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On a One-Phase Stefan Problem in Nonlinear Conduction

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Abstract

A one phase Stefan problem in nonlinear conduction is considered. The problem is shown to admit a unique solution for small times. An exact solution is obtained which is a travelling front moving with constant speed.

1 Introduction

One and two phase Stefan problems for the linear heat equation have been the subject of many studies in the past [1, 2]. Indeed these problems have a great physical relevance since they provide a mathematical model for the processes of phase changes [3, 4].

The boundary between the two phases is a free boundary: its motion has to be determined as part of the solution.

More recently the previous analysis was extended to nonlinear diffusion models. In [5, 6] exact solutions were found in parametric form for a class of Stefan problems in nonlinear heat conduction.

Moreover one and two-phase Stefan problems for the Burgers equation were solved in [7, 8] and explicit travelling wave solutions were obtained.

It is the aim of this paper to formulate and solve a one-phase Stefan problem for the nonlinear heat equation:

$$\frac{\vartheta_t}{\vartheta^2} = \vartheta_{xx}, \qquad \vartheta(x,t), \qquad t > 0, \tag{1.1}$$

on the semiinfinite domain $x \in (-\infty, s(t))$ characterized by the following set of initial and boundary data

$$\vartheta(x,0) = \vartheta_0(x), \qquad x \in (-\infty, b) \tag{1.2a}$$

and , $\vartheta_0(b) = \beta_2 < 0$, $b > 0$,

$$\vartheta(-\infty,t) = \beta_1 > 0, \quad t \geq 0, \qquad \beta_{1>}|\vartheta_0(x)| > |\beta_2|, \qquad \vartheta_x(-\infty,t) = 0, \tag{1.2b}$$

$$\vartheta(s(t),t) = \beta_2, \quad t \geq 0, \qquad \text{with} \qquad s(0) = b, \tag{1.2c}$$

$$\vartheta_x(s(t),t) = -\dot{s}(t). \tag{1.2d}$$

Equation (1.1) a well know mathematical model for heat conduction in solid crystalline hydrogen for situations with one-dimensional spatial symmetry [9]. The one-phase Stefan problem (1.1)–(1.2) is associated to a phase change (fusion) in such a system.

In the above relations β_1, β_2 and b are constants; the unknown function $s(t)$ describes the motion of the free boundary and has to be determined together with $\vartheta(x,t)$. Moreover, equation (1.2d) is a condition on the flux at the free boundary, arising from heat balance (energy) considerations.

In the following we assume $\vartheta_0(x)$ to be a continuously differentiable function of its argument.

Our analysis is based on the approach followed in [7] for the solution of a one-phase Stefan problem for the Burgers equation.

In the next Section we reduce the problem (1.1)–(1.2) to a nonlinear integral equation for the independent variable t.

In Section 3 we prove existence and uniqueness of the solution for small intervals of time; in the last Section we show that the system admits an exact solution which travels with a constant velocity proportional to the velocity of the free boundary.

Some details of the proof presented in the third Section are given in the Appendix.

2 Linearization

In order to linearize equation (1.1) we introduce the transformation

$$\psi(z,t) = \vartheta(x,t), \qquad z = z(x,t), \qquad z_x = \frac{1}{\vartheta}, \qquad z_t = -\vartheta_x, \tag{2.1}$$

whose compatibility $z_{xt} = z_{tx}$is easily proved via (1.1). Under this transformation equation (1.1) is mapped into

$$\psi_t = \psi_{zz} \tag{2.2}$$

on the domain $-\infty < z < \bar{z}(t)$ with $\bar{z}(t) = z(s(t),t)$ and $\bar{z}(0) = \bar{b}$.

(2.2) is the linear heat equation for the dependent variable $\psi(z,t)$ with initial datum given by

$$\psi(z,0) = \psi_0(z_0) = \vartheta_0(x), \tag{2.3a}$$

where

$$z_0 \equiv z_0(x) = \int_{-\infty}^{x} \frac{1}{\vartheta_0(x')} dx'. \tag{2.3b}$$

The boundary conditions (1.2b) and (1.2c) take now the form

$$\begin{aligned} &\psi(-\infty,t) = \vartheta(-\infty,t) = \beta_1, \\ &\psi_z(-\infty,t) = \vartheta_x(-\infty,t)\vartheta(-\infty,t) = 0 \end{aligned} \tag{2.3c}$$

and

$$\begin{aligned} &\psi(\bar{z}(t),t) = \beta_2, \\ &\psi_z(\bar{z}(t),t) = -\beta_2 \dot{s}(t). \end{aligned} \tag{2.3d}$$

The Stefan problem for the nonlinear equation (1.1) has then been mapped into a classical Stefan problem for the heat equation (2.2) with initial datum (2.3a), characterized by the boundary conditions (2.3d) at the free boundary.

We say $\psi(z,t)$, $\bar{z}(t)$ form a solution of the above Stefan problem for $t < \sigma$, $0 < \sigma < \infty$, when: (i) $\psi(z,t)$ is a solution of (2.2) satisfying (2.3), it exists and is continuous together with its derivatives for $-\infty < z < \bar{z}(t)$, $0 \le t < \sigma$; (ii) $s(t)$ is a continuously differentiable function for $0 \le t < \sigma$.

In the following we outline a method to prove the existence and uniqueness of the solution for small times, $t < \sigma$.

We first observe that by integrating the second relation in (2.3d) we get

$$s(t) = b - \frac{1}{\beta_2}\int_0^t \psi_z(\bar{z}(t'),t')dt', \tag{2.4a}$$

which in turn implies

$$\bar{z}(t) = h(t) - \frac{1}{\beta_2}\int_0^t \psi_z(\bar{z}(t'),t')dt', \tag{2.4b}$$

with

$$h(t) = \int_0^{s(t)} \frac{1}{\vartheta_0(x')}\,dx'. \tag{2.4c}$$

Next we turn our attention to the solution of (2.2). We introduce the fundamental kernel of the heat equation

$$K(z-\xi,t-\tau) = \frac{1}{2\sqrt{\pi}}\frac{1}{\sqrt{t-\tau}}\exp\left[-\frac{(z-\xi)^2}{4(t-\tau)}\right], \tag{2.5}$$

and integrate Green's identity for the heat equation

$$\frac{\partial}{\partial\xi}\left(K\frac{\partial\psi}{\partial\xi} - \psi\frac{\partial K}{\partial\xi}\right) - \frac{\partial}{\partial\tau}(K\psi) = 0, \tag{2.6}$$

over the domain $-\infty < \xi < \bar{z}(\tau)$, $\varepsilon < \tau < \tau - \varepsilon$ and let $\varepsilon \to 0$. Using $\psi(\bar{z}(\tau),\tau) = \beta_2$ and $K(z-\xi,0) = \delta(z-\xi)$, we obtain

$$\begin{aligned}\psi(z,t) = {}& \int_{-\infty}^{\bar{b}} K(z-\xi,t)\psi_0(\xi)d\xi - \frac{1}{\beta_2}\int_0^t K(z-\bar{z}(\tau),t-\tau)\psi_z(\bar{z}(\tau),\tau)d\tau \\ & - \beta_2\int_0^t K_\xi(\bar{z}(\tau),t-\tau)d\tau,\end{aligned} \tag{2.7}$$

with $\bar{z}(t)$ and $h(t)$ given by (2.4b) and (2.4c) respectively.

In the right hand side of (2.7) $\psi_z(\bar{z}(t),t)$ is unknown; it is convenient to take the z-derivative of both sides in (2.7) and evaluate it as $z \to z(t)^-$.

By putting $\nu(t) = \psi_z(\bar{z}(t),t)$, we obtain:

$$\begin{aligned}\nu(t) = {}& \left(1 + \frac{1}{2\beta_2}\right)^{-1}\left[-\psi_0(\bar{b})K(\bar{z}(t)-\bar{b},t) + \int_{-\infty}^{\bar{b}} K(\bar{z}(t)-\xi,t)\psi_0'(\xi)d\xi\right. \\ & \left. - \frac{1}{\beta_2}\int_0^t K_z(\bar{z}(t)-\bar{z}(\tau),t-\tau)d\tau - \beta_2\int_0^t K_\tau(\bar{z}(\tau),t-\tau)d\tau\right],\end{aligned} \tag{2.8a}$$

with

$$\bar{z}(t) = h(t) - \frac{1}{\beta_2}\int_0^t \nu(\tau)d\tau. \tag{2.8b}$$

Thus the solution of the Stefan problem (2.2), (2.3a), (2.3d) has been reduced to the solution of the nonlinear integral equation (2.8a) and (2.8b) for the independent variable t.

Once the existence and uniqueness of the function $\nu(t)$ is established for $0 \leq t < \sigma$, there follows via (2.7) the existence and uniqueness of $\psi(z,t)$ (and of $\vartheta(x,t)$) for $0 \leq t < \sigma$.

3 Contraction Mapping

In order to analyze existence properties of $v(t)$ for $0 \leq t < \sigma$, we denote by S_M the closed sphere $\|\nu\| < M$ in the Banach space of functions $\nu(t)$ continuous for $0 \leq t < \sigma$ with the uniform norm $\|\nu\| = \text{l.u.b.}\,|\nu(t)|$. On the sphere S_M define the transformation

$$w = T\nu, \tag{3.1}$$

where $T\nu$ coincides with the right hand side of (2.8a). We first prove that T is a mapping of S_M into itself. From (2.8b) we obtain

$$|\bar{z}(t)| \leq |h(t)| + \frac{1}{|\beta_2|}\left|\int_0^t \nu(\tau)d\tau\right| < \frac{M}{|\beta_2|}\left(1 + \frac{1}{|\beta_2|}\right)\sigma \equiv B_1\sigma, \tag{3.2}$$

where (2.4c) have been used. From (2.8b) we also get

$$\begin{aligned} |\bar{z}(t) - \bar{z}(t')| &\leq |h(t) - h(t')| + \frac{1}{|\beta_2|}\left|\int_0^t \nu(\tau)d\tau\right| \\ &< \frac{M}{|\beta_2|}\left(1 + \frac{1}{|\beta_2|}\right)|t - t'| \equiv B_1|t - t'|. \end{aligned} \tag{3.3}$$

We now turn our attention to the right hand side of (3.1). We first note that

$$|K(\bar{z}(t) - \bar{b}, t)| < B_2\sigma, \tag{3.4}$$

where B_2 is an appropriate constant depending on $\bar{b}$. Next we consider the integral terms in the right hand side of (3.1). We can write:

$$\begin{aligned} &\left|\int_{-\infty}^{\bar{b}} K(\bar{z}(t) - \xi, t)\psi_0'(\xi)d\xi\right| \\ &\qquad \leq \|\psi_0'\|\left|\int_{-\infty}^{\bar{b}} \frac{1}{2\sqrt{\pi}}\frac{1}{\sqrt{t}}\exp\left(-\frac{(\bar{z}(t) - \xi)^2}{4t}\right)d\xi\right| \leq \frac{\|\psi_0'\|}{\sqrt{\pi}} \equiv A_1, \end{aligned} \tag{3.5a}$$

and

$$\left|\beta_2\int_0^t K_\tau(\bar{z}(t), t - \tau)d\tau\right| \leq |\beta_2|\frac{1}{2\sqrt{\pi}}\frac{1}{\sqrt{t}}\exp\left[-\bar{z}^2(t)/4t\right] < |\beta_2|B_2\sigma. \tag{3.5b}$$

Moreover we get

$$\left|\frac{1}{\beta_2}\int_0^t K_z(\bar z(t)-\bar z(t'),t-t')\nu(t')dt'\right|$$
$$\leq \frac{M}{2|\beta_2|}\frac{1}{\sqrt{\pi}}\int_0^t \frac{|\bar z(t)-\bar z(t')|}{(t-t')^{\frac{3}{2}}}dt' < \frac{M}{|\beta_2|}\frac{1}{\sqrt{\pi}}B_1\sqrt{\sigma}, \tag{3.5c}$$

where (3.3) has been used.

We now use the inequality $\left(1+\frac{1}{2|\beta_2|}\right) > \frac{1}{2}$ and define M as $M = 2A_1+1$. When (3.1) is used together with (3.4) and (3.5) we get

$$\|w\| \leq M \tag{3.6}$$

provided we choose $\sigma = \min(\sigma_1,\sigma_2)$ with $\sigma_1 : (|\beta_2|+|\psi_0(\bar b)|)B_2\sigma_1 < \frac{1}{4}$ and $\sigma_2 : MB_1\sqrt{\sigma_2} < \frac{|\beta_2|\sqrt{\pi}}{4}$; thus the mapping is closed.

Next we wish to prove that T is a contraction; i.e. given two solutions of (3.1) with $\|\nu-\hat\nu\| = \delta$, it follows that $\|t\nu - t\hat\nu\| = \vartheta\delta$ with $0<\vartheta<1$. Using (2.8b) we have for small enough δ

$$|\bar z(t)-\hat{\bar z}(t')| \leq \frac{1}{|\beta_2|}\delta t\left|\int_{\hat s(t)}^{s(t)}\frac{1}{\vartheta_0(x')}dx'\right| \leq \frac{1}{|\beta_2|}\,\delta t\left(1+\frac{1}{|\beta_2|}\right) \equiv B_3\delta t, \tag{3.7}$$

where (2.4c) has also been used.

Similar estimates hold for $\dot{\bar z}(t)$, which will be useful in the following. From (2.8b) we see that $\dot{\bar z}(t)$ is bounded

$$|\dot{\bar z}(t)| \leq |\dot h(t)| + \frac{1}{|\beta_2|}|\nu(t)| \leq \frac{M}{|\beta_2|}\left(1+\frac{1}{|\beta_2|}\right) \equiv B_3 M, \tag{3.8a}$$

moreover it is

$$|\dot{\bar z}(t)-\dot{\hat z}(t')| \leq \frac{3}{|\beta_2|}\,\delta. \tag{3.8b}$$

¿From (3.1) we now write

$$w-\hat w = \left(1+\frac{1}{2\beta_2}\right)^{-1}\sum_{i=1}^{4}H_i \tag{3.9a}$$

with

$$H_{1=}\psi_0(\bar b)\left[K(\hat{\bar z}(t)-\bar b,t)-K(\bar z(t)-\bar b,t)\right], \tag{3.9b}$$
$$H_2 = \int_{-\infty}^{\bar b}\psi_0'(\xi)\left[K(\bar z(t)-\xi,t)-K(\hat{\bar z}(t)-\xi,t)\right]d\xi, \tag{3.9c}$$
$$H_3 = \frac{1}{\beta_2}\int_0^t\left[K_z(\hat{\bar z}(t)-\hat{\bar z}(t'),t-t')\nu(t')-K_z(\bar z(t)-\bar z(t'),t-t')\hat\nu(t')\right]dt', \tag{3.9d}$$
$$H_4 = \beta_2\int_0^t\left[K_{t'}(\hat{\bar z}(t),t-t')-K_{t'}(\bar z(t),t-t')\right]dt'. \tag{3.9e}$$

First we estimate H_1. We use the mean value theorem together with (3.7) and (3.2) in the right hand side of (3.9b); we get

$$|H_1| \le \|\psi_0\| \frac{1}{4\sqrt{\pi t}} B_1 B_3 \delta t < \frac{\|\psi_0\|}{4\sqrt{\pi t}} B_1 B_3 \sqrt{\sigma}\,\delta \equiv B_4 \sqrt{\sigma}\,\delta. \tag{3.10}$$

The estimate of H_2 in (3.9c) is obtained by writing

$$|H_2| \le \|\psi_0'\| \frac{1}{\sqrt{\pi}} \left| \int_{\bar{y}}^{\hat{\bar{y}}} e^{-y^2} dy \right|, \tag{3.11a}$$

with $\bar{y} = \frac{\bar{z}(t)-\bar{b}}{2\sqrt{t}}$ and $\hat{\bar{y}} = \frac{\hat{\bar{z}}(t)-\bar{b}}{2\sqrt{t}}$; we then obtain

$$|H_2| \le \|\psi_0'\| \frac{1}{2\sqrt{\pi}} \left|\bar{z}(t) - \hat{\bar{z}}(t)\right| < \frac{\|\psi_0'\|}{2\sqrt{\pi}} B_3 \sqrt{\sigma}\,\delta \equiv B_5 \sqrt{\sigma}\,\delta, \tag{3.11b}$$

where (3.7) has been used.

The estimate of H_3 is somewhat more cumbersome; a detailed analysis is given in the Appendix (see (A.1)–(A.6)).

There obtains

$$|H_3| < B_6 \sqrt{\sigma}\,\delta \tag{3.12}$$

where B_6 is defined in (A.7).

We finally turn our attention to the estimate of H_4 in (3.9e). When (2.5) is used, we get from the integral in the right hand side of (3.9e)

$$|H_4| \le \frac{|\beta_2|}{2\sqrt{\pi t}} \left|\exp(-\hat{\bar{z}}^2(t)/4t) - \exp(-\hat{\bar{z}}^2(t)/4t)\right| < \frac{|\beta_2|}{4\sqrt{\pi}} B_1 B_3 \sqrt{\sigma}\delta \equiv B_7 \sqrt{\sigma}\,\delta, \tag{3.13}$$

where use of the mean value theorem together with (3.2) and (3.7) has been made.

¿From (3.9a) we now write

$$|w - \bar{w}| \le \frac{2|\beta_2|}{1 + 2|\beta_2|} \sum_{i=1}^{4} |H_i|,$$

which in turn implies, when we combine together the estimates (3.10)–(3.13):

$$\frac{\|w - \bar{w}\|}{\delta} < \sqrt{\sigma} \sum_{i=4}^{7} B_i \equiv \sqrt{\sigma}\, B_8; \tag{3.14}$$

thus we conclude that if σ satisfies $\sigma < \min(\sigma_1, \sigma_2, \sigma_3)$ where

$$\sqrt{\sigma}\, B_8 < \sigma \tag{3.15}$$

it follows that T is a contraction operator on S_M, which admits a unique fixed point $v = Tv$ in S_M for $0 \le t < \sigma$.

We have then proven the existence and uniqueness of the solution of the integral equation (2.8a) for a small interval of time.

4 A Particular Solution

We now turn our attention to a particular solution of the Stefan problem (1.1), (1.2). Namely, we consider a moving front solution of equation (2.2)

$$\psi(z,t) = \beta_1 \left(1 - e^{-V(z-Vt)}\right), \tag{4.1a}$$

with

$$V < 0, \tag{4.1b}$$

which is travelling to the left with constant speed V and is compatible with the boundary conditions (2.3c). We now impose on (4.1a) the Stefan boundary conditions (2.3d): the first one implies

$$\bar{z} = \bar{b} + Vt, \tag{4.2a}$$

which in turn gives

$$\dot{\bar{z}} = V = -\frac{1}{b} \ln\left(1 + \frac{|\beta_2|}{\beta_1}\right). \tag{4.2b}$$

The boundary $\bar{z}(t)$ and the front solution (4.1a) are then both moving to the left with the same constant velocity.

When we next use the second boundary condition (2.3d), keeping into account the first one, we obtain

$$\dot{s}(t) = \frac{(\beta_2 - \beta_1)}{\beta_2} V, \tag{4.3}$$

which shows that the moving boundary $s(t)$ of the Stefan problem (1.1)–(1.2) is moving to the left with constant speed $\dot{s} = \alpha V$, $\alpha = \frac{\beta_2 \beta_1}{\beta_2} > 1$.

Finally, the solution of the one-phase Stefan problem for the nonlinear heat equation (1.1) is given by

$$\vartheta(x,t) = \left(\frac{\partial z}{\partial x}\right)^{-1}, \tag{4.4}$$

where, in virtue of (2.1), $z(x,t)$ solves

$$x = \int_0^z \psi(z',t)dz', \tag{4.5}$$

with $\psi(z,t)$ given by (4.1a) and the speed V specified by (4.2b).

We emphasize that the above solution is a very special solution of the Stefan problem (1.1), (1.2). Indeed, it corresponds to particular case when the nonlinear integral equation (2.8) reduces to a linear integral equation of Volterra type in t, as implied by substituting back (4.2a) into (2.8a).

Appendix

In order to estimate H_3, starting from (3.9d) we write

$$H_3 = \frac{1}{\beta_2}\left[-\int_0^t dt'\bar{\nu}(t')\frac{\hat{\bar{z}}(t)-\hat{\bar{z}}(t')}{t-t'}K(\hat{\bar{z}}(t)-\hat{\bar{z}}(t'),t-t') + \int_0^t dt'\nu(t')\frac{\bar{z}(t)-\bar{z}(t')}{t-t'}K(\bar{z}(t)-\bar{z}(t'),t-t')\right], \tag{A.1}$$

with $K(z-\xi, t-\tau)$ given by (2.5).

Next, we put

$$H_3 = \frac{1}{\beta_2}(J_1+J_2+J_3), \tag{A.2a}$$

$$J_1 = -\int_0^t dt'\left\{(\hat{\nu}(t')-\nu(t'))\frac{\hat{\bar{z}}(t)-\hat{\bar{z}}(t')}{t-t'}\,K(\hat{\bar{z}}(t)-\hat{\bar{z}}(t'),t-t')\right\}, \tag{A.2b}$$

$$J_2 = -\int_0^t dt'\left\{\nu(t')\left[\frac{\hat{\bar{z}}(t)-\hat{\bar{z}}(t')}{t-t'}-\frac{\bar{z}(t)-\bar{z}(t')}{t-t'}\right]K(\hat{\bar{z}}(t)-\hat{\bar{z}}(t'),t-t')\right\}, \tag{A.2c}$$

$$J_3 = -\int_0^t dt'\left\{\nu(t')\frac{\bar{z}(t)-\bar{z}(t')}{t-t'}\,K(\bar{z}(t)-\bar{z}(t'),t-t') \times\left[1-\exp\left\{-\frac{(\hat{\bar{z}}(t)-\hat{\bar{z}}(t'))^2-(\bar{z}(t)-\bar{z}(t'))^2}{4(t-t')}\right\}\right]\right\}. \tag{A.2d}$$

By using (3.3) we estimate J_1:

$$|J_1| \le \frac{1}{\sqrt{2\pi}}\int_0^t |\hat{\nu}(t')-\nu(t')|\left|\frac{\hat{\bar{z}}(t)-\hat{\bar{z}}(t')}{t-t'}\right|\frac{dt'}{\sqrt{t-t'}} \le \left(\frac{B_1}{\sqrt{\pi}}\sqrt{\sigma}\right)\delta. \tag{A.3}$$

The estimate of J_2 uses (3.8b) and the mean value theorem:

$$\begin{aligned}|J_2| &\le \frac{1}{\sqrt{2\pi}}\int_0^t |\nu(t')|\left|\frac{\hat{\bar{z}}(t)-\hat{\bar{z}}(t')}{t-t'}-\frac{\bar{z}(t)-\bar{z}(t')}{t-t'}\right|\frac{dt'}{\sqrt{t-t'}}\,\delta \\ &\le \frac{M}{\sqrt{\pi}}\int_0^\sigma |\hat{\bar{z}}(\vartheta)-\dot{z}(\vartheta)|\frac{dt'}{\sqrt{t-t'}} \le \left(3\frac{M}{\sqrt{\pi}}\frac{1}{|\beta_2|}\sqrt{\sigma}\right)\delta.\end{aligned} \tag{A.4}$$

Finally, for the estimate of J_3, we call

$$\begin{aligned}Q &= -\frac{(\hat{\bar{z}}(t)-\hat{\bar{z}}(t'))^2-(\bar{z}(t)-\bar{z}(t'))^2}{4(t-t')} \\ &= \frac{[(\bar{z}(t)-\hat{\bar{z}}(t'))-(\bar{z}(t)-\bar{z}(t'))][(\bar{z}(t)-\bar{z}(t'))-(\hat{\bar{z}}(t)-\hat{\bar{z}}(t'))]}{4(t-t')}.\end{aligned} \tag{A.5a}$$

Then from (3.3) and (3.7) we have

$$|Q| < \frac{1}{4|t-t'|}\,4B_1B_3|t-t'|\sigma\delta,$$

which gives

$$|Q| < B_1 B_3 \sigma\delta. \tag{A.5b}$$

We also note that we can estimate Q via (3.3):

$$\begin{aligned} |Q| &< \frac{1}{4|t-t'|}\left[(\hat{\bar{z}}(t)-\bar{z}(t'))^2 + (\bar{z}(t)-\bar{z}(t'))^2\right] \\ &\le \frac{1}{2}B_1^2|t-t'| < \frac{1}{2}B_1^2\sigma < B_1^2, \qquad \frac{\sigma}{2} < 1. \end{aligned} \tag{A.5c}$$

Using $\left|1-e^{-Q}\right| \le |Q|e^{|Q|}$ and (3.3) we estimate J_3:

$$\begin{aligned} |J_3| &\le \frac{1}{2\sqrt{\pi}}\int_0^\sigma |\nu(t')| \left|\frac{\bar{z}(t)-\bar{z}(t')}{t-t'}\right| \frac{1}{\sqrt{t-t'}}\left|1-e^{-Q}\right| dt' \\ &< \frac{M}{\sqrt{\pi}}\, B_1^2 B_3 e^{B_1^2}\sigma^{3/2}\delta < \left(\frac{|\beta_2|}{4} B_1 B_3 e^{B_1^2}\sqrt{\sigma}\right)\delta, \end{aligned} \tag{A.6}$$

where the definition of σ following (3.6) has also been used.

Combining the estimates of J_1, J_2 and J_3, we have from (A2.a)

$$|H_3| = \frac{1}{|\beta_2|}\left(\frac{B_1}{\sqrt{\pi}} + \frac{3M}{\sqrt{\pi}}\frac{1}{|\beta_2|} + \frac{|\beta_2|}{4}\, B_1 B_3 e^{B_1^2}\right)\sqrt{\sigma}\,\delta \equiv B_6\sqrt{\sigma}\,\delta. \tag{A.7}$$

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