, c_4 or $g(x)$ by (5.5). The constants c_3 and c_4 and the arbitrary function $g(x)$ are determined by the choice of origin of coordinates and the choice of the arbitrary additive constant in the stream function.

The connection between the conserved vector T and the total momentum flux J is that they are related through the identity

$$T^1 = \frac{1}{2\rho}\frac{\partial J}{\partial y}. \tag{5.10}$$

6 Conclusions

The solution for the stream function of a free two-dimensional jet derived by the combined work of Schlichting [7] and Bickley [1] is the group invariant solution. Only two of the five Lie point symmetries of (2.7) contribute to the generation of the group invariant solution.

The group invariant solution, given by (4.25) and (4.26), has the form (2.11) assumed by Schlichting [7, 8]. To obtain the group invariant solution, the form (2.11) was not assumed, but was derived. The Lie group analysis also shows that there are no other forms for a similarity solution of (2.7) besides (2.11).

The method used a conserved quantity to determine the constants in the linear combination of Lie point symmetries. In the problem considered here the conserved quantity was the total momentum flux in the direction of the jet. In the original problem considered by Momoniat *et. al.* [5] the conserved quantity was the total volume of the liquid drop. The method will be applicable to other problems for which conserved quantities can be derived.

We established a connection between a simple conservation law for the differential equation for the stream function and conservation of the total momentum flux. The Lie point symmetry associated with the conservation law is the same as that which generates the group invariant solution which was derived by imposing the condition that the total momentum flux is conserved.

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