Journal of Nonlinear Mathematical Physics Volume 9, Supplement 2 (2002), 110–121 **Birthday Issue**

Jacobi's Last Multiplier and the Complete Symmetry Group of the Euler–Poinsot System

M C NUCCI [†] *and P G L LEACH* [†‡]

[†] *Dipartimento di Matematica e Informatica, Università di Perugia, Perugia 06123, Italy*

[‡] *Permanent address: School of Mathematical and Statistical Sciences, University of Natal, Durban 4041, South Africa*

Received May, 2002

Abstract

The symmetry approach to the determination of Jacobi's last multiplier is inverted to provide a source of additional symmetries for the Euler–Poinsot system. These additional symmetries are nonlocal. They provide the symmetries for the representation of the complete symmetry group of the system.

1 Jacobi's last multiplier

The method of Jacobi's last multiplier [12, 13, 14] (see also [15, pp. 320, 335, 342–347] for a summary of these three papers of Jacobi) provides a means to determine an integrating factor, M, of the partial differential equation

$$Af = \sum_{i=1}^{n} a_i \frac{\partial f}{\partial x_i} = 0 \tag{1}$$

or its equivalent associated Lagrange's system

$$\frac{\mathrm{d}x_1}{a_1} = \frac{\mathrm{d}x_2}{a_2} = \cdots = \frac{\mathrm{d}x_n}{a_n}. \tag{2}$$

Provided sufficient information about the system (1)/(2) is known, the multiplier is given by

$$\frac{\partial(f, \omega_1, \omega_2, \ldots, \omega_{n-1})}{\partial(x_1, x_2, \ldots, x_n)} = MAf, \tag{3}$$

where

$$\frac{\partial(f, \omega_1, \omega_2, \ldots, \omega_{n-1})}{\partial(x_1, x_2, \ldots, x_n)} = \det \begin{bmatrix} \dfrac{\partial f}{\partial x_1} & \cdots & \dfrac{\partial f}{\partial x_n} \\ \dfrac{\partial \omega_1}{\partial x_1} & & \dfrac{\partial \omega_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial \omega_{n-1}}{\partial x_1} & \cdots & \dfrac{\partial \omega_{n-1}}{\partial x_n} \end{bmatrix} \neq 0 \tag{4}$$

and $\omega_1, \ldots, \omega_{n-1}$ are $n-1$ solutions of (1) or, equivalently, first integrals of (2). As a consequence, one can prove that each multiplier is a solution of the partial differential equation

$$\sum_{i=1}^{n} \frac{\partial \left(M a_i\right)}{\partial x_i} = 0. \tag{5}$$

A different combination of the integrals can produce a different multiplier, M'. The ratio M'/M is a solution of (1) or a first integral of (2), which may be trivial as in the application of the Poisson–Jacobi Theorem [29, 10] to the determination of additional first integrals. A scholarly essay on the history of the Poisson–Jacobi theorem which Jacobi considered [10]

> la plus profonde découverte de M. Poisson[1]

and the pervasive influence of Jacobi's work upon Lie can be found in [9].

In its original form the method of Jacobi's last multiplier required almost complete knowledge of the system under consideration. Since the existence of a first integral is consequent upon the existence of symmetry, one is not surprised that Lie [18, pp. 333–347] provided a symmetric route to the determination of Jacobi's last multiplier. A more transparent treatment is given by Bianchi [3, pp. 456–464].

Suppose that we know $n-1$ symmetries of (1)/(2)

$$X_i = \xi_{ij}\partial_{x_j}, \qquad i = 1, n-1. \tag{6}$$

Then Jacobi's last multiplier is also given by

$$M = \frac{1}{\Delta} \tag{7}$$

in the case that $\Delta \neq 0$, where now

$$\Delta = \det \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ \xi_{11} & & & \xi_{1,n} \\ \vdots & & & \vdots \\ \xi_{n-1,1} & \xi_{n-1,2} & \cdots & \xi_{n-1,n} \end{bmatrix}. \tag{8}$$

Jacobi's last multiplier provides an incestuous interrelationship between symmetries, first integrals and integrating factors for well-endowed systems. The practicality of this interrelationship was somewhat diminished in the past due to the effort required to evaluate the determinants of matrices of even moderate size. For example a Newtonian problem in three dimensions would require the evaluation of the determinant of a 6×6 matrix (7×7 if one considers time dependence, *ie* $a_1 = 1$). This possibly explains the omission of discussion of the method by postclassical authors such as Cohen [4], Dickson [5] and Eisenhart [6]. The ready availability of computer algebra systems has rendered this method an attractive alternative, say, for the determination of first integrals given symmetries. To take a trivial example the 'free particle' has the Newtonian equation

$$\ddot{y} = 0 \quad \Leftrightarrow \quad \begin{cases} \dot{u}_1 = u_2, \\ \dot{u}_2 = 0 \end{cases} \tag{9}$$

[1] 'the most profound discovery by Mr Poisson'; see also the letter of Jacobi [11].

with the Lie point symmetries

$$\begin{aligned}
&\Gamma_1 = \partial_{u_1}, && \Gamma_5 = t\partial_t - u_2\partial_{u_2},\\
&\Gamma_2 = t\partial_{u_1} + \partial_{u_2}, && \Gamma_6 = t^2\partial_t + tu_1\partial_{u_1} + (u_1 - tu_2)\,\partial_{u_2},\\
&\Gamma_3 = u_1\partial_{u_1} + u_2\partial_{u_2}, && \Gamma_7 = u_1\partial_t - u_2^2\partial_{u_2},\\
&\Gamma_4 = \partial_t, && \Gamma_8 = tu_1\partial_t + u_1^2\partial_{u_1} + (u_1 - tu_2)\,u_2\partial_{u_2}.
\end{aligned} \tag{10}$$

We find, for example, the determinants

$$\Delta_1 = \begin{vmatrix} 1 & u_2 & 0\\ 0 & t & 1\\ 0 & u_1 & u_2 \end{vmatrix} = tu_2 - u_1, \qquad \Delta_2 = \begin{vmatrix} 1 & u_2 & 0\\ 0 & u_1 & u_2\\ t^2 & tu_1 & u_1 - tu_2 \end{vmatrix} = (u_1 - tu_2)^2,$$

$$\Delta_3 = \begin{vmatrix} 1 & u_2 & 0\\ 1 & 0 & 0\\ tu_1 & u_1^2 & (u_1 - tu_2)\,u_2 \end{vmatrix} = -\,(u_1 - tu_2)\,u_2^2,$$

using Γ_2 and Γ_3; Γ_3 and Γ_6; and Γ_4 and Γ_8 respectively, and the integrals

$$I_1 = \frac{\Delta_2}{\Delta_1} = tu_2 - u_1, \qquad I_2 = \frac{\Delta_3}{\Delta_1} = u_2^2 \tag{11}$$

as expected. (Note that there are twenty-eight possible determinants to be calculated. Some of these are zero.)

2 Complete symmetry groups

In 1994 Krause [16] introduced a new concept of the complete symmetry group of a system by defining it as the group represented by the set of symmetries required to specify the system completely. There is not necessarily any relationship between the symmetries required to specify completely a system and its point symmetries. Thus Andriopoulos *et al* [1] reported the complete symmetry group of the 'free particle' to be $A_{3,3}$ $(D \oplus_s T_2)$, the semidirect sum of dilations and translations in the plane, with the symmetries being three of the usual eight Lie point symmetries of (9a), and that of the Ermakov–Pinney equation [7, 27] to be $A_{3,8}$ $(sl(2,\mathbb{R}))$ with the symmetries being the three Lie point symmetries of that equation. On the other hand Krause [16] reported that an additional three nonlocal symmetries are necessary to specify the Kepler Problem completely since the five Lie point symmetries of the three-dimensional Kepler Problem are insufficient to the purpose. This contains the implication that eight Lie symmetries are necessary to specify completely the Kepler Problem. However, a more careful analysis of sufficiency by Nucci *et al* [26], based on the method of reduction of order proposed by Nucci in 1996 [23] and her interactive code for the determination of Lie symmetries [21, 22], has revealed that the equation for the three-dimensional Kepler Problem is completely specified by six Lie symmetries with the algebra $A_1 \oplus \{A_1 \oplus_s \{2A_1 \oplus 2A_1\}\}$ $(\Leftrightarrow A_1 \oplus \{D \oplus_s \{T_2 \oplus T_2\}\})$, where the subalgebra $D \oplus_s T_2$ is that associated with the first integrals of the one-dimensional simple harmonic oscillator [19] (equally any second order ordinary differential equation with the algebra $sl(3,\mathbb{R})$) to which the Kepler Problem reduces naturally [25] under the method of reduction of order.

One assumes that, when he devised his method of the last multiplier, the original intention of Jacobi was to determine integrating factors and that the adaptation from integral to symmetries by Lie was of like intention. However, (8) in combination with (5) suggests the possibility to determine symmetries provided the multiplier is known. The general solution of (5) is equivalent to the solution of (2) unless one has the opportunity to perceive a particular solution without real effort. A particular case in point is when the functions $a_i(x)$ are independent of x_i for then (5) has the solution that M is a constant, taken to be chosen as a convenient value, which in this instance is not a 'trivial' solution. Since one now has an M, one can attempt to determine a further symmetry by solving (8) with one row of the matrix the coefficient functions of the unknown symmetry. One may infer that Lie and Bianchi had in mind point and contact symmetries in their treatments of Jacobi's last multiplier from the basis of symmetries rather than the original approach through first integrals used by Jacobi. However, as is common with many of the theoretical properties and applications of symmetries, there is no statement of the variable dependence of the coefficient functions in the method of Jacobi's last multiplier required for the method to hold. Consequently the considerations above apply equally to determination of nonlocal symmetries, in particular nonlocal symmetries of the type used by Krause, in which the nonlocality is found in the coefficient function of the independent variable, for autonomous systems. For such systems one of the known symmetries is ∂_t which is represented in the matrix of (8) by the row $(1, 0, \ldots, 0)$. In the Laplace expansion of the determinant the only possible nonzero terms must contain the first element of this row. If the unknown symmetry is

$$\Gamma_n = V\partial_t + G_i\partial_{u_i}, \tag{12}$$

V does not appear in the expression for Δ, only the G_i. These may be selected at will to satisfy the requirement that $\Delta = 1$, or other suitable constant, and for each selection a V be computed through the invariance of the system of first order ordinary differential equations under the action of $\Gamma_n^{[1]}$, the first extension of Γ_n. In principle this would permit n symmetries to be determined. However, that presumes the independence of the G_i, $i = 1, n$. This need not be the case if the u_i come from the reduction of an nth order scalar ordinary differential equation and the imposition of point symmetries is made at the level of the nth order equation.

3 The complete symmetry group of the Euler–Poinsot system

As an illustration of the ideas contained in §§ 1 and 2 we consider the simplest case of the motion of a rigid body which is the system governed by the Euler–Poinsot equations [8, 28]

$$\begin{aligned}
\dot\omega_1 &= \frac{B-C}{A}\,\omega_2\omega_3 = W_1,\\
\dot\omega_2 &= \frac{C-A}{B}\,\omega_3\omega_1 = W_2,\\
\dot\omega_3 &= \frac{A-B}{C}\,\omega_1\omega_2 = W_3
\end{aligned} \tag{13}$$

in which $\boldsymbol{\omega} := (\omega_1, \omega_2, \omega_3)^T$ is the angular velocity and A, B and C are the principal moments of inertia. It is a commonplace that the system (13) possesses respectively the two first integrals and Lie point symmetries

$$\begin{aligned} E &= \frac{1}{2}\left(A\omega_1^2 + B\omega_2^2 + C\omega_3^2\right), \qquad && L^2 = A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2, \\ \Gamma_1 &= \partial_t, && \Gamma_2 = -t\partial_t + \omega_i\partial_{\omega_i}. \end{aligned} \tag{14}$$

We note that Nucci's method of reduction of order [23] looks for first integrals in which one variable is missing [20] and provides [24]

$$\begin{aligned} I_1 &= \omega_2^2 B(A-B) - \omega_3^2 C(C-A), \\ I_2 &= \omega_3^2 C(B-C) - \omega_1^2 A(A-B), \\ I_3 &= \omega_1^2 A(C-A) - \omega_2^2 B(B-C) \end{aligned} \tag{15}$$

as three conserved quantities, which are obviously not independent and can be constructed from the two independent integrals, E and L^2, by the respective elimination of ω_1, ω_2 and ω_3 from them.

We note that ω_i is absent from the right hand side of $\dot{\omega}_i$ in (13) and so one solution of (5) for Jacobi's last multiplier is a constant. The condition that Γ_n (12) be a third symmetry of the Euler–Poinsot system (13) is

$$\Delta = \mathrm{Det}\begin{bmatrix} 1 & W_1 & W_2 & W_3 \\ 1 & 0 & 0 & 0 \\ -t & \omega_1 & \omega_2 & \omega_3 \\ V & G_1 & G_2 & G_3 \end{bmatrix} = a, \tag{16}$$

where a is the constant to which we may assign some convenient value. We obtain

$$A\omega_1 I_1 G_1 + B\omega_2 I_2 G_2 + C\omega_3 I_3 G_3 = aABC. \tag{17}$$

We take

$$G_1 = \frac{aBC}{\omega_1 I_1}, \qquad G_2 = 0, \qquad G_3 = 0 \qquad \textit{et cyc} \tag{18}$$

and set $a = I_1/BC$ *et cyc* to obtain the three sets of coefficient functions for the dependent variables

$$\left(\frac{1}{\omega_1}, 0, 0\right); \qquad \left(0, \frac{1}{\omega_2}, 0\right); \qquad \left(0, 0, \frac{1}{\omega_3}\right). \tag{19}$$

We note that the sets in (19) provide a basis and that other combinations could be taken. We keep the forms (19) simply for their present simplicity and find their subsequent utility.

It remains to determine V. Consider the first set in (19). The corresponding symmetry is written as

$$\Gamma_3 = V_3\partial_t + \frac{1}{\omega_1}\partial_{\omega_1}, \tag{20}$$

$$\Gamma_3^{[1]} = \Gamma_3 - \dot{\omega}_1\left(\frac{1}{\omega_1^2} + \dot{V}_3\right)\partial_{\dot{\omega}_1} - \dot{\omega}_2\dot{V}_3\partial_{\dot{\omega}_2} - \dot{\omega}_3\dot{V}_3\partial_{\dot{\omega}_3} \tag{21}$$

and the action of $\Gamma_3^{[1]}$ on the Euler–Poinsot system, (13), is

$$
\begin{aligned}
&-\dot{\omega}_1\left(\frac{1}{\omega_1^2}+\dot{V}_3\right)=0,\\
&\frac{\dot{\omega}_2}{\omega_1^2}=\frac{C-A}{B}\frac{\omega_3}{\omega_1},\\
&\frac{\dot{\omega}_3}{\omega_1^2}=\frac{A-B}{C}\frac{\omega_2}{\omega_1}
\end{aligned}
\tag{22}
$$

in which (22b,c) are consistent with (13) and (22a) gives

$$
V_3=-\int\frac{\mathrm{d}t}{\omega_1^2}. \tag{23}
$$

A similar calculation applies to second and third of (19).

We obtain the three nonlocal symmetries

$$
\begin{aligned}
\Gamma_3&=-\left(\int\frac{\mathrm{d}t}{\omega_1^2}\right)\partial_t+\frac{1}{\omega_1}\partial_{\omega_1},\\
\Gamma_4&=-\left(\int\frac{\mathrm{d}t}{\omega_2^2}\right)\partial_t+\frac{1}{\omega_2}\partial_{\omega_2},\\
\Gamma_5&=-\left(\int\frac{\mathrm{d}t}{\omega_3^2}\right)\partial_t+\frac{1}{\omega_3}\partial_{\omega_3}
\end{aligned}
\tag{24}
$$

for the Euler–Poinsot system (13).

Since the system (13) is of the first order and autonomous, it is in a suitable form for reduction of order. We set $y=\omega_3$ as the new independent variable. This is an arbitrary choice. Equally ω_1 or ω_2 could be chosen as the independent variable. The results do not differ. The reduced system is

$$
\begin{aligned}
\frac{\mathrm{d}\omega_1}{\mathrm{d}y}&=\frac{(B-C)C}{A(A-B)}\frac{y}{\omega_1},\\
\frac{\mathrm{d}\omega_2}{\mathrm{d}y}&=\frac{(C-A)C}{B(A-B)}\frac{y}{\omega_2}
\end{aligned}
\tag{25}
$$

and inherits the symmetries

$$
\begin{aligned}
&\tilde{\Gamma}_2=\omega_1\partial_{\omega_1}+\omega_2\partial_{\omega_2}+y\partial_y,\\
&\tilde{\Gamma}_3=\frac{1}{\omega_1}\partial_{\omega_1},\qquad \tilde{\Gamma}_4=\frac{1}{\omega_2}\partial_{\omega_2},\qquad \tilde{\Gamma}_5=\frac{1}{y}\partial_y
\end{aligned}
\tag{26}
$$

with the Lie Brackets

$$
\begin{aligned}
&\left[\tilde{\Gamma}_2,\tilde{\Gamma}_3\right]=-2\tilde{\Gamma}_3,\quad \left[\tilde{\Gamma}_3,\tilde{\Gamma}_4\right]=0,\quad \left[\tilde{\Gamma}_4,\tilde{\Gamma}_5\right]=0\\
&\left[\tilde{\Gamma}_2,\tilde{\Gamma}_4\right]=-2\tilde{\Gamma}_4,\quad \left[\tilde{\Gamma}_3,\tilde{\Gamma}_5\right]=0,\\
&\left[\tilde{\Gamma}_2,\tilde{\Gamma}_5\right]=-2\tilde{\Gamma}_5.
\end{aligned}
\tag{27}
$$

Consider a general two-dimensional system

$$\frac{\mathrm{d}\omega_1}{\mathrm{d}y} = f_1(\omega_1,\omega_2,y),$$
$$\frac{\mathrm{d}\omega_2}{\mathrm{d}y} = f_2(\omega_1,\omega_2,y), \tag{28}$$

of which the system (25) is a specific instance. We determine which of the four symmetries $\tilde{\Gamma}_2,\ldots,\tilde{\Gamma}_5$ are necessary to specify (25) given (28). (This does beg the question of the appropriateness of these four symmetries, but this will eventually become apparent.)

The actions of

$$\tilde{\Gamma}_2^{[1]} = \omega_1\partial_{\omega_1} + \omega_2\partial_{\omega_2} + y\partial_y,$$
$$\tilde{\Gamma}_3^{[1]} = \frac{1}{\omega_1}\partial_{\omega_1} - \frac{\omega_1'}{\omega_1^2}\partial_{\omega_1'},$$
$$\tilde{\Gamma}_4^{[1]} = \frac{1}{\omega_2}\partial_{\omega_2} - \frac{\omega_2'}{\omega_2^2}\partial_{\omega_2'} \tag{29}$$

on (28 a) give

$$0 = y\frac{\partial f_1}{\partial y} + \omega_1\frac{\partial f_1}{\partial \omega_1} + \omega_2\frac{\partial f_1}{\partial \omega_2},$$
$$-\frac{f_1}{\omega_1^2} = \frac{1}{\omega_1}\frac{\partial f_1}{\partial \omega_1},$$
$$0 = \frac{1}{\omega_2}\frac{\partial f_1}{\partial \omega_2} \tag{30}$$

from which it is evident that

$$f_1(\omega_1,\omega_2,y) = K\frac{y}{\omega_1} \tag{31}$$

and hence (25a) is recovered.

The same symmetries acting on (28b) lead to (25b) and so the three symmetries, $\tilde{\Gamma}_2$, $\tilde{\Gamma}_3$ and $\tilde{\Gamma}_4$ are a representation of the complete symmetry group of the system (25). By means of a similar calculation we see that the triplets of $\tilde{\Gamma}_2$, $\tilde{\Gamma}_3$ and $\tilde{\Gamma}_5$ and of $\tilde{\Gamma}_2$, $\tilde{\Gamma}_4$ and $\tilde{\Gamma}_5$ are also representations of the complete symmetry group of (25).

The listing of Lie Brackets in (27) shows that the Lie algebra in each of the three cases is $A_1 \oplus_s 2A_1$ or $D \oplus_s T_2$, a representation of the pseudo-Euclidean group $E(1,1)$ of dilations and translations in the plane.

This is not be end of the story. Consider about the actions of $\tilde{\Gamma}_3^{[1]}$, $\tilde{\Gamma}_4^{[1]}$ and $\tilde{\Gamma}_5^{[1]}$ on (28a). We obtain three constraints on f_1, *videlicet*

$$-\frac{f_1}{\omega_1^2} = \frac{1}{\omega_1}\frac{\partial f_1}{\partial \omega_1},$$
$$0 = \frac{1}{\omega_2}\frac{\partial f_1}{\partial \omega_2},$$
$$\frac{f_1}{y^2} = \frac{1}{y}\frac{\partial f_1}{\partial y} \tag{32}$$

when (28a) is taken into account. It is obvious that (25a) is recovered. Similarly (28b) reduces to (25b).

Consequently the three symmetries $\tilde{\Gamma}_3$, $\tilde{\Gamma}_4$ and $\tilde{\Gamma}_5$ provide a representation of the complete symmetry group of (25). In this case the algebra of the symmetries is the abelian $3A_1$[2] and not the $A_1 \oplus_s 2A_1$ of the previous algebras.

The system (25) is *not* the system (13) and one cannot *expect* that the complete symmetry group of (13) would be that of (25) although from the point of view of differential equations both are equally described in terms of a three-dimensional phase space. We consider the two triplets, *videlicet* Γ_3, Γ_4 and Γ_5 and Γ_2, Γ_3 and Γ_4 (equivalently Γ_2, Γ_3 and Γ_5 and Γ_2, Γ_4 and Γ_5 as we saw above) and their respective actions on the general system.

$$\begin{aligned}
\dot{\omega}_1 &= f_1(t, \omega_1, \omega_2, \omega_3),\\
\dot{\omega}_2 &= f_2(t, \omega_1, \omega_2, \omega_3),\\
\dot{\omega}_3 &= f_3(t, \omega_1, \omega_2, \omega_3)
\end{aligned} \tag{33}$$

to which class the Euler–Poinsot system (13) belongs.

The actions of the first extensions of Γ_3, Γ_4 and Γ_5, *videlicet*

$$\begin{aligned}
\Gamma_3^{[1]} &= \left(-\int \frac{\mathrm{d}t}{\omega_1^2}\right)\partial_t + \frac{1}{\omega_1}\partial_{\omega_1} + \frac{\dot{\omega}_2}{\omega_1^2}\partial_{\dot{\omega}_2} + \frac{\dot{\omega}_3}{\omega_1^2}\partial_{\dot{\omega}_3},\\
\Gamma_4^{[1]} &= \left(-\int \frac{\mathrm{d}t}{\omega_2^2}\right)\partial_t + \frac{1}{\omega_2}\partial_{\omega_2} + \frac{\dot{\omega}_1}{\omega_2^2}\partial_{\dot{\omega}_1} + \frac{\dot{\omega}_3}{\omega_2^2}\partial_{\dot{\omega}_3},\\
\Gamma_5^{[1]} &= \left(-\int \frac{\mathrm{d}t}{\omega_3^2}\right)\partial_t + \frac{1}{\omega_3}\partial_{\omega_3} + \frac{\dot{\omega}_1}{\omega_3^2}\partial_{\dot{\omega}_1} + \frac{\dot{\omega}_2}{\omega_3^2}\partial_{\dot{\omega}_2},
\end{aligned} \tag{34}$$

on (33a) are

$$\begin{aligned}
0 &= -T_1\frac{\partial f_1}{\partial t} + \frac{1}{\omega_1}\frac{\partial f_1}{\partial \omega_1},\\
\frac{f_1}{\omega_2^2} &= -T_2\frac{\partial f_1}{\partial t} + \frac{1}{\omega_2}\frac{\partial f_1}{\partial \omega_2},\\
\frac{f_1}{\omega_3^2} &= -T_3\frac{\partial f_1}{\partial t} + \frac{1}{\omega_3}\frac{\partial f_1}{\partial \omega_3}
\end{aligned} \tag{35}$$

in which we have written $T_i = \int \mathrm{d}t/\omega_i^2$. The system (35) does not contain sufficient information to reduce (33a) to the first of system (13). The same applies for (33b) and (33c). The abelian group $3A_1$ represented by Γ_3, Γ_4 and Γ_5 is not the complete symmetry group of (13).

The first extensions of Γ_2 is

$$\Gamma_2^{[1]} = -t\partial_t + \omega_1\partial_{\omega_1} + \omega_2\partial_{\omega_2} + \omega_3\partial_{\omega_3} + 2\dot{\omega}_1\partial_{\dot{\omega}_1} + 2\dot{\omega}_2\partial_{\dot{\omega}_3}. \tag{36}$$

[2] At first glance the existence of $3A_1$ for a second order system may seem to be at odds with the general result that a second order ordinary differential equation — a second order system — cannot admit $3A_1$. However, (25) cannot be written as a scalar second order ordinary differential equation. This emphasises the general point that a higher-order scalar equation may be reduced to a system of first order ordinary differential equations, but the reverse process is not always possible.

The actions of this, $\Gamma_3^{[1]}$ and $\Gamma_4^{[1]}$ on (33a) are

$$\begin{aligned}
&2f_1 = -t\frac{\partial f_1}{\partial t} + \omega_1\frac{\partial f_1}{\partial \omega_1} + \omega_2\frac{\partial f_1}{\partial \omega_2} + \omega_3\frac{\partial f_1}{\partial \omega_3},\\
&0 = -T_1\frac{\partial f_1}{\partial t} + \frac{1}{\omega_1}\frac{\partial f_1}{\partial \omega_1},\\
&\frac{f_1}{\omega_2^2} = -T_2\frac{\partial f_1}{\partial t} + \frac{1}{\omega_2}\frac{\partial f_1}{\partial \omega_2}.
\end{aligned} \tag{37}$$

From the first of (37)

$$f_1 = t^{-2}F_1(u, v, w), \tag{38}$$

where u, v and w are the three characteristics independent of f_1, *videlicet* $t\omega_1$, $t\omega_2$ and $t\omega_3$. The substitution of (38) into (37b) and (37c) gives, respectively,

$$\begin{aligned}
&\frac{1}{u}\frac{\partial F_1}{\partial u} = \frac{T_1}{t^3}\left[-2F_1 + u\frac{\partial F_1}{\partial u} + v\frac{\partial F_1}{\partial v} + w\frac{\partial F_1}{\partial w}\right],\\
&\frac{1}{v}\frac{\partial F_1}{\partial v} - \frac{F_1}{v^2} = \frac{T_2}{t^3}\left[-2F_1 + u\frac{\partial F_1}{\partial u} + v\frac{\partial F_1}{\partial v} + w\frac{\partial F_1}{\partial w}\right]
\end{aligned} \tag{39}$$

and now the situation is entirely different since the t dependence outside of the characteristics is isolated in the coefficients of the terms within crochets in both (39a) and (39b). Consequently we have the three terms separately zero, *ie*

$$\begin{aligned}
&\frac{1}{u}\frac{\partial F_1}{\partial u} = 0,\\
&\frac{1}{v}\frac{\partial F_1}{\partial v} - \frac{F_1}{v^2} = 0,\\
&u\frac{\partial F_1}{\partial u} + v\frac{\partial F_1}{\partial v} + w\frac{\partial F_1}{\partial w} = 2F_1.
\end{aligned} \tag{40}$$

We recover (13a). Like calculations recover (13b) and (13c).

The complete symmetry group of the Euler–Poinsot system, (13), is $E(1,1)$ ($\Leftrightarrow D\otimes_s T_2$). There are three equivalent representations.

4 Discussion

In this paper we have brought together several disparate ideas to arrive at something of a question mark. The symmetry-based version of Jacobi's last multiplier has been turned on its head, as it were, to provide a means to calculate nonlocal symmetries, in this instance, for the Euler–Poinsot system given the last multiplier. With these nonlocal symmetries we can identify the set of symmetries which specify completely the Euler–Poinsot system from all possible classes of systems of three first order equations. The number of symmetries required for this specification is three.

This is surprising in terms of general expectations. The general first order equation for a system with one independent and n dependent variables is

$$\dot{\omega}_i = f_i(t, \omega_1, \ldots, \omega_n). \tag{41}$$

With the application of each symmetry the number of variables in f_i is reduced by one so that, after the application of n symmetries, there remains a general function of one characteristic. The application of a further symmetry specifies that general function. Thus we would expect that the complete symmetry group would have a representation in terms of an algebra of $n+1$ elements. In fact one would regard an algebra of fewer than $n+1$ elements as being quite unusual since it would imply an unexpected degree of connectedness amongst the coefficient functions of the different variables in the symmetries. (Such an example is found most dramatically in the case of the Kepler Problem [26].)

It is already known [1] that a given system may possess more than one representation of its complete symmetry group. Indeed this is quite standard for any equation of the second order possessing eight symmetries.

In the case of the Euler–Poinsot system we have found a rather intriguing result. In the reduced two-dimensional system, (25), we found that they were two inequivalent representations of the complete symmetry group, *videlicet* $A_1 \oplus_s 2A_1$ and $3A_1$. This lack of uniqueness does not persist when one returns to the three-dimensional system, properly known as the Euler–Poinsot system, for then at the latter symmetry group, *videlicet* $3A_1$, falls away as a representation of the complete symmetry group. We do find that in conjunction with Γ_1 the three symmetries of $3A_1$ do specify (13), but the four-dimensional algebra is not a candidate as a representation of the complete symmetry group since its dimensionality is not minimal.

It would be interesting to find other examples of systems exhibiting similar properties[3]. Certainly the present result does place something of a question mark against the interpretation of the concept of a complete symmetry group as being *the* group of the symmetries which completely specify the equation although it does this without detracting from the inherent interest of the concept of a complete symmetry group. In fact one must seriously consider the identification of the characteristic system for a given problem, in this case whether it is the three-dimensional system (13), which is the standard Euler–Poinsot system, or the reduced two-dimensional system (25). Curiously enough a similar question has arisen in the case of the Painlevé Property for systems of first-order ordinary differential equations [17].

Acknowledgements

PGLL thanks the Dipartimento di Matematica e Informatica, Università di Perugia, and Dr MC Nucci for their kind hospitality while this work was undertaken and the National Research Foundation of South Africa and the University of Natal for their continuing support.

[3]For a recent instance of which see the paper by Andriopoulos *et al* [2] in the present volume.

References

[1] Andriopoulos K, Leach P G L and Flessas G P, Complete Symmetry Groups of Ordinary Differential Equations and Their Integrals: Some Basic Considerations, *J. Math. Anal. Appl.* **262** (2001), 256–273.

[2] Andriopoulos K and Leach P G L, The Economy of Complete Symmetry Groups for Linear Higher Dimensional Systems, *J. Nonlin. Math. Phys.* **9**, Suppl. 2 (2002), 10–23.

[3] Bianchi L, Lezione sulla teoria dei gruppi continui finiti di transformazioni, Enrico Spoerri, Pisa, 1918.

[4] Cohen A, An Introduction to the Lie Theory of One-Parameter Groups, Stechert, New York, reprinted 1931.

[5] Dickson LE, Differential Equations from the Group Standpoint, *Ann. Math.* **25** (1925), 287–378.

[6] Eisenhart L P, Continuous Groups of Transformations, Dover, New York, 1961.

[7] Ermakov V, Second Order Differential Equations. Conditions of Complete Integrability, *Univ. Izvestia Kiev, Ser III* **9** (1880), 1–25 (translated by A O Harin).

[8] Euler L, Theoria motus corporum solidorum seu rigidorum ex primis nostræcognitionis principiis stabilita et ad omnes motus, qui in huiusmodi corpora cadere possunt, accommodata, Rostochii et Gryphiswaldiae, Litteris et Impensis Röse AF, 1765.

[9] Hawkins T, Jacobi and the Birth of Lie's Theory of Groups, *Arch. Hist. Exact Sciences* **42** (1991), 187–278.

[10] Jacobi C G J, Sur un théorème de Poisson, *Comptes Rendus Acad. Sci. Paris* **11** (1840), 529–530.

[11] Jacobi C G J, Adressé à M. le president de l'Academie des Sciences, *J. de Math.* **V** (1840), 350–355 (including editor's notes).

[12] Jacobi C G J, Sul principio dell'ultimo moltiplicatore, e suo uso come nuovo principio generale di meccanica, *Giornale arcadico di scienze, lettere ed arti* **Tomo 99** (1844), 129–146.

[13] Jacobi C G J, Theoria novi multiplicatoris systemati æquationum differentialum vulgarium applicandi: Pars I, *J. für Math.* **27** (1844), 199.

[14] Jacobi C G J, Theoria novi multiplicatoris systemati æquationum differentialum vulgarium applicandi: Pars II, *J. für Math.* **29** (1845), 213.

[15] Koenigsberger L, Carl Gustav Jacob Jacobi, B T Teubner, Leipzig, 1904.

[16] Krause J, On the Complete Symmetry Group of the Classical Kepler System, *J. Math. Phys.* **35** (1994), 5734–5748.

[17] Leach P G L, Cotsakis S and Flessas G P, Symmetry, Singularity and Integrability in Complex Dynamics: I The Reduction Problem, *J. Nonlin. Math. Phys.* **7** (2000), 445–479.

[18] Lie S, Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformationen, B T Teubner, Leipzig, 1912.

[19] Mahomed F M and Leach P G L, Lie Algebras Associated with Scalar Second Order Ordinary Differential Equations, *J. Math. Phys.* **30** (1989), 2770–2775.

[20] Marcelli M and Nucci M C, Lie Point Symmetries and First Integrals: the Kowalevsky top, Preprint, RT 2002-1, Dipartimento di Matematica e Informatica, Università di Perugia, 2002.

[21] Nucci M C, Interactive REDUCE Programs for Calculating Classical, Non-Classical and Lie–Bäcklund Symmetries of Differential Equations, Preprint, GT Math: 0620902-051, Department of Mathematics, Georgia Institute of Technology, Atlanta, Georgia, USA, 1990.

[22] Nucci M C, Interactive REDUCE Programs for Calculating Lie Point, Nonclassical, Lie–Bäcklund and Approximate Symmetries of Differential Equations: Manual and Floppy Disk, in CRC Handbook of Lie Group Analysis of Differential Equations, Vol. 3, New Trends in Theoretical Developments and Computational Methods, Editor: Ibragimov N H, CRC Press, Boca Raton, 1996, 415–481.

[23] Nucci M C, The Complete Kepler Group Can Be Derived by Lie Group Analysis, *J. Math. Phys.* **37** (1996), 1772–1775.

[24] Nucci M C, Lorenz Integrable System Moves à la Poinsot, Preprint, RT 2002-15, Dipartimento di Matematica e Informatica, Università di Perugia, 2002.

[25] Nucci M C and Leach P G L, The Harmony in the Kepler and Related Problems, *J. Math. Phys.* **42** (2001), 746–764.

[26] Nucci M C, Andriopoulos K and Leach P G L, The Ermanno–Bernoulli Constants and Representations of the Complete Symmetry Group of the Kepler Problem, Preprint, School of Mathematical and Statistical Sciences, University of Natal, Durban 4041, Republic of South Africa, 2002.

[27] Pinney E, The Nonlinear Differential Equation $y''(x)+p(x)y+cy^{-3}=0$, *Proc. Amer. Math. Soc.* **1** (1950), 681.

[28] Poinsot L, Théorie nouvelle de la rotations des corps, Paris, 1834 (republished in *J. Math. Pures Appl.* **16** (1851), 289–336).

[29] Poisson S, Mémoire sur la variation des constantes arbitraires dans les questions de mécanique, *J. École Poly* **15** (1809), 266–344.