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Symmetries and Integrating Factors

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Abstract

Cheb-Terrab and Roche (*J. Sym. Comp.* **27** (1999), 501–519) presented what they termed a systematic algorithm for the construction of integrating factors for second order ordinary differential equations. They showed that there were instances of ordinary differential equations without Lie point symmetries which were solvable with this algorithm. We demonstrate that the existence of integrating factors is paralleled by the existence of suitable Lie symmetries which enable one to reduce the equations to quadratures thereby emphasising the fact that integrability relies upon symmetry.

1 Introduction

In their recent paper Cheb-Terrab and Roche [7] presented what they termed a systematic algorithm for the construction of integrating factors of the form $\mu(x, y)$, $\mu(x, y')$ and $\mu(y, y')$ for second order ordinary differential equations. The algorithm determines the existence and explicit form of the integrating factor without the necessity to solve any differential equations with the exception of a linear equation in a subcase of the first type of integrating factor. The algorithm was implemented in Maple and was applied to many examples taken from the book of Kamke [16]. The MAPLE routine was demonstrated not only to be very successful but to be superior to other methods for solving ordinary differential equations using symbolic manipulation packages. In this paper we consider the relationship between the actual existence of an integrating factor and the underlying symmetries which are the reason for its existence.

The history of integrating factors goes back over three centuries to the very first days of the integral and differential calculus. According to Ince [14, p. 531] the first known instance of the use of an integrating factor was reported in a letter from Fatio de Duiller to Christian Huygens in June, 1687, for the solution of the first order equation $3x\mathrm{d}y - 2y\mathrm{d}x = 0$. Ince further relates the extension of this elementary result to the general linear equation of arbitrary order by Lagrange and Laplace in the second half of the eighteenth century [pp. 536–537]. Lie [19, Kap. 6, p. 95 ff] discussed the relationship between the existence of a Lie point symmetry and the determination of an integrating factor for a first order ordinary differential equation.

As is well known, it is, in principle, always possible to determine whether a given ordinary differential equation is exact and, if it is not exact, to find the integrating factor. However, one must emphasise the 'in principle' for the determination of the integrating factor requires the solution of a linear partial differential equation of order n in $n+1$ variables in the case of an ordinary differential equation of order n. The solution of this partial differential equation is a nontrivial matter just as is the parallel problem of solving the determining equation for the Lie point symmetries of the differential equation. In general one has to make some sort of an *Ansatz* to restrict the generality inherent in the solution of the equation so that one can make some progress to the solution. Again this parallels precisely the situation in the determination of the Lie symmetries.

Most commonly one looks for the determination of point symmetries and there exists a number of symbolic manipulation codes, for example Program Lie [13, 31] and the well-known interactive code of Nucci [25, 27], which are quite efficient for the determination of these symmetries and also generalised symmetries in which the symmetry depends upon the derivatives as well. Lie's original work dealt with point [20] and contact [21] symmetries. (Some subsequent classics are the very readable text of Bianchi [5], the introductory exposition of Dickson [8] and that of Eisenhart [9].) Generalised symmetries came more into use with Noether's Theorem [24]. In the last decade of the twentieth century there has been attention paid to nonlocal symmetries in which the coefficient functions can contain integrals involving the dependent variable and its derivatives. Nonlocal symmetries have been used to explain the occurrence of the so-called 'hidden symmetries' [1, 2] and the integrability of equations devoid of any Lie point symmetries [3]. They have also been shown to be of use in the reduction of order of differential equations [10] and to be part of the group, known as the complete symmetry group, which precisely determines the structure of a given equation whether it be integrable [17] or not [18]. One of the problems associated with nonlocal symmetries is that of their determination. The paper by Krause [17] was supplemented by those of Nucci [26], Govinder *et al* [11] and of Pillay *et al* [29].

As Cheb-Terrab and Roche [7] make — quite justifiably — the point that their algorithm works in the case of second order ordinary differential equations which do not have the requisite number of Lie point symmetries, any explanation of the integrability of these equations in terms of symmetry will have to be in terms of nonlocal or generalised symmetries. To investigate these symmetries we employ the method used in Pillay *et al* [29]. We illustrate the idea with the trivial equation

$$y'' = 0. \tag{1.1}$$

Equation (1.1) possesses the Lie point symmetry,

$$G = \xi\partial_x + \eta\partial_y, \tag{1.2}$$

where we make no assumptions as to the nature of the coefficient functions ξ and η, provided

$$\eta'' - y'\xi'' = 0 \tag{1.3}$$

in which we have made use of the differential equation, (1.1), to eliminate the coefficient of ξ'. As we need only two symmetries to reduce (1.1) to quadratures, we can in (1.3) put ξ'' equal to zero and obtain, quite trivially in this case,

$$\eta = A + Bx \tag{1.4}$$

so that we have the two symmetries

$$G_1 = \partial_y \qquad \text{and} \qquad G_2 = x\partial_y \tag{1.5}$$

with the Lie Bracket $[G_1, G_2] = 0$. We recognise this as the standard representation of Lie's Type II two-dimensional algebra [19, Kap. 18, p. 412 ff]. Any second order equation with this algebra is integrable as a quadrature.

It is well known that (1.1) has eight Lie point symmetries. We have given only two by our assumption. In the solution of (1.3) with $\xi'' = 0$ we have omitted the solutions coming from this assumption which give us

$$G_3 = \partial_x \qquad \text{and} \qquad G_4 = x\partial_x. \tag{1.6}$$

In addition we have the two symmetries

$$G_5 = y\partial_x \qquad \text{and} \qquad G_6 = y\partial_y \tag{1.7}$$

which follow from the original differential equation, (1.1), by the identification of ξ and η with y respectively. The two remaining point symmetries are found by not separating (1.3) into two parts, but by considering the integral consequences of the original differential equation [29]. The two remaining Lie point symmetries are

$$G_7 = x^2\partial_x + xy\partial_y, \tag{1.8}$$
$$G_8 = xy\partial_x + y^2\partial_y. \tag{1.9}$$

The integral consequences of (1.3) are

$$\eta' = y'\xi' + A_1 \qquad \text{and} \qquad \eta = y'\xi + A_1x + A_0, \tag{1.10}$$

where the original equation has been taken into account and A_0 and A_1 are arbitrary constants. We consider G_8; G_7 is in the same way. If ξ and η are given by

$$\xi = xy \qquad \text{and} \qquad \eta = y^2, \tag{1.11}$$

(1.3) is automatically satisfied. Substitution of ξ and η into (1.10) results in

$$2yy' = y'(xy' + y) + A_1 \qquad \text{and} \qquad y^2 = y'xy + A_1x + A_0 \tag{1.12}$$

respectively. After some simplification equations (1.12) become

$$A_1 = y'(y - xy') \qquad \text{and} \qquad y = \frac{A_1x + A_0}{y - xy'} \tag{1.13}$$

Table 1. Canonical forms of Lie algebras of dimension two and their scalar second order ordinary differential equations. In each instance the function f is arbitrary.

Type	$[G_1, G_2]$	Canonical forms of G_1 and G_2	Form of equation
I	0	$G_1 = \partial_x$ $G_2 = \partial_y$	$y'' = f(y')$
II	0	$G_1 = \partial_y$ $G_2 = x\partial_y$	$y'' = f(x)$
III	G_1	$G_1 = \partial_y$ $G_2 = x\partial_x + y\partial_y$	$xy'' = f(y')$
IV	G_1	$G_1 = \partial_y$ $G_2 = y\partial_y$	$y'' = y'f(x)$

which is true since the first integrals of (1.1) are

$$I_1 = y' \qquad \text{and} \qquad I_2 = y - xy'. \tag{1.14}$$

In general the coefficient functions are related according to

$$\eta = \int \left(\int y'\xi'' \mathrm{d}x \right) \mathrm{d}x \tag{1.15}$$

and we may put any function we like for ξ and obtain a symmetry[1].

In the previous paragraph we showed how to obtain all of the point symmetries for (1.1) from the equation (1.3) simply for the sake of completeness. Since we are treating second order equations, a knowledge of two symmetries is sufficient for the reduction to quadratures. There are four two-dimensional Lie algebras [20, Kap. 18, p. 412 ff]. Their canonical forms are given in Table 1 together with the normal form of the second order equation invariant under their action. In seeking two symmetries for the general equation

$$y'' = f\left(x, y, y'\right) \tag{1.16}$$

we need to solve the equivalent of (1.3), *videlicet*

$$\eta'' - 2f\xi' - y'\xi'' = \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \left(\eta' - y'\xi'\right)\frac{\partial f}{\partial y'}. \tag{1.17}$$

[1] For a general equation and arbitrary ξ the symmetry will usually be nonlocal. However, in this case the integration in (1.15) can be carried to completion by the use of integration by parts and the original differential equation. Thus we have from (1.15)

$$\eta = \int \left[y'\xi' - \int y''\xi' \mathrm{d}x \right] \mathrm{d}x = y'\xi - \int y''\xi \mathrm{d}x = y'\xi$$

which means that the symmetry will be a generalised symmetry if ξ is a point function of x and y. One cannot expect always to be so fortunate.

This can be quite a daunting task in general if we do not wish to specify the nature of the dependence of η and ξ. We can make our task considerably easier by putting $\xi = 0$ so that we seek the two solutions of

$$\eta'' = \eta \frac{\partial f}{\partial y} + \eta' \frac{\partial f}{\partial y'}. \tag{1.18}$$

To be able to identify which one of the four types of algebra we obtain and consequently the normal form of the equation we need to express all of the symmetries in the form $\eta\partial_y$. In the cases of Types II and IV this is already the case. We also note that for these algebras the equations in normal form are linear in the dependent variable. In the case of Type I the two symmetries are

$$G_1 = y'\partial_y \qquad \text{and} \qquad G_2 = \partial_y \tag{1.19}$$

and in the case of Type III they are

$$G_1 = \partial_y \qquad \text{and} \qquad G_2 = \left(xy' + y\right)\partial_y. \tag{1.20}$$

In general one has the passage from $f\partial_x$ to $fy'\partial_y$ as a consequence of the application of the chain rule.

In the next section we show how to calculate the symmetries for a selection of the equations for which Cheb-Terrab and Roche were able to obtain a solution or a reduction of order in the case which they regard as most interesting, *ie* the ones which contain all of the variables[2] x, y or y'. We conclude the paper with some observations.

2 Computation of the symmetries

In presenting our results our selection follows the order of equations as presented in the Appendix of the paper of Cheb-Terrab and Roche [7]. In our calculations we make use of both local and nonlocal symmetries as the circumstances of each particular equation demand. We attempt to maintain the correct balance between over-detailed calculation and excessive conciseness so that the interested reader can see how the symmetries were obtained in each case for which they are used for the reduction of order of the given equation. The reference at the beginning of each equation refers to the number in Kamke [16].

1. 6.36 (p. 550). The equation is

$$y'' + 2yy' + f(x)y' + f'(x)y = 0 \tag{2.1}$$

and we seek a symmetry of the form

$$G = \eta\partial_y. \tag{2.2}$$

The application of the second extension, $G^{[2]}$, of (2.2) to (2.1) gives

$$\eta'' + 2\eta y' + 2\eta' y + f\eta' + f'\eta = 0. \tag{2.3}$$

[2] We realise that this can be somewhat artificial since a simple Kummer–Liouville transformation can make a very complicated equation from something quite simple. However, we are in the situation in which we are presented with the complicated equation.

Equation (2.3) is linear in η and is easily integrated once to give

$$\eta' + (2y + f)\eta = B, \tag{2.4}$$

where B is a constant of integration, which is a linear nonhomogeneous first order equation and is easily solved to give

$$\eta = \exp\left[-\int (2y+f)\mathrm{d}x\right]\left[A + B\int \exp\left[\int (2y+f)\mathrm{d}x\right]\right]. \tag{2.5}$$

The two symmetries are

$$\begin{aligned} G_1 &= \exp\left[-\int (2y+f)\mathrm{d}x\right]\partial_y, \\ G_2 &= \exp\left[-\int (2y+f)\mathrm{d}x\right]\left[\int \exp\left[\int (2y+f)\mathrm{d}x\right]\right]\partial_y. \end{aligned} \tag{2.6}$$

By inspection of the two symmetries in (2.6) it is evident that the algebra is of Lie's Type IV and that (2.1) is really a linear equation. We may determine this result in a more orderly fashion by using G_1 to produce a reduction of order. The zeroth order and first order differential invariants are obtained from the solution of the associated Lagrange's system

$$\frac{\mathrm{d}x}{0} = \frac{\mathrm{d}y}{1} = \frac{\mathrm{d}y'}{-(2y+f)} \tag{2.7}$$

and are

$$u = x \qquad \text{and} \qquad v = y' + y^2 + fy \tag{2.8}$$

so that the reduced equation is

$$\frac{\mathrm{d}v}{\mathrm{d}u} = 0 \quad \Rightarrow \quad v = C, \tag{2.9}$$

where C is a constant of integration. Hence (2.1) possesses the first integral

$$y' + y^2 + fy = C \tag{2.10}$$

which, when considered as a differential equation, is an equation of Riccati type and under the standard transformation $y = w'/w$ becomes the linear second order equation

$$w'' + fw' - Cw = 0. \tag{2.11}$$

The integrability of (2.1) has been demonstrated by its transformation to the linear equation (2.11) via reduction of order by means of one of its nonlocal symmetries. We note that under the reduction of order due to G_1 the second nonlocal symmetry G_2 becomes the rather obvious symmetry ∂_v of (2.9). The very nonlocality of the two symmetries and the Riccati transformation is unsurprising when one considers that the natural dependent variable for the two symmetries is $Y = \exp\left[\int (2y+f)\mathrm{d}x\right]$.

In the case of (2.1) we have given a fairly detailed treatment to enable the reader to assimilate our methodology. In subsequent examples our treatment is more succinct.

2. 6.51 (p. 554). We have

$$y'' + f(y)y'^2 + g(x)y' = 0 \tag{2.12}$$

and the two symmetries

$$\begin{aligned} G_1 &= \exp\left[-\int f(y)y'\mathrm{d}x\right]\partial_y \qquad \text{and} \\ G_2 &= \exp\left[-\int f(y)y'\mathrm{d}x\right]\int y'\exp\left[\int f(y)y'\mathrm{d}x\right]\mathrm{d}x\partial_y \end{aligned} \tag{2.13}$$

obtained from the solution of

$$\left(\frac{\eta'}{y'}\right)' + (\eta f)' = 0. \tag{2.14}$$

We note that both G_1 and G_2 are point symmetries. The invariants of G_1 are $u = x$ and $v = y'\exp\left[\int f(y)\mathrm{d}y\right]$ and the reduced equation is

$$\frac{\mathrm{d}v}{\mathrm{d}u} = -g(u)v \tag{2.15}$$

which is linear. In fact we note that under the transformation $w = \exp\left[\int f(y)y'\mathrm{d}x\right]$ the original equation, (2.12), becomes the very linear equation

$$w'' + g(x)w' = 0 \tag{2.16}$$

which theoretically is quite trivial. The linearity of (2.12) becomes obvious when we realise that the two symmetries in (2.13) constitute a representation of Lie's Type IV algebra.

3. 6.169 (p. 582). The equation

$$xyy'' + xy'^2 - yy' = 0 \tag{2.17}$$

is of Euler type, has the two obvious Lie point symmetries

$$G_1 = x\partial_x \qquad \text{and} \qquad G_2 = y\partial_y \tag{2.18}$$

and, since it can be written in the form

$$x\left(y^2\right)'' - \left(y^2\right)' = 0, \tag{2.19}$$

is a linear equation and so trivial in the context of our discussion. The equation for η is

$$x(\eta y)'' - (\eta y)' = 0 \tag{2.20}$$

and, in addition to G_2, gives the two symmetries

$$G_3 = \frac{1}{y}\partial_y \qquad \text{and} \qquad G_4 = \frac{x^2}{y}\partial_y. \tag{2.21}$$

(The very existence of four point symmetries reveals the inherent linearity of the original second order equation.) The invariants of G_3 are $u = x$ and $v = yy'$ which gives the reduced equation

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{v}{u}. \tag{2.22}$$

The first integral following from (2.22) is $I_1 = yy'/x$ and this is readily integrated to give the solution of the equation. Since the Lie Bracket of G_3 and G_4 is zero, G_4 can equally be used for the reduction of order. The corresponding integral is $I_2 = xyy' - y^2$. The solution of (2.17) may be obtained by the elimination of y' from the two first integrals and is

$$y^2 = x^2 I_1 - I_2 \tag{2.23}$$

which represents an ellipse or an hyperbola or the degeneracies thereof depending upon the values of the integrals.

4. 6.203 (p. 589). The equation

$$ay(y-1)y'' - (a-1)(2y-1)y'^2 + f(x)y(y-1)y' = 0 \tag{2.24}$$

has a more suggestive appearance when written in the form

$$a\frac{y''}{y'} - (a-1)\frac{(2y-1)y'}{y^2 - y} + f(x) = 0. \tag{2.25}$$

The second order equation for the coefficient function η, *videlicet*

$$a\left(\frac{\eta'}{y'}\right)' - (a-1)\left\{\eta\frac{2y-1}{y^2-y}\right\}' = 0, \tag{2.26}$$

is easily solved to give the two Lie point symmetries

$$\begin{aligned} G_1 &= \left(y^2 - y\right)^{\frac{a-1}{a}} \partial_y \qquad \text{and} \\ G_2 &= \left[\left(y^2 - y\right)^{\frac{a-1}{a}} \int \frac{\mathrm{d}y}{\left(y^2 - y\right)^{\frac{a-1}{a}}}\right] \partial_y \end{aligned} \tag{2.27}$$

so that the Lie algebra of the two symmetries is obviously of Type IV and under the transformation

$$X = x, \qquad Y = \int \frac{\mathrm{d}y}{\left(y^2 - y\right)^{\frac{a-1}{a}}} \tag{2.28}$$

one obtains the normal form

$$\frac{\mathrm{d}^2 Y}{\mathrm{d}X^2} + \frac{f(X)}{a}\frac{\mathrm{d}Y}{\mathrm{d}X} = 0. \tag{2.29}$$

Equation (2.29) can be integrated by quadratures.

5. 6.206 (p. 590). The equation

$$\left(x^2 - a^2\right)\left(y^2 - a^2\right)y'' - \left(x^2 - a^2\right)yy'^2 + x\left(y^2 - a^2\right)y' = 0 \tag{2.30}$$

admits the obvious first integral

$$I_1 = y'\left(\frac{x^2 - a^2}{y^2 - a^2}\right)^{1/2} \tag{2.31}$$

from which the solution of (2.30)

$$y = a\cosh\left[I_1 \mathrm{arch}\left(\frac{x}{a}\right) + I_2\right] \tag{2.32}$$

follows by a simple quadrature. The equation for η is easily integrated to give the first order linear equation

$$\eta' - \frac{yy'}{y^2 - a^2}\eta = Ky' \tag{2.33}$$

for η from which we obtain the two Lie point symmetries

$$\begin{aligned} G_1 &= \left[\left(y^2 - a^2\right)^{1/2}\right]\partial_y \qquad \text{and} \\ G_2 &= \left[\left(y^2 - a^2\right)^{1/2} \mathrm{arch}\left(\frac{y}{a}\right)\right]\partial_y. \end{aligned} \tag{2.34}$$

The two symmetries are a representation of Lie's Type IV algebra. The transformation to the normal form

$$\frac{\mathrm{d}^2 Y}{\mathrm{d}X^2} - \frac{X}{X^2 - a^2}\frac{\mathrm{d}Y}{\mathrm{d}X} = 0 \tag{2.35}$$

is given by

$$X = x, \qquad Y = \mathrm{arch}\left(\frac{y}{a}\right). \tag{2.36}$$

6. 6.66 (p. 557). The equation is

$$y'' = 2a(y + bx + c)\left(y'^2 + 1\right)^{3/2} \tag{2.37}$$

and is presented in [7] as an instance of an equation which contains both x and y. This is misleading as with a new dependent variable obtained by replacing $y + bx + c$ by y we obtain the autonomous equation

$$y''\left[(y' - b)^2 + 1\right]^{-3/2} = 2ay \tag{2.38}$$

which is reduced to the quadrature

$$\int \frac{v\mathrm{d}v}{\left[(v - b)^2 + 1\right]^{3/2}} = \int 2au\mathrm{d}u \tag{2.39}$$

by the use of the obvious point symmetry ∂_x and the variables $u = y$ and $v = y'$. The quadrature in (2.39) is not trivial, but after some effort one obtains the first integral

$$I = ay^2 - \frac{b(y'-b)-1}{\left[(y'-b)^2+1\right]^{1/2}}. \tag{2.40}$$

From (2.40) we can obtain the solution in terms of the quadrature

$$x - x_0 = \int \frac{\left[\left(I-ay^2\right)^2 - b^2\right] \mathrm{d}y}{b\left\{(I-ay^2)^2 - b^2 - 1\right\} \pm \sqrt{\left\{-\left(I-ay^2\right)\left[(I-ay^2)-b^2-1\right]\right\}}} \tag{2.41}$$

which is not a very attractive integral and one would not expect to be able to invert it to obtain y as a function of x, but the solution of the original differential equation has been reduced to the quadrature.

We note that there is a nonlocal symmetry

$$G_2 = \exp\left[2a\int \frac{y}{y'}\left[\left(y'-b\right)^2+1\right]\mathrm{d}x\right]\partial_y \tag{2.42}$$

which also gives v as the characteristic.

7. 6.108 (p. 570). The equation is

$$yy'' + y^2 = ax + b. \tag{2.43}$$

According to Cheb-Terrab and Roche [7] this equation has the integrating factor y. According to Kamke [16] the equation has its origins in the works of Braude [6] and Muller [22, 23] on the motions of electrons in electric and magnetic fields. Muller [22] states that to his knowledge the equation has not been integrated in terms of elementary functions. We have not been able to determine any symmetries for it when $a \neq 0$. When $a = 0$, there is the obvious symmetry ∂_x which permits reduction to a first order equation. The first integral is

$$I_1 = y'^2 + y^2 - 2b\log y \tag{2.44}$$

with an obvious reduction to quadratures. If, in the case $a = 0$, one persues the standard method of this paper, the two symmetries are

$$G_1 = y'\partial_y \qquad \text{and} \qquad G_2 = \left[y'\int\frac{\mathrm{d}x}{yy'^2}\right]\partial_y. \tag{2.45}$$

Reduction by G_1 leads to the characteristics $u = x$ and $v = I_1$ of (2.44) so that the reduced first order equation is simply $\mathrm{d}v/\mathrm{d}u = 0$. The second symmetry remains as nonlocal being $(1/y)\partial_v$ which is an obvious symmetry of the first order equation.

8. 6.133 (p. 575). Under the transformation $w = y + x$

$$(y+x)y'' + y'^2 - y' = 0 \tag{2.46}$$

becomes

$$ww'' + w'^2 - 3w' = 0 \tag{2.47}$$

which, apart from being trivially integrable once, possesses the two obvious Lie point symmetries $G_1 = \partial_x$ and $G_2 = x\partial_x + w\partial_w$. Reduction using G_1 leads to the obvious integral

$$I = ww' - 3w \tag{2.48}$$

which can be easily integrated. Two nonlocal symmetries of (2.47) are

$$\begin{aligned} G_3 &= \left\{\frac{1}{w}\exp\left[\int\frac{3}{w}\mathrm{d}x\right]\right\}\partial_w \qquad \text{and} \\ G_4 &= \left\{\frac{1}{w}\exp\left[\int\frac{3}{w}\mathrm{d}x\right]\int\exp\left[-\int\frac{3}{w}\mathrm{d}x\right]\right\}\partial_w. \end{aligned} \tag{2.49}$$

The invariants of G_3 are $u = x$ and $v = ww' - 3w$. The reduced equation is trivially integrated to give the integral in (2.48). Under this reduction G_4 becomes ∂_v.

9. 6.136 (p. 576). The two equations, 6.134 and 6.135, are subsumed in

$$(y - x)y'' + f(y') = 0 \tag{2.50}$$

and a transformation, $y = w + x$, brings us to

$$\frac{XW''}{W'^3 f\left(\frac{1}{W'} + 1\right)} - 1 = 0 \tag{2.51}$$

which has the two symmetries

$$G_1 = \partial_W \qquad \text{and} \qquad G_2 = (XW' - W)\partial_W. \tag{2.52}$$

The first integral may be written formally as

$$I = X\exp\left[\int^{1/W'}\frac{\mathrm{d}s}{f(s+1)}\right], \tag{2.53}$$

in which form it is difficult to accept the claim of Cheb-Terrab and Roche [7] that this equation is solved for a general function f.

10. 6.226 (p. 594). The equation is

$$y'y''x - x^2yy' - xy^2 = 0 \tag{2.54}$$

and has the obvious point symmetry $G_1 = y\partial_y$. The invariants of G_1 are $u = x$ and $v = y'/y$. Under this reduction of order (2.54) becomes the Abel's equation of the second type

$$v\left(v' + v^2\right) - x^2v - x = 0 \tag{2.55}$$

from which no joy can be expected. Assuming a symmetry of the form $\eta\partial_y$ we find that η satisfies the equation

$$(\eta' y')' - (x^2\eta y)' = 0 \tag{2.56}$$

which, in addition to the solution given by G_1, gives the two nonlocal symmetries

$$G_2 = \exp\left[\int \frac{x^2 y}{y'}\mathrm{d}x\right]\partial_y \qquad \text{and}$$
$$G_3 = \left\{\exp\left[\int \frac{x^2 y}{y'}\mathrm{d}x\right]\int \frac{1}{y'}\exp\left[-\int \frac{x^2 y}{y'}\mathrm{d}x\right]\mathrm{d}x\right\}\partial_y. \tag{2.57}$$

From G_2 we obtain the invariants $u = x$ and $v = y'^2 - x^2y^2$. The reduced equation is

$$\frac{\mathrm{d}v}{\mathrm{d}u} = 0 \tag{2.58}$$

which is trivially integrated. We note that under this reduction G_1 and G_3 become $v\partial_v$ and ∂_v which are obvious symmetries of (2.58). Unfortunately neither of these two symmetries helps in the solution of

$$y'^2 - x^2y^2 = K \tag{2.59}$$

which is the first order equation which results from the integration of (2.58). As Cheb-Terrab and Roche [7] observe, (2.54) can be reduced to a first order equation the integration of which is not obvious.

11. 6.235 (p. 596). This equation is

$$f(y')y'' + g(y)y' + h(x) = 0 \tag{2.60}$$

and the two symmetries are

$$G_1 = \exp\left[-\int \frac{g}{f}\mathrm{d}x\right]\partial_y \qquad \text{and}$$
$$G_2 = \left\{\exp\left[-\int \frac{g}{f}\mathrm{d}x\right]\int \frac{\mathrm{d}x}{f}\exp\left[\int \frac{g}{f}\mathrm{d}x\right]\right\}\partial_y \tag{2.61}$$

obtained from the solution of

$$(\eta' f(y'))' + (\eta g)' = 0. \tag{2.62}$$

The invariants of G_1 are

$$u = x \qquad \text{and} \qquad v = F(y') + G(y), \tag{2.63}$$

where $F' = f$ and $G' = g$. The reduced equation is

$$\frac{\mathrm{d}v}{\mathrm{d}u} = -h(u) \tag{2.64}$$

from which one obtains the first integral

$$I = F(y') + G(y) + H(x), \tag{2.65}$$

where, in an obvious notation, $H' = h$. The arbitrariness of the functions in (2.65) makes it quite obvious that (2.60) can only have a single reduction of order, as Cheb-Terrab and Roche also observed.

3 Comments and observations

In the previous section we have seen that the integrability or otherwise of some of the equations discussed by Cheb-Terrab and Roche [7] in the context of the existence or otherwise of integrating factors can be explained in terms of the existence of symmetries. We note that these symmetries need not be point symmetries, but may include generalised and nonlocal symmetries. If one restricts the consideration of symmetry to point symmetries only, there is the apparent contradiction of the existence of an integrating factor and the lack of existence of the symmetry which is the source of the integrating factor. The inadvisability of ignoring symmetries which are not point symmetries has found expression not only in the question of the solution of differential equations [3, 12, 29] but also in the very question of the nature of the complete symmetry group of a differential equation [4, 17, 18, 28]. It is interesting to note that the reason for nonlocal symmetries to be of use in the reduction of order is that the nonlocality in the coefficient functions is not carried over to the invariants associated with the symmetry. In the case of an exponential nonlocal symmetry — a symmetry in which the nonlocal part is an exponential term common to both coefficient functions — the nonlocal part cancels from the associated Lagrange's system used to calculate the invariants. This is usually the way a nonlocal symmetry can provide local functions for the invariants. It is not invariably the situation as was observed for one of the integrable cases of the Hénon–Heiles system [30].

In this paper we have been concerned with the demonstration of the necessity for the existence of symmetry as a prelude to the possibility of the reduction of order and/or integration by means of integrating factors. As a practical tool the algorithm devised by Cheb-Terrab and Roche can only be expected to be of benefit for those who need to solve a differential equation in the course of their scientific investigations.

We pause for a moment to consider the case of third order equations and choose our examples from Chapter 7 of Kamke's book. As in the previous section the numbers refer to the equation number in Kamke's book and the page on which it occurs.

1. 7.6 (p. 601). Equation

$$x^2y''' + x(y-1)y'' + xy'^2 + (1-y)y' = 0 \tag{3.1}$$

is of Euler type and so we write it in the standard form

$$y''' + (y-4)y'' + y'(y' - 2y + 4) = 0. \tag{3.2}$$

The equation for η is

$$\eta''' - 4\eta'' + 4\eta' + \left(\eta''y + 2\eta'y' + \eta y''\right) - 2\left(\eta'y + \eta y'\right) = 0 \tag{3.3}$$

which we have written in this form to make the first integration to

$$\eta'' - 4\eta' + 4\eta + (\eta y)' - 2(\eta y) = A \tag{3.4}$$

obvious. It is evident that (3.4) is a candidate for the application of the shift theorem [15, Chapter 5] and, if we multiply it by $\exp[-2x]$, we obtain

$$(\eta\exp[-2x])'' + (\eta y\exp[-2x])' = A\exp[-2x]. \tag{3.5}$$

We obtain the three nonlocal symmetries

$$
\begin{aligned}
G_1 &= \left\{\exp[\int (2-y)\mathrm{d}x]\right\} \partial_y,\\
G_2 &= \left\{\exp[\int (2-y)\mathrm{d}x] \left[\int \exp[\int (y-2)\mathrm{d}x]\mathrm{d}x\right]\right\} \partial_y,\\
G_3 &= \left\{\exp[\int (2-y)\mathrm{d}x] \left[\int \exp[\int y\mathrm{d}x]\mathrm{d}x\right]\right\} \partial_y.
\end{aligned}
\tag{3.6}
$$

The first differential invariant of G_1 is $v = y' + \frac{1}{2}(y-2)^2$ and the reduced equation is

$$
\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} - 2\frac{\mathrm{d}v}{\mathrm{d}x} = 0. \tag{3.7}
$$

Under this reduction of order the other two nonlocal symmetries become the solution symmetries

$$
G_2 \Rightarrow \partial_v \qquad \text{and} \qquad G_3 \Rightarrow \mathrm{e}^{2x}\partial_v \tag{3.8}
$$

and, since $[G_2, G_3] = 0$, the two symmetries have Lie's Type II algebra.

2. 7.10 (p. 602). The equation

$$
2y'y''' - 3y''^2 = 0, \tag{3.9}
$$

known as the Kummer–Schwarz equation, has a symmetry of the desired form if the coefficient function, η, is a solution of the third order linear equation

$$
\eta' y''' + \eta''' y' - 3\eta'' y'' = 0. \tag{3.10}
$$

When one multiplies (3.10) by y'' and invokes (3.9), the resulting equation

$$
\left(\eta'' y' - \eta' y''\right) y''' - \left(\eta''' y'' - \eta'' y'''\right) y' = 0 \tag{3.11}
$$

is easily integrated to give a first order linear equation in η'

$$
\eta'' - \frac{3}{2}\frac{y''}{y'}\eta' = 2By'' \tag{3.12}
$$

so that

$$
\eta = A + By + C\int \mathrm{d}x y'^{3/2}. \tag{3.13}
$$

Our procedure gives the three symmetries

$$
G_1 = \partial_y, \qquad G_2 = y\partial_y, \qquad G_3 = \left[\int \mathrm{d}x y'^{3/2}\right]\partial_y. \tag{3.14}
$$

Under reduction of order using G_1 (3.9) becomes

$$
2vv'' - 3v'^2 = 0, \tag{3.15}
$$

where $u = x$ and $v = y'$ are the invariants of G_1. In terms of the new variables

$$G_2 = v\partial_v \qquad \text{and} \qquad G_3 = v^{3/2}\partial_v \tag{3.16}$$

and under reduction by G_3, for which the new variables are u and $w = v'/v^{3/2}$, (3.15) becomes

$$w' = 0 \quad \Leftrightarrow \quad \left(\frac{1}{v^{1/2}}\right)'' = 0. \tag{3.17}$$

The linearity in the second equation in (3.17) is to be expected since G_2 and G_3 constitute a representation of Lie's Type IV algebra.

Equation (3.9) is well-known to possess ten contact symmetries and so be equivalent under contract transformation to $y''' = 0$. Of the ten contact symmetries six are point symmetries giving a double representation of $sl(2, \mathbb{R})$. One of these representations is G_1, G_2 and $G_4 = y^2\partial_y$. It is amusing that our procedure yields G_3 and not G_4. Under the transformation $X = x$, $Y = 1/y$ (3.9) is invariant and G_1 and G_4 are interchanged. However, (3.10) is not invariant under this transformation.

3. 7.16 (p. 604). As an application of our procedure to what is formally a higher order equation the coefficient function for the equation

$$3y''y^{\text{iv}} - 5y'''^2 = 0 \tag{3.18}$$

is found from the solution of

$$3\left(\frac{\eta'''}{y'''}\right)' - 5\left(\frac{\eta''}{y''}\right)' = 0. \tag{3.19}$$

The four symmetries are

$$\begin{aligned} &G_1 = \partial_y, \qquad G_2 = x\partial_y, \\ &G_3 = y\partial_y, \qquad G_4 = \left[\int\int y''^{5/3}\mathrm{d}x\mathrm{d}x\right]\partial_y. \end{aligned} \tag{3.20}$$

If we reduce by G_1 with the variables $u = x$, $v = y'$, (3.18) and (3.20) become respectively

$$\begin{aligned} &3v'v''' - 5v''^2 = 0 \qquad \text{and} \\ &G_2 \Rightarrow \partial_v, \qquad G_3 \Rightarrow v\partial_v, \qquad G_4 \Rightarrow \left[\int v'^{5/3}\mathrm{d}x\right]\partial_v. \end{aligned} \tag{3.21}$$

The invariants associated with G_2 are u and $w = v'$ and we obtain

$$\begin{aligned} &3ww'' - 5w'^2 = 0 \qquad \text{and} \\ &G_3 \Rightarrow w\partial_w \qquad \text{and} \qquad G_4 \Rightarrow w^{5/3}\partial_w. \end{aligned} \tag{3.22}$$

The obvious symmetry for the next reduction is G_4 for which the variables are u and $z = w'/w^{5/3}$. The reduced equation is simply

$$\frac{\mathrm{d}z}{\mathrm{d}u} = 0. \tag{3.23}$$

Were we to use G_3, the variables would be u and $s = w'/w$ with the reduced equation

$$\frac{\mathrm{d}s}{\mathrm{d}u} = \frac{5}{3}s^2 \tag{3.24}$$

which is an equation of both Riccati and Bernoulli types. The difference in the ease of integrability using the two routes of reduction is slight.

In these few examples we have seen that the procedure proposed for the explanation, in terms of Lie symmetries, of the existence of integrating factors for the second order equations considered by Cheb-Terrab and Roche [7] is equally applicable to third and fourth order equations. Indeed we could have extended the range of examples to include the fifth order equation given by Kamke (7.17, p. 604), but, as this is a trivial extension of 7.9, there is nothing new to it. Of the few equations which we have considered here the most interesting one from the point of view of group theoretical properties of differential equations and reduction of order is doubtless the first, 7.6, for in this case all of the symmetries provided by our procedure were nonlocal.

Provided that the algorithm developed by Cheb-Terrab and Roche can be extended in a manageable fashion to include higher order equations, we have no doubt that integrating factors would be found for each of these equations. The very reason for the existence of the integrating factors is based in the existence of the symmetries which we have demonstrated above.

The examples which we have considered above are not particularly complicated, but then those of the previous section were not always very difficult. These examples were simply to act by way of illustration. In the practical context of the solution of third order equations the existence of an algorithm to calculate integrating factors as has been provided by Cheb-Terrab and Roche for second order equations would be a boon. One can be quite certain that the existence of these integrating factors is a consequence of the underlying symmetry of the equation, be that symmetry point, generalised or nonlocal.

To this happy conclusion we must recall that there is one marked discrepancy and that is Kamke's 6.108 [16, p. 570], which, according to Cheb-Terrab and Roche [7], was solvable even by two of the three less successful differential equation solvers. Braude [6] and Muller [22, 23] were unable to integrate it in terms of elementary functions. We have been unable to determine a Lie symmetry and yet Cheb-Terrab and Roche [7, p. 519] give the simple integrating factor y so that one should be able to write some first integral, $I(x, y, y')$, of (2.43) with the property that

$$\frac{\mathrm{d}I}{\mathrm{d}x} = y^2y'' + y^3 - y(ax + b). \tag{3.25}$$

Alternatively one should be able to write the right hand side of

$$I = \int \left[y^2y'' + y^3 - y(ax + b)\right] \mathrm{d}x \tag{3.26}$$

in terms of x, y and y' only. Our inability to do this should be a matter of some concern to those who rely upon packages such as that of Cheb-Terrab and Roche to solve the differential equations which arise in the course of their work.

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